VALUES OF INTEGRALS IN TERMS OF k^2.

By *J. W. L. Glaisher.*

§ 1. The investigations contained in the present paper, which relates principally to the expansion of definite integrals in powers of h, were suggested by some of the formulæ in the preceding paper, in which corresponding expansions in powers of λ were obtained.

The derivative of $\int_0^K \operatorname{sn}^m u \operatorname{cn}^n u \operatorname{dn}^r u \, du$, §§ 2–4.

§ 2. The formula

$$\frac{d}{dh}\int_0^K \operatorname{sd}^n u\, du = \frac{n+1}{2}\int_0^K \operatorname{sd}^{n+2} u \, du,$$

which was proved in § 2 of the preceding paper (p. 110), may be generalised in the following manner.

Putting $\operatorname{sn} u = x^{\frac{1}{2}}$, we find

$$\int_0^K \operatorname{sn}^m u \operatorname{cn}^n u \operatorname{dn}^p u \, du = \tfrac{1}{2}\int_0^1 x^{\frac{1}{2}m-\frac{1}{2}}(1-x)^{\frac{1}{2}n-\frac{1}{2}}(1-hx)^{\frac{1}{2}p-\frac{1}{2}}dx.$$

Expanding $(1-hx)^{\frac{1}{2}p-\frac{1}{2}}$ in ascending powers of h, the coefficient of h^r in this integral

$$=(-1)^r\frac{(\frac{1}{2}p-\frac{1}{2})(\frac{1}{2}p-\frac{3}{2})\dots\{\frac{1}{2}p-\frac{1}{2}(2r-1)\}}{r!}\int_0^1 x^{r+\frac{1}{2}m-\frac{1}{2}}(1-x)^{\frac{1}{2}n-\frac{1}{2}}dx$$

$$=(-1)^r\frac{(p-1)(p-3)\dots(p-2r+1)}{2^r . r!}\,\frac{\Gamma\left(r+\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(r+\frac{m+n}{2}+1\right)}.$$

We thus find

$$\int_0^K \operatorname{sn}^m u \operatorname{cn}^n u \operatorname{dn}^p u\, du = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n}{2}+1\right)}$$

$$\times\left\{1-\frac{(m+1)(p-1)}{2(m+n+2)}h+\frac{(m+1)(m+3)(p-1)(p-3)}{2.4(m+n+2)(m+n+4)}h^2\right.$$

$$\left.-\frac{(m+1)(m+3)(m+5)(p-1)(p-3)(p-5)}{2.4.6(m+n+2)(m+n+4)(m+n+6)}h^3+\&c.\right\}.$$

The series in brackets is equal to the hypergeometric series $F(\alpha, \beta, \gamma, h)$, if

$$\alpha = \frac{m+1}{2}, \quad \beta = \frac{1-p}{2}, \quad \gamma = \frac{m+n}{2}+1,$$

so that we may write the formula

$$\int_0^K \mathrm{sn}^m u \, \mathrm{cn}^n u \, \mathrm{dn}^p u \, du$$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n}{2}+1\right)} F\left(\frac{m+1}{2}, \frac{1-p}{2}, \frac{m+n}{2}+1, h\right).$$

Now

$$\frac{d}{dx} F(\alpha, \beta, \gamma, x) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1, x),$$

so that

$$\frac{d}{dh}\int_0^K \mathrm{sn}^m u \, \mathrm{cn}^n u \, \mathrm{dn}^p u \, du = \frac{(m+1)(1-p)}{2(m+n+2)} \frac{\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n}{2}+1\right)}$$

$$\times F\left(\frac{m+3}{2}, \frac{3-p}{2}, \frac{m+n}{2}+2, h\right)$$

$$= \frac{1-p}{2} \frac{\Gamma\left(\frac{m+3}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n}{2}+2\right)} F\left(\frac{m+3}{2}, \frac{3-p}{2}, \frac{m+n}{2}+2, h\right).$$

Putting $m+2$ and $p-2$ for m and p in the formula proved above, we have

$$\int_0^K \mathrm{sn}^{m+2} u \, \mathrm{cn}^n u \, \mathrm{dn}^{p-2} u \, du$$

$$= \frac{\Gamma\left(\frac{m+3}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n}{2}+2\right)} F\left(\frac{m+3}{2}, \frac{3-p}{2}, \frac{m+n}{2}+2, h\right),$$

whence we find

$$\frac{d}{dh}\int_0^K \mathrm{sn}^m u \, \mathrm{cn}^n u \, \mathrm{dn}^p u \, du = \frac{1-p}{2}\int_0^K \mathrm{sn}^{m+2} u \, \mathrm{cn}^n u \, \mathrm{dn}^{p-2} u \, du,$$

or, since $d\lambda = -2dh$,

$$\frac{d}{d\lambda}\int_0^K \mathrm{sn}^m u\,\mathrm{cn}^n u\,\mathrm{dn}^p u\,du = \frac{p-1}{4}\int_0^K \mathrm{sn}^{m+2} u\,\mathrm{cn}^n u\,\mathrm{dn}^{p-2} u\,du.$$

In order that the integrals may be finite it is only necessary that m and n should both be > -1.

It is curious that the external factor on the right-hand side of the equation should be independent of m and n, and that the exponent of $\mathrm{cn}\,u$ should be the same in both integrals.

§ 3. By putting $p = -m$, the formula becomes

$$\frac{d}{d\lambda}\int_0^K \mathrm{sd}^m u\,\mathrm{cn}^n u\,du = -\frac{m+1}{4}\int_0^K \mathrm{sd}^{m+2} u\,\mathrm{cn}^n u\,du.$$

We thus see that the formula in § 2 is unaltered by introducing the function $\mathrm{cn}^n u$ under the integral sign.

§ 4. The formula in § 2 may also be expressed in the forms:

$$\frac{d}{dh}\int_0^K \mathrm{sd}^m u\,\mathrm{cd}^n u\,\mathrm{nd}^p u\,du = \frac{m+n+p+1}{2}\int_0^K \mathrm{sd}^{m+2} u\,\mathrm{cd}^n u\,\mathrm{nd}^p u\,du,$$

m and n being both > -1;

$$\frac{d}{dh}\int_0^K \mathrm{sc}^m u\,\mathrm{nc}^n u\,\mathrm{dc}^p u\,du = \frac{1-p}{2}\int_0^K \mathrm{sc}^{m+2} u\,\mathrm{nc}^n u\,\mathrm{dc}^{p-2} u\,du,$$

m being > -1 and $m + n + p < 1$;

$$\frac{d}{dh}\int_0^K \mathrm{ns}^m u\,\mathrm{cs}^n u\,\mathrm{ds}^p u\,du = \frac{1-p}{2}\int_0^K \mathrm{ns}^m u\,\mathrm{cs}^n u\,\mathrm{ds}^{p-2} u\,du,$$

n being > -1 and $m + n + p < 1$.

Particular cases of the general series, §§ 5–10.

§ 5. By putting $p = -m$ in the general formula of § 2, we find

$$\int_0^K \mathrm{sd}^m u\,\mathrm{cn}^n u\,du = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n}{2}+1\right)}\left\{1 + \frac{(m+1)^2}{2\,(m+n+2)}h + \frac{(m+1)^2\,(m+3)^2}{2.4\,(m+n+2)\,(m+n+4)}h^2 + \&c.\right\},$$

whence, putting $n = -m$ and replacing m by n,

$$\int_0^K \left(\frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u}\right)^n du = \frac{\pi}{2\cos\frac{1}{2}n\pi} \times \left\{1 + \frac{(n+1)^2}{2^2} h + \frac{(n+1)^2(n+3)^2}{2^2.4^2} h^2 + \&c.\right\}.$$

In this formula n must lie between -1 and $+1$.

§ 6. Similarly, by putting $p = m$ in the general formula, we find

$$\int_0^K \operatorname{sn}^m u \operatorname{cn}^n u \operatorname{dn}^m u\, du = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n}{2}+1\right)}$$

$$\times \left\{1 + \frac{1^2 - m^2}{2(m+n+2)} h + \frac{(1^2-m^2)(3^2-m^2)}{2.4(m+n+2)(m+n+4)} h^2 + \&c.\right\},$$

whence, putting $m = -n$,

$$\int_0^K \left(\frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u}\right)^n du = \frac{\pi}{2\cos\frac{1}{2}n\pi}\left\{1 + \frac{1^2-n^2}{2^2} h + \frac{(1^2-n^2)(3^2-n^2)}{2^2.4^2} h^2 + \&c.\right\}.$$

In this formula also n must lie between $+1$ and -1.

§ 7. The series in the last formula is unaltered by changing the sign of n. This is as it should be, for by substituting $K - u$ for u in the integral, we find

$$\int_0^K \left(\frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u}\right)^n du = \int_0^K \left(\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\right)^n du.$$

§ 8. By putting $K - u$ for u in the last formula of § 4, we find

$$\int_0^K \left(\frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}\right)^n du = \frac{\pi}{2\cos\frac{1}{2}n\pi} k'^n \left\{1 + \frac{(n+1)^2}{2^2} h + \frac{(n+1)^2(n+3)^2}{2^2.4^2} h^2 + \&c.\right\},$$

and also, by changing the sign of n in the same formula,

$$\int_0^K \left(\frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}\right)^n du = \frac{\pi}{2\cos\frac{1}{2}n\pi}\left\{1 + \frac{(1-n)^2}{2^2}h + \frac{(1-n)^2(3-n)^2}{2^2.4^2}h^2 + \&c.\right\}.$$

In these formulæ n must lie between $+1$ and -1.

§ 9. It is easily shown that these two results are consistent with each other, for by putting

$$\alpha = \frac{n+1}{2},\ \beta = \frac{n+1}{2},\ \gamma = 1,\ x = h$$

in the formula

$$F(\gamma-\alpha,\ \gamma-\beta,\ \gamma,\ x) = (1-x)^{\alpha+\beta-\gamma}F(\alpha,\beta,\gamma,x),$$

we deduce

$$F\left(\frac{1-n}{2},\ \frac{1-n}{2},\ 1,\ h\right) = h'^n F\left(\frac{1+n}{2},\ \frac{1+n}{2},\ 1,\ h\right),$$

which is the result obtained by equating the two values of the integral.

§ 10. In connexion with the results in §§ 5–8, we may also notice the formulæ

$$\int_0^K \left(\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}\right)^n du = \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{2\Gamma(n+1)}\left\{1 + \frac{(n+1)^2}{4(n+1)}h + \frac{(n+1)^2(n+3)^2}{4.8(n+1)(n+2)}h^2 + \frac{(n+1)^2(n+3)^2(n+5)^2}{4.8.12(n+1)(n+2)(n+3)}h^3 + \&c.\right\},$$

$$\int_0^K \left(\frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u}\right)^n du = \frac{\Gamma^2\left(\frac{1-n}{2}\right)}{2\Gamma(1-n)}\left\{1 + \frac{(1-n)^2}{4(1-n)}h + \frac{(1-n)^2(3-n)^2}{4.8(1-n)(2-n)}h^2 + \frac{(1-n)^2(3-n)^2(5-n)^2}{4.8.12(1-n)(2-n)(3-n)}h^3 + \&c.\right\}.$$

In the first of these equations n must be > -1, and in the second n must be < 1. Both integrals are unaltered by substituting $K-u$ for u.

Series for $\int_0^K \mathrm{sn}^n u\, du$, *&c. in ascending powers of* h, §§ 11, 12.

§ 11. By giving special values to m, n, p in the general formula of § 2, we obtain the following system of results, which is perhaps worth placing on record ;

$$\int_0^K \mathrm{sn}^n u\, du = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n}{2}+1\right)}\left\{1+\frac{n+1}{2(n+2)}h + \frac{1.3(n+1)(n+3)}{2.4(n+2)(n+4)}h^2 + \&c.\right\},$$

$$\int_0^K \mathrm{cn}^n u\, du = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n}{2}+1\right)}\left\{1+\frac{1^2}{2(n+2)}h + \frac{1^2.3^2}{2.4(n+2)(n+4)}h^2 + \&c.\right\},$$

$$\int_0^K \mathrm{dn}^n u\, du = \tfrac{1}{2}\pi\left\{1-\frac{n-1}{2^2}h + \frac{1.3(n-1)(n-3)}{2^2.4^2}h^2 - \&c.\right\},$$

$$\int_0^K \mathrm{cd}^n u\, du = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n}{2}+1\right)}\left\{1+\frac{n+1}{2(n+2)}h + \frac{1.3(n+1)(n+3)}{2.4(n+2)(n+4)}h^2 + \&c.\right\},$$

$$\int_0^K \mathrm{sd}^n u\, du = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n}{2}+1\right)}\left\{1+\frac{(n+1)^2}{2(n+2)}h + \frac{(n+1)^2(n+3)^2}{2.4(n+2)(n+4)}h^2 + \&c.\right\},$$

$$\int_0^K \mathrm{nd}^n u\, du = \tfrac{1}{2}\pi \left\{1 + \frac{n+1}{2^2} h + \frac{1.3\,(n+1)(n+3)}{2^2.4^2} h^2 + \&c.\right\},$$

$$\int_0^K \mathrm{dc}^n u\, du = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma\left(1-\frac{n}{2}\right)} \left\{1 + \frac{n-1}{2\,(n-2)} h + \frac{1.3\,(n-1)(n-3)}{2.4\,(n-2)(n-4)} h^2 + \&c.\right\},$$

$$\int_0^K \mathrm{nc}^n u\, du = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma\left(1-\frac{n}{2}\right)} \left\{1 - \frac{1^2}{2\,(n-2)} h + \frac{1^2.3^2}{2.4\,(n-2)(n-4)} h^2 - \&c.\right\},$$

$$\int_0^K \mathrm{sc}^n u\, du = \frac{2\cos\frac{1}{2}n\pi}{\pi} \left\{1 + \frac{n+1}{2^2} h + \frac{1.3\,(n+1)(n+3)}{2^2.4^2} h^2 + \&c.\right\},$$

$$\int_0^K \mathrm{ns}^n u\, du = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma\left(1-\frac{n}{2}\right)} \left\{1 + \frac{n-1}{2\,(n-2)} h + \frac{1.3\,(n-1)(n-3)}{2.4\,(n-2)(n-4)} h^2 + \&c.\right\}.$$

$$\int_0^K \mathrm{ds}^n u\, du = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma\left(1-\frac{n}{2}\right)} \left\{1 - \frac{(n-1)^2}{2\,(n-2)} h + \frac{(n-1)^2(n-3)^2}{2.4\,(n-2)(n-4)} h^2 - \&c.\right\},$$

$$\int_0^K \operatorname{cs}^n u\, du = \frac{\pi}{2\cos\frac{1}{2}n\pi}\left\{1 - \frac{n-1}{2^2}h + \frac{1.3\,(n-1)(n-3)}{2^2.4^2}h^2 - \&c.\right\}.$$

In the formulæ involving sn, cn, cd, sd, n must be > -1; in those involving dc, nc, ns, ds, n must be < 1; in those involving dn and nd, n is unrestricted; and in those involving sc and cs, n must lie between $+1$ and -1. These conditions must be satisfied in order that the integrals may be finite. It will be noticed that when n is intermediate to $+1$ and -1 all the twelve integrals are finite.

The case $n = 0$, § 12.

§ 12. When $n = 0$, all the twelve formulæ reduce to

$$K = \frac{\pi}{2}\left\{1 + \frac{1^2}{2^2}h + \frac{1^2.3^2}{2^2.4^2}h^2 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}h^3 + \&c.\right\}.$$

This is also the case with the series for $\int_0^K \left(\frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u}\right)^n du$, &c. in §§ 5–10.

All of these series may therefore be regarded as generalisations of the above well-known series for K.

The case $n = 1$, § 13.

§ 13. Since

$$\int_0^K \operatorname{sn} u\, du = \frac{1}{2k}\log\frac{1+k}{1-k}, \quad \int_0^K \operatorname{sd} u\, du = \frac{\gamma}{kk'},$$

$$\int_0^K \operatorname{cn} u\, du = \frac{\gamma}{k}, \quad \int_0^K \operatorname{cd} u\, du = \frac{1}{2k}\log\frac{1+k}{1-k},$$

$$\int_0^K \operatorname{dn} u\, du = \frac{\pi}{2}, \quad \int_0^K \operatorname{nd} u\, du = \frac{\pi}{2k'},$$

we find, by putting $n = 1$ in the formulæ of § 10,

$$\frac{1}{2k}\log\frac{1+k}{1-k} = 1 + \frac{1}{3}h + \frac{1}{5}h^2 + \frac{1}{7}h^3 + \&c.,$$

$$\frac{\gamma}{k} = 1 + \frac{1^2}{3!}h + \frac{1^2.3^2}{5!}h^2 + \frac{1^2.3^2.5^2}{7!}h^3 + \&c.,$$

$$\frac{\gamma}{kk'} = 1 + \frac{2^2}{3!}h + \frac{2^2.4^2}{5!}h^2 + \frac{2^2.4^2.6^2}{7!}h^3 + \&c.,$$

$$\frac{\pi}{2k'} = \frac{\pi}{2}\left\{1 + \frac{2!}{2^2}h + \frac{4!}{2^2.4^2}h^2 + \frac{6!}{2^2.4^2.6^2}h^3 + \&c.\right\},$$

which are very easily verified.

The case $n=2$, § 14.

§ 14. Similarly, since

$$\int_0^K \mathrm{sn}^2 u\, du = -\frac{I}{h}, \quad \int_0^K \mathrm{sd}^2 u\, du = \frac{G}{hh'},$$

$$\int_0^K \mathrm{cn}^2 u\, du = \frac{G}{h}, \quad \int_0^K \mathrm{cd}^2 u\, du = -\frac{I}{h},$$

$$\int_0^K \mathrm{dn}^2 u\, du = E, \quad \int_0^K \mathrm{nd}^2 u\, du = \frac{E}{h'},$$

we find, by putting $n=2$,

$$-\frac{I}{h} = \frac{\pi}{4}\left\{1 + \frac{1.3}{2.4}h + \frac{1.3^2.5}{2.4^2.6}h^2 + \&c.\right\},$$

$$\frac{G}{h} = \frac{\pi}{4}\left\{1 + \frac{1^2}{2.4}h + \frac{1^2.3^2}{2.4^2.6}h^2 + \&c.\right\},$$

$$E = \frac{\pi}{2}\left\{1 - \frac{1}{2^2}h - \frac{1^2\,3}{2^2.4^2}h^2 - \frac{1^2.3^2.5}{2^2.4^2.6^2}h^3 - \&c.\right\},$$

$$\frac{G}{hh'} = \frac{\pi}{4}\left\{1 + \frac{1^2.3^2}{2.4}h + \frac{1^2.3^2.5^2}{2.4^2.6}h^2 + \&c.\right\},$$

$$\frac{E}{h'} = \frac{\pi}{2}\left\{1 + \frac{1^2.3}{2^2}h + \frac{1^2.3^2.5}{2.4^2}h^2 + \frac{1^2.3^2.5^2.7}{2^2.4^2.6^2}h^3 + \&c.\right\},$$

which are all known formulæ.

The case $m=n=p$, § 15.

§ 15. The formula obtained by putting $m=n=p$ in the general theorem is perhaps worth notice. It may be written

$$\int_0^K (\mathrm{sn}\, u \,\mathrm{cn}\, u \,\mathrm{dn}\, u)^n du = \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{2\Gamma(n+1)}\left\{1 - \frac{n^2-1^2}{4(n+1)}h\right.$$

$$\left. + \frac{(n^2-1^2)(n^2-3^2)}{4.8(n+1)(n+2)}h^2 - \frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{4.8.12(n+1)(n+2)(n+3)}h^3 + \&c.\right\},$$

n being > -1. Thus the value of the integral can be expressed in a finite form in powers of h, when n is an uneven integer.

When $n=0$ this series also reduces to the well-known expression for K, § 12.

The general hypergeometric series, §§ 16, 17.

§ 16. In the general formula of § 2 let

$$\frac{m+1}{2}=\alpha, \quad \frac{-p+1}{2}=\beta, \quad \frac{m+n}{2}+1=\gamma,$$

whence $m=2\alpha-1$, $p=-2\beta+1$, $n=2\gamma-2\alpha-1$.

Thus the general formula becomes

$$\int_0^K (\operatorname{sn} u)^{2\alpha-1}(\operatorname{cn} u)^{2\gamma-2\alpha-1}(\operatorname{dn} u)^{-2\beta+1}du$$

$$=\frac{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}{2\Gamma(\gamma)}\left\{1+\frac{\alpha\beta}{1.\gamma}h+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1.2\gamma(\gamma+1)}h^2+\&c.\right\},$$

and we see that the hypergeometric series

$$F(\alpha, \beta, \gamma, h)$$

is represented by the expression

$$\frac{2\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}\int_0^K \frac{(\operatorname{sn} u)^{2\alpha-1}(\operatorname{cn} u)^{2\gamma-2\alpha-1}}{(\operatorname{dn} u)^{2\beta-1}}du.$$

In order that this integral may be finite, α must be positive and $\gamma>\alpha$, but there is no restriction attached to β.

Since the series is unaltered by the interchange of α and β the series may also be represented by the expression

$$\frac{2\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\int_0^K \frac{(\operatorname{sn} u)^{2\beta-1}(\operatorname{cn} u)^{2\gamma-2\beta-1}}{(\operatorname{dn} u)^{2\alpha-1}}du,$$

in which α is unrestricted, but β and γ must be positive and $\gamma>\beta$.

§ 17. Taking the first form of the integral, it is evident that it may be written in the form

$$\int_0^K (\operatorname{sc} u)^{2\alpha-1}(\operatorname{nd} u)^{2\beta-1}(\operatorname{cn} u)^{2\gamma-2}du,$$

in which the exponents α, β, γ are attached to separate functions.

Putting $K-u$ for u in this integral, it becomes

$$k'^{\gamma-\alpha-\beta}\int_0^K (\mathrm{cs}\,u)^{2\alpha-1}(\mathrm{dn}\,u)^{2\beta-1}(\mathrm{sd}\,u)^{2\gamma-2}du.$$

This form of the integral may also be obtained from the series by means of the transformation

$$F(\gamma-\alpha,\ \gamma-\beta,\ \gamma,\ x)=(1-x)^{\alpha+\beta-\gamma}F(\alpha,\ \beta,\ \gamma,\ x).$$

I have not examined in any detail the other expressions to which the well-known transformations of a hypergeometric series give rise.

Particular cases of the general result, §§ 18–21.

§ 18. By means of the formulæ

$$\frac{\sin nt}{n\sin t}=F\left(\frac{n+1}{2},\ \frac{1-n}{2},\ \frac{3}{2},\ \sin^2 t\right),$$

$$\frac{\sin nt}{n\sin t\cos t}=F\left(\frac{n+2}{2},\ \frac{2-n}{2},\ \frac{3}{2},\ \sin^2 t\right),$$

$$\cos nt=F\left(\frac{n}{2},\ -\frac{n}{2},\ \frac{1}{2},\ \sin^2 t\right),$$

$$\frac{\cos nt}{\cos t}=F\left(\frac{n+1}{2},\ \frac{1-n}{2},\ \frac{1}{2},\ \sin^2 t\right),$$

we may deduce from the general formula of § 15 the following results:

$$\int_0^K\left(\frac{\mathrm{sn}\,u\,\mathrm{dn}\,n}{\mathrm{cn}\,u}\right)^n\mathrm{sn}\,u\,du=\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\frac{n}{2}\right)}{\pi^{\frac{1}{2}}}\ \frac{\sin n\gamma}{n\sin\gamma}\ ,$$

$$\int_0^K\left(\frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u}\right)^n\mathrm{sn}\,u\,du=\frac{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{1-n}{2}\right)}{\pi^{\frac{1}{2}}}\ \frac{\sin n\gamma}{n\sin\gamma\cos\gamma},$$

$$\int_0^K\left(\frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u}\right)^n\frac{\mathrm{dn}\,u}{\mathrm{sn}\,u}du=\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2\pi^{\frac{1}{2}}}\cos n\gamma.$$

$$\int_0^K \left(\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\right)^n \frac{1}{\operatorname{cn} u}\, du = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(-\frac{n}{2}\right)}{2\pi^{\frac{1}{2}}}\, \frac{\cos n\gamma}{\cos\gamma},$$

γ being the modular angle.

In these formulæ, in order that the integral may be finite it is necessary that n should lie between -1 and 2 in the first, between -2 and 1 in the second, between 0 and 1 in the third, and between 0 and -1 in the fourth.

§ 19. By putting $n=\frac{1}{2}$ in the first formula, and $n=-\frac{1}{2}$ in the last, we find

$$\int_0^K \sqrt{(\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u)}\, du = \frac{\Gamma^2\left(\frac{3}{4}\right)}{\pi^{\frac{1}{2}}}\, \frac{1}{\cos\frac{1}{2}\gamma},$$

$$\int_0^K \frac{du}{\sqrt{(\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u)}} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{\pi^{\frac{1}{2}}}\, \frac{\cos\frac{1}{2}\gamma}{2\cos\gamma}.$$

The right-hand members of these equations may also be expressed by

$$\frac{2G^0}{\cos\frac{1}{2}\gamma} \text{ and } 2K^0\frac{\cos\frac{1}{2}\gamma}{\cos\gamma}$$

respectively (p. 112).

§ 20. These two results may also be deduced without difficulty from § 15, for, putting $n=\frac{1}{2}$ in the formula of that section, we have

$$\int_0^K \sqrt{(\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u)}\, du = \frac{\Gamma^2\left(\frac{3}{4}\right)}{2\Gamma\left(\frac{3}{2}\right)}$$
$$\times\left\{1+\frac{h}{8}+\frac{1.7}{2\,!}\left(\frac{h}{8}\right)^2+\frac{1.9.11}{3\,!}\left(\frac{h}{8}\right)^3+\&\text{c.}\right\}.$$

Now, in the well-known expansion

$$\{1+\sqrt{(1-4t)}\}^n = 2^n\left\{1-nt+\frac{n(n-3)}{2\,!}t^2-\frac{n(n-4)(n-5)}{3\,!}t^3+\&\text{c.}\right\},$$

let $4t=h$ and $n=-\frac{1}{2}$: we thus find

$$(1+h')^{-\frac{1}{2}} = 2^{-\frac{1}{2}}\left\{1+\frac{h}{8}+\frac{1.7}{2\,!}\left(\frac{h}{8}\right)^2+\frac{1.9.11}{3\,!}\left(\frac{h}{8}\right)^3+\&\text{c.}\right\},$$

so that the value of the integral

$$=\frac{\Gamma^2(\frac{3}{4})}{\Gamma(\frac{1}{2})}\,\frac{\sqrt{2}}{\sqrt{(1+\cos\gamma)}}=\frac{\Gamma^2(\frac{3}{4})}{\sqrt{\pi}}\,\frac{1}{\cos\frac{1}{2}\gamma}$$

as before.

Similarly, putting $n=-\frac{1}{2}$,

$$\int_0^K \frac{du}{\sqrt{(\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u)}}=\frac{\Gamma^2(\frac{1}{4})}{2\Gamma(\frac{1}{2})}$$

$$\times\left\{1+3\,\frac{h}{8}+\frac{5.7}{2\,!}\left(\frac{h}{8}\right)^2+\frac{7.9.11}{3\,!}\left(\frac{h}{8}\right)^3+\&c.\right\},$$

and differentiating the expansion of $\{1+\sqrt{(1-4t)}\}^n$,

$$\frac{\{1+\sqrt{(1-4t)}\}^{n-1}}{\sqrt{(1-4t)}}=2^{n-1}\left\{1-(n-3)\,t+\frac{(n-4)(n-5)}{2\,!}\,t^2-\&c.\right\},$$

whence, putting $4t=h$, and $n=\frac{3}{2}$,

$$\frac{(1+h')^{\frac{1}{2}}}{h'}=2^{\frac{1}{2}}\left\{1+3\,\frac{h}{8}+\frac{5.7}{2\,!}\left(\frac{h}{8}\right)^2+\&c.\right\}.$$

Thus, the integral

$$=\frac{\Gamma^2(\frac{1}{4})}{2\Gamma(\frac{1}{2})}\,\frac{\sqrt{(1+\cos\gamma)}}{\sqrt{2}\,.\cos\gamma}=\frac{\Gamma^2(\frac{1}{4})}{\pi^{\frac{1}{2}}}\,\frac{\cos\frac{1}{2}\gamma}{2\cos\gamma}$$

as above.

§ 21. In connexion with the results in § 18, the formula

$$\int_0^{2K} \operatorname{sd} u\left(\frac{1-\operatorname{cn} u}{1+\operatorname{cn} u}\right)^i du=\frac{\pi}{kk'}\,\frac{\sin 2i\gamma}{\sin i\pi},$$

which was obtained in § 18 of the preceding paper (p. 118), may be noticed.

Putting $2u$ for u, and n for $2i$, this result gives

$$\int_0^K \left(\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\right)^n \operatorname{sd} 2u\, du=\frac{\pi}{2kk'}\,\frac{\sin n\gamma}{\sin\frac{1}{2}n\gamma},$$

$$=\frac{\pi}{\sin\frac{1}{2}n\pi}\,\frac{\sin n\gamma}{\sin\gamma}.$$

Value of $\int_0^K \frac{\text{sn}^{n-1}u}{\text{dn}^n u}\,du$, § 22.

§ 22. By putting $n-1$, 0, $-n$ for m, n, p in the general formula of § 2, we find

$$\int_0^K \frac{\text{sn}^{n-1}u}{\text{dn}^n u}\,du = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)}\left\{1+\frac{n}{2}h+\frac{n(n+2)}{2.4}h^2+\&\text{c.}\right\}$$

$$=\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)}(1-h)^{-\frac{1}{2}n}=\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)}\frac{1}{k'^n}.$$

In § 21 (p. 120) of the previous paper the same integral was shown to be equal to

$$2^{\frac{1}{2}n-2}\frac{\Gamma\left(\frac{n}{4}\right)\Gamma\left(\frac{n}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\frac{1}{k'^n}.$$

The two results are easily seen to be in argreement, for, putting $n=2$ in the general formula,

$$\Gamma(x)\,\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\ldots\Gamma\left(x+\frac{n-1}{n}\right)$$
$$=\Gamma(nx)\,(2\pi)^{\frac{1}{2}(n-1)}n^{\frac{1}{2}-nx},$$

we have

$$\Gamma(x)\,\Gamma(x+\tfrac{1}{2})=2^{1-2x}\pi^{\frac{1}{2}}\Gamma(2x),$$

giving, when $x=\frac{1}{4}n$,

$$\Gamma\left(\frac{n}{4}\right)\Gamma\left(\frac{n}{4}+\frac{1}{2}\right)=2^{1-\frac{1}{2}n}\pi^{\frac{1}{2}}\Gamma\left(\frac{n}{2}\right),$$

which is the formula obtained by equating the two values of the integral.
