These last three expressions, in which all positive values of n are admissible differ from the corresponding formulæ in §§ 3 and 4 only by the factor $\dfrac{1}{n}$, which occurs outside the whole expression, and also in the argument of the Gamma function.

Thus, taking the first double integral, we see that its value when the exponent $u^2 + v^2 + 2xuv$ is raised to the power n bears to its value when the exponent is raised to the power m the ratio of

$$\frac{1}{n}\,\Gamma\left(\frac{\alpha + \beta + 2}{2n}\right) \ \text{ to } \ \frac{1}{m}\,\Gamma\left(\frac{\alpha + \beta + 2}{2m}\right).$$

DEVELOPMENTS IN POWERS OF $k'^2 - k^2$.

By *J. W. L. Glaisher.*

The general theorem, § 1.

§ 1. In §§ 9 and 10 of the preceding paper it was shown that

$$16\int_0^\infty \int_0^\infty e^{-s^4-t^4-2\lambda s^2 t^2}\, s^n t^n ds\, dt = \frac{\Gamma\left(\dfrac{n+1}{2}\right)}{2^{n-2}} \int_0^K \mathrm{sd}^n u\, du$$

$$= \ \Gamma^2\left(\frac{n+1}{4}\right)\left\{1 + \frac{(n+1)^2}{2.4}\lambda^2 + \frac{(n+1)^2(n+5)^2}{2.4.6.8}\lambda^4 + \&\text{c.}\right\}$$

$$- 4\Gamma^2\left(\frac{n+3}{4}\right)\left\{\frac{1}{2}\lambda + \frac{(n+3)^2}{2.4.6}\lambda^3 + \frac{(n+3)^2(n+7)^2}{2.4\ldots10}\lambda^5 + \&\text{c.}\right\},$$

where $\lambda = h' - h$, h and h' denoting k^2 and k'^2 respectively.

The letter n is not restricted to integral values: it may have any value > -1.

Differential relation between P_n *and* P_{n+2}, § 2.

§ 2. By differentiating the first relation

$$\int_0^\infty \int_0^\infty e^{-s^4-t^4-2\lambda s^2 t^2}\, s^n t^n ds\, dt = \frac{\Gamma\left(\dfrac{n+1}{2}\right)}{2^{n-6}} \int_0^K \mathrm{sd}^n u\, du,$$

with respect to λ, the left-hand side becomes

$$- 2 \int_0^\infty \int_0^\infty e^{-s^4 - t^4 - 2\lambda s^2 t^2} s^{n+2} t^{n+2} \, ds \, dt,$$

which

$$= - 2 \frac{\Gamma\left(\dfrac{n+3}{2}\right)}{2^{n-4}} \int_0^K \mathrm{sd}^{n+2} u \, du,$$

whence $\qquad \dfrac{d}{d\lambda} \displaystyle\int_0^K \mathrm{sd}^n u \, du = - \frac{1}{2} \frac{n+1}{2} \int_0^K \mathrm{sd}^{n+2} u \, du \,;$

so that, if

$$P_n = \int_0^K \mathrm{sd}^n u \, du,$$

then $\qquad \dfrac{dP_n}{d\lambda} = - \dfrac{n+1}{4} P_{n+2},$

or, since $d\lambda = - 2dh$,

$$\frac{dP_n}{dh} = \frac{n+1}{2} P_{n+2}.$$

It is to be noticed that this result (which may be obtained also by differentiating the series) is true not only for positive and integral values of n, but also for fractional values > -1.

The cases $n = 0$, 2, 4, &c., §§ 3–5.

§ 3. Putting $n = 0$,

$$P_0 = \int_0^K du = K;$$

whence $\qquad P_2 = 2 \dfrac{dP_0}{dh} = 2 \dfrac{dK}{dh} = \dfrac{G}{hh'}.$

Similarly

$$P_4 = \frac{2}{3} \frac{d}{dh} P_2 = \frac{K}{3hh'} - \frac{2(h' - h)}{3h^2 h'^2} G,$$

$$P_6 = \frac{2}{5} \frac{d}{dh} P_4 = \frac{8h^2 + 8h'^2 - 7hh'}{15 h^2 h'^3} G - \frac{4(h' - h)}{15 h^2 h'^2} K,$$

and so on.

§ 4. Putting $n = 0, 2, 4, 6$, &c. in the last two equalities of § 1, we find

$$4\,\Gamma\left(\tfrac{1}{2}\right) P_0 = \Gamma^2\left(\tfrac{1}{4}\right)\left\{1 + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \text{&c.}\right\}$$
$$- 4\Gamma^2\left(\tfrac{3}{4}\right)\left\{\frac{1}{2}\lambda + \frac{3^2}{2.4.6}\lambda^3 + \frac{3^2.7^2}{2.4\ldots10}\lambda^5 + \text{&c.}\right\},$$

$$\Gamma\left(\tfrac{3}{2}\right) P_2 = \Gamma^2\left(\tfrac{3}{4}\right)\left\{1 + \frac{3^2}{2.4.}\lambda^2 + \frac{3^2.7^2}{2.4.6.8}\lambda^4 + \text{&c.}\right\}$$
$$- 4\Gamma^2\left(\tfrac{5}{4}\right)\left\{\frac{1}{2}\lambda + \frac{5^2}{2.4.6}\lambda^3 + \frac{5^2.9^2}{2.4\ldots10}\lambda^5 + \text{&c.}\right\},$$

$$\tfrac{1}{4}\Gamma\left(\tfrac{5}{2}\right) P_4 = \Gamma^2\left(\tfrac{5}{4}\right)\left\{1 + \frac{5^2}{2.4}\lambda^2 + \frac{5^2.9^2}{2.4.6.8}\lambda^4 + \text{&c.}\right\}$$
$$- 4\Gamma^2\left(\tfrac{7}{4}\right)\left\{\frac{1}{2}\lambda + \frac{7^2}{2.4.6}\lambda^3 + \frac{7^2.11^2}{2.4\ldots10}\lambda^5 + \text{&c.}\right\},$$

$$\tfrac{1}{16}\Gamma\left(\tfrac{7}{2}\right) P_6 = \Gamma^2\left(\tfrac{7}{4}\right)\left\{1 + \frac{7^2}{2.4}\lambda^2 + \frac{7^2.11^2}{2.4.6.8}\lambda^4 + \text{&c.}\right\}$$
$$- 4\Gamma^2\left(\tfrac{9}{4}\right)\left\{\frac{1}{2}\lambda + \frac{9^2}{2.4.6}\lambda^3 + \frac{9^2.13^2}{2.4\ldots10}\lambda^5 + \text{&c.}\right\},$$

$$\text{&c.,} \qquad\qquad \text{&c.,}$$

giving

$$\Gamma^2\left(\tfrac{1}{4}\right)\left\{1 + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \text{&c.}\right\}$$
$$- 4\Gamma^2\left(\tfrac{3}{4}\right)\left\{\frac{1}{2}\lambda + \frac{3^2}{2.4.6}\lambda^3 + \frac{3^2.7^2}{2.4\ldots10}\lambda^5 + \text{&c.}\right\} = 4\pi^{\frac{1}{2}}K,$$

$$4\Gamma^2\left(\tfrac{3}{4}\right)\left\{1 + \frac{3^2}{2.4}\lambda^2 + \frac{3^2.7^2}{2.4.6.8}\lambda^4 + \text{&c.}\right\}$$
$$- \Gamma^2\left(\tfrac{1}{4}\right)\left\{\frac{1}{2}\lambda + \frac{5^2}{2.4.6}\lambda^3 + \frac{5^2.9^2}{2.4\ldots10}\lambda^5 + \text{&c.}\right\} = 2\pi^{\frac{1}{2}}\frac{G}{hh'},$$

$$\Gamma^2\left(\tfrac{1}{4}\right)\left\{1 + \frac{5^2}{2.4}\lambda^2 + \frac{5^2.9^2}{2.4.6.8}\lambda^4 + \text{&c.}\right\}$$
$$- 4.3^2\Gamma^2\left(\tfrac{3}{4}\right)\left\{\frac{1}{2}\lambda + \frac{7^2}{2.4.6}\lambda^3 + \frac{7^2.11^2}{2.4\ldots10}\lambda^6 + \text{&c.}\right\}$$
$$= \pi^{\frac{1}{2}}\left\{\frac{K}{hh'} - \frac{2(h'-h)}{h^2h'^2}G\right\},$$

$$4.3\Gamma^2\left(\tfrac{3}{4}\right)\left\{1 + \frac{7^2}{2.4}\lambda^2 + \frac{7^2.11^2}{2.4.6.8}\lambda^4 + \&c.\right\}$$

$$- 5^2\Gamma^2\left(\tfrac{1}{4}\right)\left\{\frac{1}{2}\lambda + \frac{9^2}{2.4.6}\lambda^3 + \frac{9^2.13^2}{2.4...10}\lambda^5 + \&c.\right\}$$

$$= \tfrac{1}{2}\pi^2\left\{\frac{8h^2 + 8h'^2 - 7hh'}{h^3h'^3}\,G - \frac{4\,(h' - h)}{hh'}\,K\right\},$$

$$\&c., \qquad\qquad\qquad \&c.$$

The right-hand members of these equations may easily be expressed in terms of λ by means of the equations

$$h' - h = \lambda, \quad hh' = \tfrac{1}{4}\left(1 - \lambda^2\right).$$

§ 5. Since

$$\Gamma^2\left(\tfrac{1}{4}\right) = 4\pi^{\frac{1}{2}}K^0, \ \ \Gamma^2\left(\tfrac{3}{4}\right) = 2\pi^{\frac{1}{2}}G^0,$$

where K^0 and G are the values of K and G when $\lambda = 0$ $\left(\text{that is when } h = \dfrac{1}{\sqrt{2}}\right)$, it is evident that the first formula is the same as that given on p. 147 of vol. xix, and that the second is the same as that from which the series for G was deduced by multiplication by $1 - \lambda^2$ on p. 148 of the same volume.

It will be noticed that in all the formulæ for even values of n greater than 2, the right-hand members will necessarily consist of two terms, one of which is a multiple of K and the other a multiple of G.

The cases $n = 1, 3, 5, \&c.,$ §§ 6–9.

§ 6. Considering now the case when n is an uneven integer, we find

$$P_1 = \frac{\gamma}{kk'},$$

where γ is the modular angle; for

$$\int \mathrm{sd}\, u\, du = \frac{1}{kk'}\tan^{-1}\left(\frac{k'}{k\,\mathrm{cn}\,u}\right),^*$$

so that $\displaystyle \int_0^K \mathrm{sd}\, u\, du = \frac{1}{kk'}\left\{\tfrac{1}{2}\pi - \tan^{-1}\frac{k'}{k}\right\} = \frac{\gamma}{kk'}.$

By differentiating, we find

$$P_3 = \frac{1}{2hh'} - \frac{h' - h}{2hh'}\frac{\gamma}{kk'},$$

$$P_4 = \frac{3h^2 + 3h'^2 - 2hh'}{8h^2h'^2}\frac{\gamma}{kk'} - \frac{3(h' - h)}{8h^2h'^2},$$

&c. &c.,

whence, proceeding as in § 4,

$$\pi\left\{1 + \frac{1^2}{2!}\lambda^2 + \frac{1^2 \cdot 3^2}{4!}\lambda^4 + \&c.\right\} - 2\left\{\lambda + \frac{2^2}{3!}\lambda^3 + \frac{2^2 \cdot 4^2}{5!}\lambda^5 + \&c.\right\}$$
$$= \frac{2\gamma}{kk'},$$

$$1 + \frac{2^2}{2 \cdot 4}\lambda^2 + \frac{2^2 \cdot 4^2}{4!}\lambda^4 + \&c. - \tfrac{1}{2}\pi\left\{\lambda + \frac{3^2}{3!}\lambda^3 + \frac{3^2 \cdot 5^2}{5!}\lambda^5 + \&c.\right\}$$
$$= \frac{1}{4hh} - \frac{h' - h}{4hh'}\frac{\gamma}{kk},$$

$$\tfrac{1}{4}\pi\left\{1 + \frac{3^2}{2!}\lambda^2 + \frac{3^2 \cdot 5^2}{4!}\lambda^4 + \&c.\right\} - 2\left\{\lambda + \frac{4^2}{3!}\lambda^3 + \frac{4^2 \cdot 6^2}{5!}\lambda^5 + \&c.\right\}$$
$$= \frac{3h^2 + 3h'^2 - 2hh'}{32h^2h'^2}\frac{\gamma}{kk'} - \frac{3(h' - h)}{32h^2h'^2},$$

&c. &c.

§ 7. We can easily verify these results, for

$$1 + \frac{1^2}{2!}\lambda^2 + \frac{1^2 \cdot 3^2}{4!}\lambda^4 + \&c. = \frac{1}{\sqrt{(1 - \lambda^2)}} = \frac{1}{2kk'},$$

and

$$\lambda + \frac{2^2}{3!}\lambda^3 + \frac{2^2 \cdot 4^2}{5!}\lambda^5 + \&c. = \frac{\sin^{-1}\lambda}{\sqrt{(1 - \lambda^2)}} = \frac{\sin^{-1}(1 - 2k^2)}{2kk'}$$
$$= \frac{\tfrac{1}{2}\pi - 2\gamma}{2kk'},$$

so that the first equation becomes

$$\frac{\pi}{2kk'} - \frac{\pi - 4\gamma}{2kk'} = \frac{2\gamma}{kk'},$$

which is an identity. The other results follow from the first by differentiation.

§ 8. By integrating the first equation with respect to λ, we find

$$\gamma^2 = \frac{\pi^2}{16} - \frac{\pi}{4} \left\{ \lambda + \frac{1}{3!} \lambda^3 + \frac{1^2 . 3^2}{5!} \lambda^5 + \&c. \right\}$$
$$+ \tfrac{1}{2} \left\{ \frac{1}{2!} \lambda^2 + \frac{2^2}{4!} \lambda^4 + \frac{2^2 . 4^2}{6!} \lambda^6 + \&c. \right\},$$

which expresses the square of the modular angle as a series proceeding by ascending powers of λ.

§ 9. It may be remarked that the expansion of γ^2 in powers of k is easily obtained; for

$$\frac{\sin^{-1} k}{\sqrt{(1 - k^2)}} = k + \frac{2}{3} k^3 + \frac{2.4}{3.5} k^5 + \frac{2 \, 4 \, 6}{3.5.7} k^7 + \&c.,$$

whence, by integration,

$$\gamma^2 = h + \frac{2}{3} \frac{h^2}{2} + \frac{2.4}{3.5} \frac{h^3}{3} + \frac{2.4.6}{3.5.7} \frac{h^4}{4} + \&c.$$

The case of n fractional, § 10.

§ 10. The systems of expansions in §§ 4 and 6 might be obtained by differentiation from the first equation in each system, viz. those in § 4 by repeatedly differentiating the series for K and those in § 6 by repeatedly differentiating the series for $\frac{\gamma}{kk'}$. The general formula in § 1 is, however, true for fractional as well as integral values of n; and it is on this account that it seems to be specially deserving of attention.

For example, putting $n = -\frac{1}{2}$ and $\frac{1}{2}$, it gives

$$2^2 \Gamma \left(\tfrac{1}{4} \right) \int_0^K \frac{du}{\sqrt{(\mathrm{sd}\, u)}}$$
$$= \Gamma^2 \left(\tfrac{5}{8} \right) \left\{ 1 + \frac{1^2}{4.8} \lambda^2 + \frac{1^2 . 9^2}{4.8.12.16} \lambda^4 + \frac{1^2 . 9^2 . 17^2}{4.8 \dots 24} \lambda^6 + \&c. \right\}$$
$$- 8\Gamma^2 \left(\tfrac{5}{8} \right) \left\{ \frac{1}{4} \lambda + \frac{5^2}{4.8.12} \lambda^3 + \frac{5^2 . 13^2}{4.8 \dots 20} \lambda^5 + \&c. \right\},$$

$$2^2 \Gamma \left(\tfrac{3}{4} \right) \int_0^K \sqrt{(\mathrm{sd}\, u)}\, du$$
$$= \Gamma^2 \left(\tfrac{3}{8} \right) \left\{ 1 + \frac{3^2}{4.8} \lambda^2 + \frac{3^2 . 11^2}{4.8.12.16} \lambda^4 + \frac{3^2 . 11^2 . 19^2}{4.8 \dots 24} \lambda^6 + \&c. \right\}$$
$$- 8\Gamma^2 \left(\tfrac{7}{8} \right) \left\{ \frac{1}{4} \lambda + \frac{7^2}{4.8.12} \lambda^3 + \frac{7^2 . 15^2}{4.8 \dots 20} \lambda^5 + \&c. \right\}.$$

Formula of reduction for P_n, § 11.

§ 11. The indefinite integral $\int \mathrm{sd}^n u\, du$ satisfies the formula of reduction

$$(n+1)\, hh' \int \mathrm{sd}^{n+2}u\, du + n\, (h'-h) \int \mathrm{sd}^n u\, du - (n-1) \int \mathrm{sd}^{n-2}u\, du$$
$$+ \mathrm{sd}^{n-1}u\, \mathrm{nd}\, u\, \mathrm{cd}\, u = 0,^*$$

in which n is unrestricted.

Supposing n to be < 2 we have, therefore, by taking the limits of integration to be 0 and K,

$$(n+1)\, hh'\, P_{n+2} + n\, (h'-h)\, P_n - (n-1)\, P_{n-2} = 0,$$

where, as in § 2, P_n denotes $\int_0^K \mathrm{sd}^n u\, du$.

By putting $n = 0, 2, \ldots$, in this formula we can calculate P_4 from P_2 and P_0, P_6 from P_4 and P_2, and so on, thus obtaining the values of P_4, P_6, ..., which were found in § 3 by means of the differential relation in § 2. Similarly from P_1 and P_3 we may calculate P_5, and so on.

Differential relations satisfied by P_0 and P_1, §§ 12, 13.

§ 12. From the differential relation in § 2 we deduce that, if n be any positive integer,

$$P_{2n} = \frac{2^n}{1.3.5\ldots(2n-1)} \left(\frac{d}{dh} \right)^n P_0.$$

Putting for P_0 its value K and substituting in the formul

$$(2n+1)\, hh'\, P_{2n+2} + 2n\, (h'-h)\, P_{2n} - (2n-1)\, P_{2n-2} = 0,$$

we find

$$4hh' \left(\frac{d}{dh} \right)^{n+1} K + 4n\, (h'-h) \left(\frac{d}{dh} \right)^n K - (2n-1)^2 \left(\frac{d}{dh} \right)^{n-1} K = 0,$$

which is the same result as would be obtained by operating with $\left(\dfrac{d}{dh} \right)^{n-1}$ upon the differential equation of the second order satisfied by K.

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* *Messenger*, vol. xi., p. 125 (1881).

§13. Similarly, taking the case in which the suffixes are uneven, we have

$$(2n + 2) \, hh' \, P_{2n+3} + (2n + 1) \, (h' - h) \, P_{2n+1} - 2n \, P_{2n-1} = 0,$$

which, since

$$P_{2n+1} = \frac{1}{n!} \left(\frac{d}{dh} \right)^n P_1,$$

gives

$$2hh' \left(\frac{d}{dh} \right)^{n+1} P_1 + (2n + 1) \, (h' - h) \left(\frac{d}{dh} \right)^n P_1 - 2n^2 \left(\frac{d}{dh} \right)^{n-1} P_1 = 0,$$

where

$$P_1 = \frac{\gamma}{kk'}.$$

This is the same result as would be obtained by operating with $\left(\dfrac{d}{dh} \right)^{n-1}$ upon the differential equation

$$2hh' \frac{d^2u}{dh^2} + 3 \, (h' - h) \frac{du}{dh} - 2u = 0,$$

which is satisfied by $u = \dfrac{\gamma}{kk'}$.

Corresponding formulæ involving cn u, §14.

§14. Since

$$\text{sd} \, (K - u) = \frac{1}{k'} \, \text{cn} \, u,$$

we see that

$$\int_0^K \text{cn}^n u \, du = k'^n \int_0^K \text{sd}^n u \, du,$$

so that in the preceding results we may attribute to P_n the value $\dfrac{1}{k'^n} \displaystyle\int_0^K \text{cn}^n u \, du$. The formulæ are however more conveniently expressed by means of the function sd u, as we should expect, the group of functions cd u, sd u, nd u being generally more simple and regular as regard its formulæ than either of the two corresponding groups sn u, cn u, dn u or dc u, nc u, sc u.[*]

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[*] One very convenient property of sd u distinguishes it from all the other elliptic functions, viz. sd $iu = i$ sd $(u, \, k')$, where i denotes one of the square roots of -1.

Definite integrals involving elliptic functions, §§ 15–21.

§ 15. From § 7 of the preceding paper we obtain a result which may be written in the form:

$$2\pi^{\frac{1}{2}} \int_0^{2K} \left(\frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}\right)^i du$$

$$= \Gamma\left(\frac{1}{4} + \frac{i}{2}\right)\Gamma\left(\frac{1}{4} - \frac{i}{2}\right)\left\{1 - \frac{i^2 - \frac{1}{4}}{2!}\lambda^2 + \frac{(i^2 - \frac{1}{4})(i^2 - \frac{9}{4})}{4!}\lambda^4 - \&c.\right\}$$

$$-2\Gamma\left(\frac{3}{4} + \frac{i}{2}\right)\Gamma\left(\frac{3}{4} - \frac{i}{2}\right)\left\{\lambda - \frac{i^2 - \frac{9}{4}}{3!}\lambda^3 + \frac{(i^2 - \frac{9}{4})(i^2 - \frac{25}{4})}{5!}\lambda^5 - \&c.\right\},$$

where i lies between $-\frac{1}{2}$ and $\frac{1}{2}$.

Transforming the definite integral by putting $u = 2v$, we have

$$2\pi^{\frac{1}{2}} \int_0^{2K} \left(\frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}\right)^i du = 4\pi^{\frac{1}{2}} \int_0^{K} \left(\frac{1 - \operatorname{cn} 2u}{1 + \operatorname{cn} 2u}\right)^i du$$

$$= 4\pi^{\frac{1}{2}} \int_0^{K} \frac{\operatorname{sn}^{2i} u\, \operatorname{dn}^{2i} u}{\operatorname{cn}^{2i} u}\, du = 4\pi^{\frac{1}{2}} \int_0^{K} \frac{\operatorname{cn}^{2i} u}{\operatorname{sn}^{2i} u\, \operatorname{dn}^{2i} u}\, du,$$

so that the above series represents the value of these integrals.

§ 16. We may obtain a still more general result by putting

$$\alpha = n + i - \tfrac{1}{2}, \quad \beta = n - i - \tfrac{1}{2}$$

in the theorem in § 4 of the preceding paper (p. 99).
We thus find

$$\frac{\Gamma\left(n + \frac{1}{2}\right)}{2^{2n-1}} \int_0^{2K} \frac{(1 - \operatorname{cn} u)^{n+i}(1 + \operatorname{cn} u)^{n-i}}{\operatorname{dn}^{2i} u}\, du$$

$$= \Gamma\left(\frac{n+i}{2} + \frac{1}{4}\right)\Gamma\left(\frac{n-i}{2} + \frac{1}{4}\right)\left\{1 - \frac{i^2 - (n + \frac{1}{2})^2}{2!}\lambda^2\right.$$

$$\left. + \frac{\{i^2 - (n + \frac{1}{2})^2\}\{i^2 - (n + \frac{5}{2})^2\}}{4!}\lambda^4 - \&c.\right\}$$

$$-2\Gamma\left(\frac{n+i}{2} + \frac{3}{4}\right)\Gamma\left(\frac{n-i}{2} + \frac{3}{4}\right)\left\{\lambda - \frac{i^2 - (n + \frac{3}{2})^2}{3!}\lambda^3\right.$$

$$\left. + \frac{\{i^2 - (n + \frac{3}{2})^2\}\{i^2 - (n + \frac{7}{2})^2\}}{5!}\lambda^5 - \&c.\right\},$$

where n and i are any quantities, subject only to the conditions

$$n + i > -\tfrac{1}{2}, \quad n - i > -\tfrac{1}{2}.$$

It is evident that the integral may also be written in the form

$$\frac{\Gamma\left(n + \tfrac{1}{2}\right)}{2^{2n-1}} \int_0^{2K} \mathrm{sd}^{2n} u \left(\frac{1 - \mathrm{cn}\, u}{1 + \mathrm{cn}\, u}\right)^i du.$$

§ 17. Putting in this formula $\tfrac{1}{2}n$ for n, we have

$$\frac{\Gamma\left(\dfrac{n+1}{2}\right)}{2^{n-1}} \int_0^{2K} \mathrm{sd}^n u \left(\frac{1 - \mathrm{cn}\, u}{1 + \mathrm{cn}\, u}\right)^i du$$

$$= \Gamma\left(\frac{n+1}{4} + \frac{i}{2}\right) \Gamma\left(\frac{n+1}{4} - \frac{i}{2}\right) \left\{ 1 - \frac{i^2 - \left(\dfrac{n+1}{2}\right)^2}{2!} \lambda^2 \right.$$

$$\left. + \frac{\left\{i^2 - \left(\dfrac{n+1}{2}\right)^2\right\} \left\{i^2 - \left(\dfrac{n+5}{2}\right)^2\right\}}{4!} \lambda^4 - \&c. \right\}$$

$$- 2\Gamma\left(\frac{n+3}{4} + \frac{i}{2}\right) \Gamma\left(\frac{n+3}{4} - \frac{i}{2}\right) \left\{ \lambda - \frac{i^2 - \left(\dfrac{n+3}{2}\right)^2}{3!} \lambda^3 \right.$$

$$\left. + \frac{\left\{i^2 - \left(\dfrac{n+3}{2}\right)^2\right\} \left\{i^2 - \left(\dfrac{n+7}{2}\right)^2\right\}}{5!} \lambda^5 - \&c. \right\},$$

where n and i are subject to the conditions

$$n + 2i > -1, \quad n - 2i > -1.$$

This result includes both the formulæ of §§ 1 and 15 as particular cases, the former corresponding to $i = 0$ and the latter to $n = 0$.

§ 18. The formula contained in § 14 of the previous paper (p. 104) corresponds to the case $n = 1$. It was shown in that section that the series then admit of summation, the resulting formula being

$$\int_0^{2K} \mathrm{sd}\, u \left(\frac{1 - \mathrm{cn}\, u}{1 + \mathrm{cn}\, u}\right)^i du = \frac{\pi}{kk'} \frac{\sin 2i\gamma}{\sin i\pi},$$

where γ is the modular angle. In this formula i must be between -1 and $+1$.

§ 19. This result may be easily verified in the case of $i = \frac{1}{2}$, for the right-hand member of the equation then becomes

$$\frac{\pi}{kk'} \sin \gamma = \frac{\pi}{k'},$$

and the integral

$$= \int_0^{2K} \operatorname{sd} u \left(\frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}\right)^{\frac{1}{2}} du = \int_0^{2K} \frac{1 - \operatorname{cn} u}{\operatorname{dn} u} du$$

$$= 2 \int_0^K \operatorname{nd} u \, du = \frac{2}{k'} \left[\tan^{-1}\left(k' \frac{\operatorname{sd} u}{\operatorname{cd} u}\right)\right]_0^K = \frac{2}{k'} \cdot \frac{\pi}{2} = \frac{\pi}{k'}.$$

§ 20. Putting $\alpha = n - 1, \quad \beta = n,$

in the general theorem of § 4 of the preceding paper (p. 99), we find that the series becomes

$$\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \left\{1 + \frac{n(n+1)}{2!} \lambda^2 + \frac{n(n+1)\dots(n+3)}{4!} \lambda^4 + \&c.\right\}$$

$$- \frac{1}{2} \Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n}{2}\right) \left\{(n-1)n\lambda + \frac{(n-1)n(n+1)(n+2)}{3!} \lambda^3 + \&c.\right\}$$

$$= \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \left\{1 - n\lambda + \frac{n(n+1)}{2!} \lambda^2 - \frac{n(n+1)(n+2)}{3!} \lambda^3 + \&c.\right\}$$

$$= \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \frac{1}{(1+\lambda)^n} = \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2^n h'^n},$$

and the integral

$$= \frac{1}{2^{2n-1}} \Gamma\left(n + \frac{1}{2}\right) \int_0^{2K} \frac{\operatorname{sn}^{2n-1} u \, (1 + \operatorname{cn} u)}{\operatorname{dn}^{2n} u} du$$

$$= \frac{1}{2^{2n-2}} \Gamma\left(n + \frac{1}{2}\right) \int_0^K \frac{\operatorname{sn}^{2n-1} u}{\operatorname{dn}^{2n} u} du;$$

whence

$$\int_0^K \frac{\operatorname{sn}^{2n-1} u}{\operatorname{dn}^{2n} u} du = 2^{n-2} \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(n + \frac{1}{2}\right)} \frac{1}{h'^n},$$

where n is any quantity > -1.

This result includes the formula given in § 17 of the preceding paper (p. 106), in which n was restricted to positive and integral numbers.

§ 21. Replacing n by $\frac{1}{2}n$ in the above formula, it becomes

$$\int_0^K \frac{\mathrm{sn}^{n-1}u}{\mathrm{dn}^n u}\, du = 2^{\frac{1}{2}n-1} \frac{\Gamma\left(\dfrac{n}{4}\right)\Gamma\left(\dfrac{n}{4}+\dfrac{1}{2}\right)}{\Gamma\left(\dfrac{n+1}{2}\right)} \frac{1}{k'^n},$$

where n is any quantity greater than $-\frac{1}{2}$.

Integrals involving elliptic functions with special values of the modulus, §§ 22—29.

§ 22. In vol. XII., p. 98, of the *Proc. Lond. Math. Soc.* it was shown that for modulus $\dfrac{\sqrt{3}}{2}$,

$$\int_0^K \sqrt[3]{(\mathrm{dn}\, u)}\, du = \frac{1}{6}\frac{\Gamma^2\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}.$$

This result may be readily generalised for the same value of the modulus, as follows:

By putting $u = r\sin\theta$, $v = r\cos\theta$, we find

$$\int_0^\infty \int_0^\infty e^{-u^6-v^6} u^\alpha v^\beta\, du\, dv$$

$$= \int_0^{\frac{1}{2}\pi} \int_0^\infty e^{-r^6(1-\frac{3}{4}\sin^2 2\theta)}\, r^{\alpha+\beta+1}\sin^\alpha\theta\cos^\beta\theta\, dr\, d\theta$$

$$= \frac{1}{6}\Gamma\left(\frac{\alpha+\beta+2}{6}\right)\int_0^{\frac{1}{2}\pi} \frac{\sin^\alpha\theta\cos^\beta\theta\, d\theta}{\left(1-\frac{3}{4}\sin^2 2\theta\right)^{\frac{\alpha+\beta+2}{6}}}$$

$$= \frac{1}{12}\Gamma\left(\frac{\alpha+\beta+2}{6}\right)\int_0^\pi \frac{\sin^\alpha\frac{1}{2}\theta\cos^\beta\frac{1}{2}\theta\, d\theta}{\left(1-\frac{3}{4}\sin^2\theta\right)^{\frac{\alpha+\beta+2}{6}}}$$

$$= \frac{1}{2^{\frac{1}{2}(\alpha+\beta)}}\frac{1}{12}\Gamma\left(\frac{\alpha+\beta+2}{6}\right)\int_0^{2K} \frac{(1-\mathrm{cn}\,u)^{\frac{1}{2}\alpha}(1+\mathrm{cn}\,u)^{\frac{1}{2}\beta}}{(\mathrm{dn}\,u)^{\frac{\alpha+\beta-1}{3}}}\, du,$$

the modulus being $\dfrac{\sqrt{3}}{2}$, whence

$$\int_0^{2K} \frac{(1 - \operatorname{cn} u)^{\frac{1}{2}\alpha} (1 + \operatorname{cn} u)^{\frac{1}{2}\beta}}{(\operatorname{dn} u)^{\frac{\alpha + \beta - 1}{3}}} \, du = \frac{2^{\frac{1}{2}(\alpha + \beta)}}{3} \frac{\Gamma\left(\dfrac{\alpha + 1}{6}\right) \Gamma\left(\dfrac{\beta + 1}{6}\right)}{\Gamma\left(\dfrac{\alpha + \beta + 2}{6}\right)},$$

where α and β are any quantities > -1, and the modulus $= \dfrac{\sqrt{3}}{2}$.

§ 23. If $\beta = \alpha$, this equation becomes

$$\int_0^K \frac{\operatorname{sn}^\alpha u}{\operatorname{dn}^{\frac{2\alpha - 1}{3}} u} \, du = \frac{2^{\alpha - 1}}{3} \frac{\Gamma^2\left(\dfrac{\alpha + 1}{6}\right)}{\Gamma\left(\dfrac{\alpha + 1}{3}\right)} \quad \left(\text{mod.} = \frac{\sqrt{3}}{2}\right),$$

which includes the formula quoted at the beginning of the preceding section as the particular case of $\alpha = 0$.

By putting $\alpha = \frac{1}{2}$, we find

$$\int_0^K \sqrt{(\operatorname{sn} u)} \, du = \frac{1}{3\sqrt{2}} \frac{\Gamma^2\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \quad \left(\text{mod.} = \frac{\sqrt{3}}{2}\right)$$

$$= \frac{2\sqrt{2}}{3} K^0,$$

where K^0 is the value of the K corresponding to $k = \dfrac{1}{\sqrt{2}}$.

§ 24. If $\beta = \alpha + 2$, the integral becomes

$$\int_0^{2K} \frac{\operatorname{sn}^\alpha u \, (1 + \operatorname{cn} u)}{\operatorname{dn}^{\frac{2\alpha + 1}{3}} u} \, du = 2 \int_0^K \frac{\operatorname{sn}^\alpha u}{\operatorname{dn}^{\frac{2\alpha + 1}{3}} u} \, du,$$

so that

$$\int_0^K \frac{\operatorname{sn}^\alpha u}{\operatorname{dn}^{\frac{2\alpha + 1}{3}} u} \, du = \frac{2^\alpha}{3} \frac{\Gamma\left(\dfrac{\alpha + 1}{6}\right) \Gamma\left(\dfrac{\alpha + 3}{6}\right)}{\Gamma\left(\dfrac{\alpha + 2}{3}\right)} \quad \left(\text{mod.} = \frac{\sqrt{3}}{2}\right),$$

the only condition with respect to α being that it must be > -1.

By putting $\alpha = -\frac{1}{2}$, we find

$$\int_0^K \frac{du}{\sqrt{(\operatorname{sn} u)}} \, du = \frac{1}{3\sqrt{2}} \frac{\Gamma\left(\frac{1}{12}\right) \Gamma\left(\frac{5}{12}\right)}{\Gamma\left(\frac{1}{2}\right)} \quad \left(\text{mod.} = \frac{\sqrt{3}}{2}\right).$$

§ 25. It is evident that

$$\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right) = \int_0^\infty \int_0^\infty e^{-m-n} m^{\frac{1}{2}(\alpha-1)} n^{\frac{1}{2}(\beta-1)} dm\,dn$$

$$= 4\int_0^\infty \int_0^\infty e^{-u^2-v^2} u^\alpha v^\beta du\,dv$$

$$= 16\int_0^\infty \int_0^\infty e^{-s^4-t^4} s^{2\alpha+1} t^{2\beta+1} ds\,dt$$

$$= 36\int_0^\infty \int_0^\infty e^{-p^6-q^6} p^{3\alpha+2} q^{3\beta+2} dp\,dq.$$

Putting m, u, s, p equal to $r\sin\theta$, and n, v, t, q equal to $r\cos\theta$ in the different integrals, we find

$$\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right) = \Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{\frac{1}{2}\pi} \frac{\sin^{\frac{1}{2}(\alpha-1)}\theta \cos^{\frac{1}{2}(\beta-1)}\theta}{(\sin\theta+\cos\theta)^{\frac{1}{2}(\alpha+\beta+2)}}$$

$$= 2\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{\frac{1}{2}\pi} \sin^\alpha\theta \cos^\beta\theta\, d\theta$$

$$= 4\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{\frac{1}{2}\pi} \frac{\sin^{2\alpha+1}\theta \cos^{2\beta+1}\theta\, d\theta}{(1-\frac{1}{2}\sin^2 2\theta)^{\frac{1}{2}(\alpha+\beta+2)}}$$

$$= 6\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{\frac{1}{2}\pi} \frac{\sin^{3\alpha+2}\theta \cos^{3\beta+2}\theta\, d\theta}{(1-\frac{3}{4}\sin^2 2\theta)^{\frac{1}{2}(\alpha+\beta+2)}},$$

in which α and β must both be > -1.

§ 26. The last two results show that

$$\frac{1}{2^{\alpha+\beta}}\int_0^{2K} \frac{(1-\operatorname{cn} u)^{\alpha+\frac{1}{2}}(1+\operatorname{cn} u)^{\beta+\frac{1}{2}}}{\operatorname{dn}^{\alpha+\beta+1} u} du \qquad \left(\text{mod.} = \frac{1}{\sqrt{2}}\right)$$

$$= \frac{3}{2^{\frac{3}{2}\alpha+\frac{3}{2}\beta+2}}\int_0^{2K} \frac{(1-\operatorname{cn} u)^{\frac{3}{2}\alpha+1}(1+\operatorname{cn} u)^{\frac{3}{2}\beta+1}}{\operatorname{dn}^{\alpha+\beta+1} u} du \qquad \left(\text{mod.} = \frac{\sqrt{3}}{2}\right)$$

$$= \frac{\Gamma\left(\dfrac{\alpha+1}{2}\right)\Gamma\left(\dfrac{\beta+1}{2}\right)}{\Gamma\left(\dfrac{\alpha+\beta+2}{2}\right)};$$

or, as we may write the formula

$$\int_0^{2K} \frac{\operatorname{sn} u \, (1 - \operatorname{cn} u)^\alpha \, (1 + \operatorname{cn} u)^\beta}{\operatorname{dn}^{\alpha+\beta+1} u} \, du \quad \left(\text{mod.} = \frac{1}{\sqrt{2}}\right)$$

$$= \frac{3}{2^{\frac{1}{2}(\alpha+\beta)+2}} \int_0^{2K} \frac{\operatorname{sn}^2 u \, (1 - \operatorname{cn} u)^{\frac{1}{2}\alpha} \, (1 + \operatorname{cn} u)^{\frac{1}{2}\beta}}{\operatorname{dn}^{\alpha+\beta+1} u} \, du \quad \left(\text{mod.} = \frac{\sqrt{3}}{2}\right)$$

$$= 2^{\alpha+\beta} \, \frac{\Gamma\left(\dfrac{\alpha+1}{2}\right) \Gamma\left(\dfrac{\beta+1}{2}\right)}{\Gamma\left(\dfrac{\alpha+\beta+2}{2}\right)},$$

in which α and β must both be > -1.

§ 27. Putting $\alpha = \beta$, the formula becomes

$$\int_0^K \frac{\operatorname{sn}^{2n+1} u}{\operatorname{dn}^{2n+1} u} \, du \quad \left(\text{mod.} = \frac{1}{\sqrt{2}}\right)$$

$$= \frac{3}{2^{n+2}} \int_0^K \frac{\operatorname{sn}^{3n+2} u}{\operatorname{dn}^{2n+1} u} \, du \quad \left(\text{mod.} = \frac{\sqrt{3}}{2}\right) = 2^{2n-1} \frac{\Gamma^2\left(\dfrac{n+1}{2}\right)}{\Gamma(n+1)}.$$

The first of these results may be verified by putting $\lambda = 0$ in the formula of § 1; we thus find that, for modulus $\dfrac{1}{\sqrt{2}}$,

$$\int_0^K \operatorname{sd}^n u \, du = 2^{n-2} \, \frac{\Gamma^2\left(\dfrac{n+1}{4}\right)}{\Gamma\left(\dfrac{n+1}{2}\right)},$$

which agrees with the preceding result on putting $2n + 1$ for n. The second result is the same as the first formula in § 23.

§ 28. In all the expansions in powers of λ, by putting $\lambda = 0$, so that the series reduces to its first term, we obtain a finite expression for an integral involving elliptic functions in which the modulus is $\dfrac{1}{\sqrt{2}}$. For example from § 17, we see

that

$$\int_0^{2K} \operatorname{sd}^n u \left(\frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}\right)^i du \quad \left(\text{mod.} = \frac{1}{\sqrt{2}}\right)$$

$$= 2^{n-1} \frac{\Gamma\left(\dfrac{n+1}{4} + \dfrac{i}{2}\right) \Gamma\left(\dfrac{n+1}{4} - \dfrac{i}{2}\right)}{\Gamma\left(\dfrac{n+1}{2}\right)}.$$

The case when the modulus is $\dfrac{\sqrt{3}}{2}$ corresponds to $\lambda = -\frac{1}{2}$.

§ 29. It may be remarked that if, in the system of equal integrals in § 25, we include the integral

$$64 \int_0^\infty \int_0^\infty e^{-f^4 - g^4} f^{4\beta+3} g^{4\beta+3} df dg,$$

we do not thus obtain a result involving elliptic functions, but we find

$$\int_0^\pi \frac{\sin^3\theta \, (1 - \cos\theta)^{2\alpha} (1 + \cos\theta)^{2\beta}}{(\sin^4\theta + 8 \cos^2\theta)^{\frac{1}{4}(1+\beta+2)}} \, d\theta$$

$$= 2^{\frac{1}{2}(\alpha+\beta-2)} \frac{\Gamma\left(\dfrac{\alpha+1}{2}\right) \Gamma\left(\dfrac{\beta+1}{2}\right)}{\Gamma\left(\dfrac{\alpha+\beta+2}{2}\right)},$$

in which α and β must both be > -1.

From this result, by putting $\alpha = \beta$, we deduce

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^{4n+3}\theta \, d\theta}{(1 + 6\sin^2\theta + \sin^4\theta)^{n+1}} = 2^{n-3} \frac{\Gamma^2\left(\dfrac{n+1}{2}\right)}{\Gamma(n+1)},$$

α being any quantity > -1.

Expressions for $\Gamma^3\left(\frac{1}{6}\right)$ *as a single definite integral*, §§ 30, 31.

§ 30. In connexion with the method employed in §25, it is interesting to investigate the results to which we are led by treating in a similar manner the formula

$$\int_0^\infty \int_0^\infty \int_0^\infty e^{-x^6 - y^6 - z^6} dx \, dy \, dz = \frac{1}{6^3} \Gamma^3\left(\frac{1}{6}\right).$$

Transforming the triple integral by putting

$$x = r \sin\theta \cos\phi, \ y = r \sin\theta \sin\phi, \ z = r \cos\theta,$$

it becomes

$$\int_0^\infty \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} e^{-r^6\{\sin^6\theta(\sin^6\phi+\cos^6\phi)+\cos^6\theta\}} r^2 \sin\theta \, dr \, d\theta \, d\phi,$$

which

$$= \frac{1}{6} \, \Gamma\left(\tfrac{1}{2}\right) \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{\sin\theta \, d\theta \, d\phi}{\sqrt{\{\sin^6\theta \, (\sin^6\theta + \cos^6\theta) + \cos^6\theta\}}} \, .$$

Now

$$\sqrt{\{\sin^6\theta \, (\sin^6\phi + \cos^6\phi) + \cos^6\theta\}} = \sqrt{\{\sin^6\theta \, (1 - \tfrac{3}{4} \sin^2 2\phi) + \cos^6\theta\}}$$

$$= \sqrt{(\sin^6\theta + \cos^6\theta)} \ \sqrt{\left\{1 - \frac{\tfrac{3}{4}\sin^6\theta}{\sin^6\theta + \cos^6\theta} \sin^2 2\phi\right\}}$$

$$= \sqrt{(\sin^6\theta + \cos^6\theta)} \ \sqrt{(1 - k^2 \sin^2 2\phi)},$$

where

$$k^2 = \frac{\tfrac{3}{4}\sin^6\theta}{\sin^6\theta + \cos^6\theta} \, .$$

Thus, the integral

$$= \frac{1}{6} \, \Gamma\left(\tfrac{1}{2}\right) \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{\sin\theta \, d\theta \, d\phi}{\sqrt{(\sin^6\theta + \cos^6\theta)} \ \sqrt{\{(1 - k^2 \sin^2 2\phi)\}}}$$

$$= \frac{1}{6} \, \Gamma\left(\tfrac{1}{2}\right) \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{\sin\theta \, d\theta \, d\phi}{\sqrt{(\sin^6\theta + \cos^6\theta)} \ \sqrt{(1 - k^2 \sin^2 \phi)}} \, ;$$

whence we find

$$\Gamma\left(\tfrac{1}{2}\right) \int_0^{\frac{1}{2}\pi} \frac{K \sin\theta \, d\theta}{\sqrt{(\sin^6\theta + \cos^6\theta)}} = \frac{1}{6^2} \, \Gamma^3\left(\tfrac{1}{6}\right) = 6\Gamma^3\left(\tfrac{7}{6}\right).$$

the modulus k being equal to

$$\frac{\sqrt{3}}{2} \ \frac{\sin^3\theta}{\sqrt{(\sin^6\theta + \cos^6\theta)}} \, .$$

§ 31. This value of k gives

$$\frac{3}{4k^2} = 1 + \frac{\cos^6\theta}{\sin^6\theta} \, ;$$

whence

$$\frac{dk}{4k^3} = \frac{\cos^6\theta}{\sin^7\theta} \, d\theta.$$

The integral therefore

$$= \frac{1}{4} \Gamma\left(\tfrac{1}{2}\right) \int_0^{\frac{\sqrt{3}}{2}} \frac{K \sin^8\theta}{\cos^5\theta \sqrt{(\sin^6\theta + \cos^6\theta)}} \frac{dk}{k^3}$$

$$= \frac{\Gamma\left(\tfrac{1}{2}\right)}{2\sqrt{3}} \int_0^{\frac{\sqrt{3}}{2}} \frac{\sin^5\theta}{\cos^5\theta} \frac{K\,dk}{k^3}.$$

Now
$$\frac{\sin^6\theta}{\cos^6\theta} = \frac{4k^2}{3 - 4k^2},$$

so that
$$\frac{\sin^5\theta}{\cos^5\theta} = \frac{2^{\frac{3}{2}}k^{\frac{5}{3}}}{(3 - 4k^2)^{\frac{5}{6}}},$$

Substituting in the integral, we find

$$\frac{2^{\frac{2}{3}}\Gamma\left(\tfrac{1}{2}\right)}{\sqrt{3}} \int_0^{\frac{\sqrt{3}}{2}} \frac{K\,dk}{k^{\frac{1}{3}}(3 - 4k^2)^{\frac{5}{6}}} = \frac{1}{6^2} \Gamma^3\left(\tfrac{1}{6}\right),$$

whence
$$\int_0^{\frac{\sqrt{3}}{2}} \frac{K\,dk}{k^{\frac{1}{3}}(3 - 4k^2)^{\frac{5}{6}}} = \frac{\Gamma^3\left(\tfrac{1}{6}\right)}{2^{\frac{4}{3}}3^{\frac{3}{2}}\pi^{\frac{1}{2}}}.$$

This curious result may also be written in the form

$$\int_0^{\frac{1}{2}\pi} \frac{K \sin 2\gamma\, d\gamma}{(\sin\gamma)^{\frac{2}{3}}(\sin 3\gamma)^{\frac{5}{6}}} = \frac{\Gamma^3\left(\tfrac{1}{6}\right)}{2^{\frac{2}{3}}3^{\frac{2}{3}}\pi^{\frac{1}{2}}}.$$

Expressions for $\Gamma^4\left(\tfrac{1}{4}\right)$, § 32.

§ 32. The result obtained by a similar treatment of the formula

$$\int_0^\infty \int_0^\infty \int_0^\infty e^{-x^4 - y^4 - z^4} dx\,dy\,dz = \frac{1}{4^3} \Gamma^3\left(\tfrac{1}{4}\right)$$

is perhaps also worth notcie.

We thus find

$$\Gamma\left(\tfrac{3}{2}\right) \int_0^{\frac{1}{2}\pi} \int_0^K \frac{\sin\theta\, d\theta\, du}{(\sin^4\theta + \cos^4\theta)^{\frac{3}{4}} \sqrt{(\mathrm{dn}\, u)}} = \frac{1}{4^3} \Gamma^3\left(\tfrac{1}{4}\right),$$

the modulus k being equal to

$$\frac{1}{\sqrt{2}} \frac{\sin^2\theta}{\sqrt{(\sin^4\theta + \cos^4\theta)}}.$$

Transforming the second variable from θ to k, we obtain the formula

$$\frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt{2}} \int_0^1 \int_0^K \frac{dk\,du}{(1-2k^2)^{\frac{3}{4}} \sqrt{(\mathrm{dn}\,u)}} = \frac{1}{4^2}\,\Gamma^3\left(\tfrac{1}{4}\right),$$

whence $$\int_0^1 \int_0^K \frac{dk\,du}{(1-2k^2)^{\frac{3}{4}} \sqrt{(\mathrm{dn}\,u)}} = \frac{1}{8\sqrt{2}} \frac{\Gamma^3\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\Gamma^4\left(\frac{1}{4}\right)}{16\pi} = \frac{1}{16}\frac{\Gamma^4\left(\frac{1}{4}\right)}{\Gamma^2\left(\frac{1}{2}\right)}.$$

Putting $k = \sin\gamma$, so that γ is the modular angle, this result takes the form

$$\int_0^{\frac{1}{2}\pi} \int_0^K \frac{\cos\gamma\,d\gamma\,du}{(\cos 2\gamma)^{\frac{3}{4}} (\mathrm{dn}\,u)^{\frac{1}{2}}} = \frac{\Gamma^4\left(\frac{1}{4}\right)}{16\pi},$$

or, by substituting $K - u$ for u,

$$\int_0^{\frac{1}{2}\pi} \int_0^K \frac{(\cos\gamma)^{\frac{1}{2}} (\mathrm{dn}\,u)^{\frac{1}{2}}}{(\cos 2\gamma)^{\frac{3}{4}}} \, d\gamma\,du = \frac{\Gamma^4\left(\frac{1}{4}\right)}{16\pi}.$$

The right-hand member of this equation is also the value of $(K^0)^2$, where K^0 is the value of K corresponding to the modulus $\frac{1}{\sqrt{2}}$.

Value of a series with squared factorials in the denominators.
§§ 33—36.

§ 33. The series

$$1 + \frac{x^2}{1^2} + \frac{x^4}{1^2 \cdot 2^2} + \frac{x^6}{1^2 \cdot 2^2 \cdot 3^2} + \&c.,$$

may be represented by the definite integral $\dfrac{1}{\pi} \displaystyle\int_0^\pi e^{-2x\cos\psi}\,d\psi$; so that the triple integral

$$\int_0^\pi \int_0^\infty \int_0^\infty e^{-s^4 - t^4 - 2xs^2t^2\cos\psi} s^{2\alpha-1} t^{2\beta-1}\, ds\,dt\,d\psi$$

$$= \pi \int_0^\infty \int_0^\infty e^{-s^4 - t^4} s^{2\alpha-1} t^{2\beta-1} \left(1 + \frac{x^2 s^4 t^4}{1^2} + \frac{x^4 s^8 t^8}{1^2 \cdot 2^2} + \&c.\right) ds\,dt$$

$$= \frac{\pi}{16}\,\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\beta}{2}\right) \left\{ 1 + \frac{\alpha\beta}{2^2}\,x^2 + \frac{\alpha(\alpha+2)\,\beta(\beta+2)}{2^2 \cdot 4^2}\,x^4 \right.$$

$$\left. + \frac{\alpha(\alpha+2)(\alpha+4)\,\beta(\beta+2)(\beta+4)}{2^2 \cdot 4^2 \cdot 6^2}\,x^6 + \&c. \right\}.$$

Transforming the triple integral by putting
$$s = r \sin\theta, \quad t = r \cos\theta,$$
we see that it

$$= \frac{1}{4} \, \Gamma\left(\frac{\alpha+\beta}{2}\right) \int_0^\pi \int_0^{\frac{1}{2}\pi} \frac{(\sin\theta)^{2\alpha-1} (\cos\theta)^{2\beta-1} \, d\theta \, d\psi}{(\sin^4\theta + \cos^4\theta + 2x \sin^2\theta \cos^2\theta \cos\psi)^{\frac{1}{2}(\alpha+\beta)}}$$

$$= \frac{1}{8} \, \Gamma\left(\frac{\alpha+\beta}{2}\right) \int_0^\pi \int_0^\pi \frac{(\sin\frac{1}{2}\theta)^{2\alpha-1} (\cos\frac{1}{2}\theta)^{2\beta-1} \, d\theta \, d\psi}{(1 - h \sin^2\theta)^{\frac{1}{2}(\alpha+\beta)}} \, ,$$

where $h = \frac{1}{2} - \frac{1}{2}x \cos\psi$,

$$= \frac{1}{2^{\alpha+\beta+1}} \, \Gamma\left(\frac{\alpha+\beta}{2}\right) \int_0^\pi \int_0^{2K} \frac{(1 - \operatorname{cn} u)^{\alpha-\frac{1}{2}} (1 + \operatorname{cn} u)^{\beta-\frac{1}{2}}}{(\operatorname{dn} u)^{\alpha+\beta-1}} \, du \, d\psi,$$

where ψ is connected with the modulus of the elliptic functions by the relation
$$\cos\psi = \frac{\lambda}{x},$$

λ being, as before, $= k'^2 - k^2$.

If in this double integral we transform the variable from ψ to λ, we have to replace

$$\int_0^\pi \ldots \, d\psi \quad \text{by} \quad \int_{-x}^x \ldots \frac{d\lambda}{\sqrt{(x^2 - \lambda^2)}},$$

or, if $v = \frac{\lambda}{x}$, by $\int_{-1}^1 \ldots \frac{dv}{\sqrt{(1 - v^2)}}$.

§ 34. Taking the first and last forms of the integral, the result may be written

$$\int_0^\pi \int_0^{2K} \frac{(1 - \operatorname{cn} u)^{\alpha-\frac{1}{2}} (1 + \operatorname{cn} u)^{\beta-\frac{1}{2}}}{(\operatorname{dn} u)^{\alpha+\beta-1}} \, du \, d\psi \quad (\text{mod.})^2 = \frac{1}{2} - \frac{1}{2}x \cos\psi,$$

$$= \int_{-1}^1 \int_0^{2K} \frac{(1 - \operatorname{cn} u)^{\alpha-\frac{1}{2}} (1 + \operatorname{cn} u)^{\beta-\frac{1}{2}} \, du \, dv}{(\operatorname{dn} u)^{\alpha+\beta-1} \sqrt{(1 - v^2)}} \quad (\text{mod.})^2 = \frac{1}{2} - \frac{1}{2}xv,$$

$$= 2^{\alpha+\beta-2} \pi \, \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\alpha+\beta}{2}\right)} \left\{ 1 + \frac{\alpha\beta}{2^2} \, x^2 + \frac{\alpha(\alpha+2)\beta(\beta+2)}{2^2.4^2} \, x^4 + \&c. \right\}.$$

In this formula α and β must both be positive quantities, and x must not exceed unity. If x is equal to unity, α and β must be less than 2.

§ 35. The preceding result might of course have been deduced from the general formula of § 4 of the preceding paper (p. 99) by putting $x \cos \psi$ for x, and integrating with respect to ψ from 0 to π.

We may also readily obtain the value of the integral when the limits of integration with respect to ψ are 0 and $\frac{1}{2}\pi$, the resulting series, which then contains also uneven powers of x, being

$$2^{\alpha+\beta-3}\pi \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\alpha+\beta}{2}\right)} \left\{ 1 + \frac{\alpha\beta}{2^2}x^2 + \frac{\alpha(\alpha+2)}{2^2}\frac{\beta(\beta+2)}{4^2}x^4 + \&c. \right\}$$

$$- 2^{\alpha+\beta-1} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\frac{\alpha+\beta}{2}\right)} \left\{ x + \frac{(\alpha+1)(\beta+1)}{1^2 \cdot 3^2}x^3 \right.$$

$$\left. + \frac{(\alpha+1)(\alpha+3)(\beta+1)(\beta+3)}{1^2 \cdot 3^2 \cdot 5^2}x^5 + \&c. \right\}.$$

§ 36. By putting $\alpha - \frac{1}{2} = m$, $\beta - \frac{1}{2} = n$, the formula of § 34 becomes

$$\int_0^\pi \int_0^{2K} \frac{(1 - \operatorname{cn} u)^m (1 + \operatorname{cn} u)^n}{(\operatorname{dn} u)^{m+n}} du\, d\psi, \quad (\text{mod.})^2 = \frac{1}{2} - \frac{1}{2}x \cos \psi$$

$$= 2^{m+n-1}\pi \frac{\Gamma\left(\frac{m}{2} + \frac{1}{4}\right)\Gamma\left(\frac{n}{2} + \frac{1}{4}\right)}{\Gamma\left(\frac{m+n+1}{2}\right)} \left\{ 1 + \frac{(m+\frac{1}{2})(n+\frac{1}{2})}{2^4}x^2 \right.$$

$$+ \frac{(m+\frac{1}{2})(m+\frac{5}{2})(n+\frac{1}{2})(n+\frac{5}{2})}{2^2 \cdot 4^2}x^4$$

$$\left. + \frac{(m+\frac{1}{2})(m+\frac{5}{2})(m+\frac{9}{2})(n+\frac{1}{2})(n+\frac{5}{2})(n+\frac{9}{2})}{2^2 \cdot 4^2 \cdot 6^2}x^6 + \&c. \right\},$$

where m and n must be $> -\frac{1}{2}$ and x not > 1. If $x = 1$, $m+n$ must satisfy the further condition of being > 1.

As a particular case, putting $m = n$ and writing $\frac{1}{2}n$ for n,

$$\int_0^\pi \int_0^{2K} (\mathrm{sd}\, u)^r \, du \, d\psi, \quad (\mathrm{mod.})^2 = \tfrac{1}{2} - \tfrac{1}{2}x \cos\psi$$

$$= 2^{n-1}\pi \frac{\Gamma^2\left(\dfrac{n+1}{4}\right)}{\Gamma\left(\dfrac{n+1}{2}\right)} \left\{ 1 + \frac{(n+1)^2}{4^2} x^2 + \frac{(n+1)^2 (n+5)^2}{4^2 . 8^2} x^4 + \&\mathrm{c.} \right\}.$$

In this formula n must be > -1 and x not > 1. If $x = 1$, n must in addition be < 1.

The case $x = 1$, §§ 37–39.

§ 37. When $x = 1$,
$$h = \tfrac{1}{2} - \tfrac{1}{2}\cos\psi,$$

so that $\cos 2\gamma = \cos\psi$, and therefore $\psi = 2\gamma$, where γ is the modular angle.

Thus from § 36,

$$\int_0^{\frac{1}{2}\pi} \int_0^{2K} \frac{(1 - \mathrm{cn}\, u)^m (1 + \mathrm{cn}\, u)^n}{(\mathrm{dn}\, u)^{m+n}} \, du \, d\gamma$$

$$= 2^{m+n-1}\pi \frac{\Gamma\left(\dfrac{m}{2} + \dfrac{1}{4}\right) \Gamma\left(\dfrac{n}{2} + \dfrac{1}{4}\right)}{\Gamma\left(\dfrac{m+n+1}{2}\right)} \left\{ 1 + \frac{(m + \frac{1}{2})(n + \frac{1}{2})}{2^2} \right.$$

$$\left. + \frac{(m + \frac{1}{2})(m + \frac{5}{2})(n + \frac{1}{2})(n + \frac{5}{2})}{2^2 . 4^2} + \&\mathrm{c.} \right\},$$

and in particular

$$\int_0^{\frac{1}{2}\pi} \int_0^K (\mathrm{sd}\, u)^n \, du \, d\gamma$$

$$= 2^{n-3}\pi \frac{\Gamma^2\left(\dfrac{n+1}{4}\right)}{\Gamma\left(\dfrac{n+1}{2}\right)} \left\{ 1 + \frac{(n+1)^2}{4^2} + \frac{(n+1)^2 (n+5)^2}{4^2 . 8^2} + \&\mathrm{c.} \right\},$$

the value of n being > -1 and < 1.

This result may be obtained also by integrating the formula in § 1 of the present paper.

§ 38. Putting $n = 0$ in this formula it becomes

$$\int_0^{\frac{1}{2}\pi} K d\gamma = \frac{\pi}{8} \frac{\Gamma^2(\frac{1}{4})}{\Gamma(\frac{1}{2})} \left\{ 1 + \frac{1^2}{4^2} + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2} + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} + \&c. \right\}.$$

The series in brackets was shown by Gauss[*] to be equal to $\frac{2K^0}{\pi}$, so that we find

$$\int_0^{\frac{1}{2}\pi} K d\gamma = \int_0^1 \frac{K}{k'} dk = (K^\bullet)^2.$$

§ 39. Since

$$\frac{2K}{\pi} = 1 + \frac{1^2}{2^2} k^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} k^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} k^6 + \&c.,$$

it follows that

$$\int_0^{\frac{1}{2}\pi} K d\gamma = \frac{\pi^2}{4} \left\{ 1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1.3}{2.4}\right)^3 + \left(\frac{1.3.5}{2.4.6}\right)^3 + \&c. \right\}.$$

The series in brackets is therefore equal to $\left(\frac{2K^0}{\pi}\right)^2$, as was shown by Gauss in the investigation just referred to.[†]

Second mode of reduction of the double integral in § 33 when
$$\alpha = \beta, \ \S 40.$$

§ 40. Putting $\alpha = \beta = n$ in the first form of the double integral in § 33, it becomes

$$\frac{\Gamma(n)}{2^{2n+1}} \int_0^\pi \int_0^{\frac{1}{2}\pi} \frac{(\sin 2\theta)^{2n-1} d\theta \, d\psi}{(\sin^4\theta + \cos^4\theta + 2x \sin^2\theta \cos^2\theta \cos\psi)^n}.$$

The denominator

$$= (1 - 2\sin^2\theta \cos^2\theta + 2x \sin^2\theta \cos^2\theta - 4x \sin^2\theta \cos^2\theta \sin^2\tfrac{1}{2}\psi)^n$$
$$= \{1 - (\tfrac{1}{2} - \tfrac{1}{2}x) \sin^2 2\theta - x \sin^2 2\theta \sin^2 \tfrac{1}{2}\psi\}^n.$$

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[*] Werke, vol. iii., p. 425.

[†] Gauss's process amounts to an independent proof that $\int_0^{\frac{1}{2}\pi} K d\gamma$ is equal to $(K^0)^2$. For he points out that the series $1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1.3}{2.4}\right)^3 + \&c.$

$$= \frac{4}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{d\phi \, d\psi}{\sqrt{(1 - \cos^2\phi \cos^2\psi)}},$$

and that this expression $\left(\text{which obviously} = \frac{4}{\pi^2} \int_0^{\frac{1}{2}\pi} K d\gamma\right)$

$$= \frac{2}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{df \, dg}{\sqrt{(\sin f \sin g)}} = \left(\frac{2K^0}{\pi}\right)^2.$$

Thus the integral

$$= \frac{\Gamma(n)}{2^{2n+1}} \int_0^\pi \int_0^{\frac{1}{2}\pi} \frac{(\sin\theta)^{2n-1} d\theta \, d\psi}{\{1 - (\frac{1}{2} - \frac{1}{2}x)\sin^2\theta\}^n \{1 - h\sin^2 \frac{1}{2}\psi\}^n}$$

$$= \frac{\Gamma(n)}{2^{2n}} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{(\sin\theta)^{2n-1} d\theta \, d\psi}{\{1 - (\frac{1}{2} - \frac{1}{2}x)\sin^2\theta\}^n \{1 - h\sin^2\psi\}^n}$$

$$= \frac{\Gamma(n)}{2^{2n}} \int_0^{\frac{1}{2}\pi} \int_0^K \frac{(\sin\theta)^{2n-1} du \, d\theta}{(\operatorname{dn} u)^{2n-1} \{1 - (\frac{1}{2} - \frac{1}{2}x)\sin^2\theta\}^n},$$

where
$$h = \frac{x\sin^2\theta}{1 - (\frac{1}{2} - \frac{1}{2}x)\sin^2\theta}.$$

This last equation gives

$$\sin^2\theta = \frac{2h}{(1+h')x + h}, \quad \cos^2\theta = \frac{(1+h')x - h}{(1+h')x + h},$$

$$1 - (\frac{1}{2} - \frac{1}{2}x)\sin^2\theta = \frac{2x}{(1+h')x + h},$$

$$\frac{d\theta}{\sin\theta} = \frac{x\,dh}{h\sqrt{\{(1+h')^2 x^2 - h^2\}}}.$$

Thus the integral

$$= \frac{\Gamma(n)}{2^{2n} x^{n-1}} \int_0^{\frac{2x}{1+x}} \int_0^K \frac{h^{n-1} du \, dh}{(\operatorname{dn} u)^{2n-1} \sqrt{\{(1+h')^2 x^2 - h^2\}}}.$$

General theorem, §§ 41, 42.

§ 41. In § 33 the same double integral was reduced to a form which, in the case $\alpha = \beta = n$, becomes

$$\frac{\Gamma(n)}{2^{2n+1}} \int_{-x}^x \int_0^K \frac{(\operatorname{sd} u)^{2n-1} du \, d\lambda}{\sqrt{(x^2 - \lambda^2)}},$$

so that we have found that

$$\int_{-x}^x \int_0^K \frac{(\operatorname{sd} u)^{2n-1} du \, d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \frac{2}{x^{n-1}} \int_0^{\frac{2x}{1+x}} \int_0^K \frac{h^{n-1} (\operatorname{nd} u)^{2n-1} du \, dh}{\sqrt{\{(1+h')^2 x^2 - h^2\}}}$$

$$= 2^{2n-3}\pi \frac{\Gamma^2(\frac{1}{2}n)}{\Gamma(n)} \left\{ 1 + \frac{n^2}{2^2} x^2 + \frac{n^2(n+2)^2}{2^2 \cdot 4^2} x^4 + \frac{n^2(n+2)^2(n+4)^2}{2^2 \cdot 4^2 \cdot 6^2} x^6 + \&c. \right\}.$$

§ 42. By taking h as the variable in the first integral, this general formula may be written

$$\int_{\frac{1}{2}(1-x)}^{\frac{1}{2}(1+x)} \int_0^K \frac{(\operatorname{sd} u)^{2n-1} du\, dh}{\sqrt{\{x^2 - (h'-h)^2\}}} = \frac{1}{x^{n-1}} \int_0^{\frac{2x}{1+x}} \int_0^K \frac{h^{n-1}(\operatorname{nd} u)^{2n-1} du\, dh}{\sqrt{\{(1+h')^2 x^2 - h^2\}}}$$

$$= 2^{2n-4} \pi \frac{\Gamma^2(\frac{1}{2}n)}{\Gamma(n)} \left\{ 1 + \frac{n^2}{2^2} x^2 + \frac{n^2(n+2)^2}{2^2 \cdot 4^2} x^4 \right.$$

$$\left. + \frac{n^2(n+2)^2(n+4)^2}{2^2 \cdot 4^2 \cdot 6^2} x^6 + \&c. \right\}.$$

The case $n = \frac{1}{2}$, §§ 43, 44.

§ 43. In the special case $n = \frac{1}{2}$, the equation in § 41 becomes

$$\int_{-x}^{x} \frac{K d\lambda}{\sqrt{(x^2 - \lambda^2)}} = 2x^{\frac{1}{2}} \int_0^{\frac{2x}{1+x}} \frac{K dh}{h^{\frac{1}{2}} \sqrt{\{(1+h')^2 x^2 - h^2\}}}$$

$$= \frac{\pi}{4} \frac{\Gamma^2(\frac{1}{4})}{\Gamma(\frac{1}{2})} \left\{ 1 + \frac{1^2}{4^2} x^2 + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2} x^4 + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} x^6 + \&c. \right\}$$

$$= \pi K^0 \left\{ 1 + \frac{1^2}{4^2} x^2 + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2} x^4 + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} x^6 + \&c. \right\}$$

§ 44. When $x = 1$,

$$\int_0^x \frac{K d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \int_0^1 \frac{K dh}{h^{\frac{1}{2}} h'^{\frac{1}{2}}} = 2 \int_0^{\frac{1}{4}\pi} K d\gamma,$$

$$2x^{\frac{1}{2}} \int_0^{\frac{2x}{1+x}} \frac{K dh}{h^{\frac{1}{2}} \sqrt{\{(1+h')^2 x^2 - h^2\}}} = \int_0^1 \frac{K dh}{h^{\frac{1}{2}} h'^{\frac{1}{2}}} = 2 \int_0^{\frac{1}{2}\pi} K d\gamma,$$

and the series becomes

$$\pi K^0 \left(1 + \frac{1^2}{4^2} + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2} + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} + \&c. \right)$$

$$= \pi K^0 \frac{2 K^0}{\pi} = 2 (K^0)^2.$$

The formula therefore in this case agrees with the result in § 38.

The case $n = 1$, §§ 45, 46.

§ 45. Putting $n = 1$ in the general formula of § 41, it becomes

$$\int_{-x}^{x} \int_{0}^{K} \frac{\operatorname{sd} u\, du\, d\lambda}{\sqrt{(x^2 - \lambda^2)}} = 2 \int_{0}^{\frac{\pi}{1+x}} \int_{0}^{K} \frac{\operatorname{nd} u\, du\, dh}{\sqrt{\{(1 + h')^2 x^2 - h^2\}}}$$

$$= \frac{\pi^2}{2} \left\{ 1 + \frac{1^2}{2^2} x^2 + \frac{1^2 . 3^2}{2^2 . 4^2} x^4 + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} x^6 + \&c. \right\}$$

$$= \pi K, \quad (\text{mod.} = x).$$

This result may be verified, for the first integral

$$= \int_{-x}^{x} \frac{d\lambda}{\sqrt{(x^2 - \lambda^2)}} \int_{0}^{K} \operatorname{sd} u\, du,$$

and, by § 6, $\int_{0}^{K} \operatorname{sd} u\, du = \dfrac{\gamma}{kk'} = \dfrac{\frac{1}{2} \cos^{-1} \lambda}{\frac{1}{2} \sqrt{(1 - \lambda^2)}}$,

so that the integral

$$= \int_{-x}^{x} \frac{\cos^{-1} \lambda\, d\lambda}{\sqrt{\{(x^2 - \lambda^2)(1 - \lambda^2)\}}}$$

$$= \int_{-x}^{x} \frac{(\frac{1}{2}\pi - \sin^{-1}\lambda)\, d\lambda}{\sqrt{\{(x^2 - \lambda^2)(1 - \lambda^2)\}}} = \pi \int_{0}^{x} \frac{d\lambda}{\sqrt{\{(x^2 - \lambda^2)(1 - \lambda^2)\}}}$$

$$= \pi \int_{0}^{1} \frac{d\mu}{\sqrt{\{(1 - \mu^2)(1 - x^2\mu^2)\}}}, \quad \text{by putting } \lambda = x\mu,$$

$$= \pi K, \quad (\text{mod.} = x).$$

§ 46. Since

$$\int_{0}^{K} \operatorname{nd} u\, du = \frac{\pi}{2k'},$$

the second integral

$$= \pi \int_{0}^{\frac{\pi}{1+x}} \frac{dh}{h'^2 \sqrt{\{(1 + h')^2 x^2 - h^2\}}}$$

$$= 2\pi \int_{\sqrt{\left(\frac{1-x}{1+x}\right)}}^{1} \frac{dv}{\sqrt{\{(1 + v^2)^2 x^2 - (1 - v^2)^2\}}},$$

where $\qquad v = \sqrt{(1 - h)} = k'.$

The denominator in this integral

$$= \sqrt{[\{(1+v^2)\,x - 1 + v^2\}\{(1+v^2)\,x + 1 - v^2\}]}$$

$$= \sqrt{[\{(1+x)\,v^2 - (1-x)\}\{(1+x) - (1-x)\,v^2\}]}$$

$$= i\,\sqrt{(1-x^2)}\,\sqrt{\left[\left\{1 - \frac{1+x}{1-x}\,v^2\right\}\left\{1 - \frac{1-x}{1+x}\,v^2\right\}\right]}$$

$$= i\,\sqrt{(1-x^2)}\,\sqrt{\left[(1-w^2)\left\{1 - \left(\frac{1-x}{1+x}\right)^2 w^2\right\}\right]} \text{ if } w^2 = \frac{1+x}{1-x}\,v^2.$$

Thus the integral itself

$$= \frac{2\pi}{i\,(1+x)} \int_1^{\sqrt{\left(\frac{1+x}{1-x}\right)}} \frac{dw}{\sqrt{\left[(1-w^2)\left\{1 - \left(\frac{1-x}{1+x}\right)^2 w^2\right\}\right]}},$$

which, since

$$\int_0^{\frac{1}{\sqrt{k}}} \frac{dx}{\sqrt{\{(1-x^2)(1-k^2 x^2)\}}} = i \int_0^{\frac{1}{\sqrt{(1+k)}}} \frac{dx}{\sqrt{\{(1-x^2)(1-k'^2 x^2)\}}} = \tfrac{1}{2} i K''$$

$$= \frac{\pi}{1+x}\,K, \quad \left(\text{mod.} = \frac{2\sqrt{x}}{1+x}\right) = \pi K, \quad (\text{mod.} = x).$$

The case $n = \tfrac{3}{2}$, § 47.

§ 47. Putting $n = \tfrac{3}{2}$ in the formula of § 41, it becomes

$$\int_{-x}^{x} \frac{P_2\,d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \frac{2}{x^2} \int_0^{\frac{2x}{1+x}} \frac{Q_2 h^{\frac{1}{2}} dh}{\sqrt{\{(1+h')^2 x^2 - h^2\}}}$$

$$= \pi\,\frac{\Gamma^2(\tfrac{3}{4})}{\Gamma(\tfrac{3}{2})}\left\{1 + \frac{3^2}{4^2}\,x^2 + \frac{3^2 \cdot 7^2}{4^2 \cdot 8^2}\,x^4 + \frac{3^2 \cdot 7^2 \cdot 11^2}{4^2 \cdot 8^2 \cdot 12^2}\,x^6 + \&c.\right\},$$

where

$$P_2 = \int_0^{K} \mathrm{sd}^2 u\,du = \frac{G}{hh'} = \frac{4G}{1 - \lambda^2},$$

$$Q_2 = \int_0^{K} \mathrm{nd}^2 u\,du = \frac{E}{h'},$$

and

$$\frac{\Gamma^2(\tfrac{3}{4})}{\Gamma(\tfrac{3}{2})} = 4G^0.$$

We thus obtain the result

$$\int_{-x}^{x} \frac{G\,d\lambda}{(1-\lambda^2)\sqrt{(x^2 - \lambda^2)}} = \frac{1}{2x^2} \int_0^{\frac{2x}{1+x}} \frac{E h^{\frac{1}{2}} dh}{h'\,\sqrt{\{(1+h')^2 x^2 - h^2\}}}$$

$$= \pi G_0\left\{1 + \frac{3^2}{4^2}\,x^2 + \frac{3^2 \cdot 7^2}{4^2 \cdot 8^2}\,x^4 + \frac{3^2 \cdot 7^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2}\,x^6 + \&c.\right\}.$$

The equality of the first integral and the series may of course be at once derived by integration from the formula

$$\frac{G}{1-\lambda^2} = G^0 \left\{ 1 + \frac{3^2}{2.4} \lambda^2 + \frac{3^2.7^2}{2.4.6.8} \lambda^4 + \frac{3.7^2.11^2}{2.4...12} \lambda^6 + \&c. \right\}$$
$$ - \tfrac{1}{2} K^0 \left\{ \frac{1^2}{2} \lambda + \frac{1^2.5^2}{2.4.6} \lambda^3 + \frac{1^2.5^2.9^2}{2.4.6.8.10} \lambda^5 + \&c. \right\}.$$

The case $n = 2$, §§ 48, 49.

§ 48. From § 6 we have

$$\int_0^K \mathrm{s\mathring{d}}^2 u\, du = \frac{1}{2hh'} - \frac{h'-h}{2hh'} \frac{\gamma}{kk'}$$
$$= \frac{2}{1-\lambda^2} - \frac{2\lambda \cos^{-1}\lambda}{(1-\lambda^2)^{\frac{3}{2}}};$$

whence, putting $n = 2$ in § 41 and taking the first integral only, we find

$$2 \int_{-x}^{x} \frac{d\lambda}{(1-\lambda^2)(x^2-\lambda^2)^{\frac{1}{2}}} - 2\int_{-x}^{x} \frac{\lambda \cos^{-1}\lambda\, d\lambda}{(1-\lambda^2)^{\frac{3}{2}}(x^2-\lambda^2)^{\frac{1}{2}}}$$
$$= 2\pi\,(1 + x^2 + x^4 + x^6 + \&c.) = \frac{2\pi}{1-x^2}.$$

Replacing $\cos^{-1}\lambda$ by $\tfrac{1}{2}\pi - \sin^{-1}\lambda$, this result becomes

$$\int_0^x \frac{d\lambda}{(1-\lambda^2)(x^2-\lambda^2)^{\frac{1}{2}}} + \int_0^x \frac{\lambda \sin^{-1}\lambda\, d\lambda}{(1-\lambda^2)^{\frac{3}{2}}(x^2-\lambda^2)^{\frac{1}{2}}} = \frac{\pi}{2(1-x^2)}.$$

The first integral is easily shown to be equal to $\dfrac{\pi}{2\sqrt{(1-x^2)}}$, so that we find

$$\int_0^x \frac{\lambda \sin^{-1}\lambda\, d\lambda}{(1-\lambda^2)^{\frac{3}{2}}(x^2-\lambda^2)^{\frac{1}{2}}} = \frac{\pi}{2} \left\{ \frac{1}{1-x^2} - \frac{1}{\sqrt{(1-x^2)}} \right\}.$$

The integral may also be written in the forms

$$x \int_0^{\frac{1}{2}\pi} \frac{\sin\phi \sin^{-1}(x\sin\phi)}{(1-x^2\sin^2\phi)^{\frac{3}{2}}} d\phi = x \int_0^K \frac{\mathrm{sn}\, u \sin^{-1}(x\,\mathrm{sn}\, u)}{\mathrm{dn}^2 u} du,$$
$$(\text{mod.} = x).$$

§49. Putting $x = \sin \alpha$ in the first form of the integral, we may also express the result by the equation

$$\int_0^\alpha \frac{\theta \sin \theta \, d\theta}{\cos^2 \theta \, (\sin^2 \alpha - \sin^2 \theta)^{\frac{1}{2}}} = \pi \, \frac{\sin^2 \frac{1}{2}\alpha}{\cos^2 \alpha} \, .$$

Values of integrals and series, §§ 50–54.

§50. It is easy to prove that

$$\int_0^x \frac{A_0 + A_1 \lambda^2 + A_2 \lambda^4 + \&c.}{\sqrt{(x^2 - \lambda^2)}} \, d\lambda$$

$$= \frac{\pi}{2} \left\{ A_0 + \frac{1}{2} A_1 x^2 + \frac{1.3}{2.4} A_2 x^4 + \frac{1.3 \, 5}{2.4.6} A_3 x^6 + \&c. \right\},$$

and by applying this result to the expression of G on p. 148 of vol xix., we find

$$\int_0^x \frac{G \, d\gamma}{\sqrt{(x^2 - \lambda^2)}} = \pi G^0 \left\{ 1 + \frac{1}{4^2} x^2 + \frac{3^2}{4^2 \cdot 8^2} x^4 + \frac{3^2 \cdot 7^2}{4^2 \cdot 8^2 \cdot 12^2} x^6 + \&c. \right\},$$

and

$$\int_0^x \frac{G\lambda \, d\lambda}{\sqrt{(x^2 - \lambda^2)}} = -\frac{1}{2}\pi K^0 \left\{ \frac{1}{4} x^2 + \frac{1^2}{4^2 \cdot 8} x^4 + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2 \cdot 12} x^6 \right.$$
$$\left. + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2 \cdot 16} x^8 + \&c. \right\}.$$

§51. Similarly from the series for I and E we find

$$\int_{-x}^x \frac{I \, d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \pi G^0 \left\{ 1 - \frac{3}{4^2} x^2 - \frac{3^2 \cdot 7}{4^2 \cdot 8^2} x^4 - \frac{3^2 \cdot 7^2 \cdot 11}{4^2 \cdot 8^2 \cdot 12^2} x^6 - \&c. \right\}$$
$$- \frac{1}{2}\pi K^0 \left\{ 1 + \frac{1^2}{4^2} x^2 + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2} x^4 + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} x^6 + \&c. \right\},$$

$$\int_{-x}^x \frac{E \, d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \pi G^0 \left\{ 1 - \frac{3}{4^2} x^2 - \frac{3^2 \cdot 7}{4^2 \cdot 8^2} x^4 - \frac{3^2 \cdot 7^2 \cdot 11}{4^2 \cdot 8^2 \cdot 12^2} x^6 + \&c. \right\}$$
$$+ \frac{1}{2}\pi K^0 \left\{ 1 + \frac{1^2}{4^2} x^2 + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2} x^4 + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} x^6 + \&c. \right\},$$

with corresponding series in which $d\lambda$ is replaced by $\lambda d\lambda$.

§52. By putting
$$\alpha = \tfrac{1}{4}, \ \beta = \tfrac{1}{4}, \ \gamma = 1$$
in Gauss's well-known formula
$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma) \, \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \, \Gamma(\gamma - \beta)},$$
we find
$$1 + \frac{1^2}{4^2} + \frac{1^2 . 5^2}{4^2 . 8^2} + \frac{1^2 . 5^2 . 9^2}{4^2 . 8^2 . 12^2} + \&\text{c.} = \frac{\Gamma(\tfrac{1}{2})}{\Gamma^2(\tfrac{3}{4})} = \pi^{\frac{1}{2}} \frac{\Gamma^2(\tfrac{1}{4})}{\Gamma^2(\tfrac{1}{4}) \, \Gamma^2(\tfrac{3}{4})}$$
$$= \pi^{\frac{1}{2}} \frac{\Gamma^2(\tfrac{1}{4})}{2\pi^{\frac{3}{2}}} = \frac{2K^0}{\pi},$$

agreeing with the result quoted in §38.

Similarly, putting
$$\alpha = -\tfrac{1}{4}, \ \beta = -\tfrac{1}{4}, \ \gamma = 1,$$
we find
$$1 + \frac{1}{4^2} + \frac{3^2}{4^2 . 8^2} + \frac{3^2 . 7^2}{4^2 . 8^2 . 12^2} + \&\text{c.} = \frac{\Gamma(\tfrac{3}{2})}{\Gamma^2(\tfrac{5}{4})} = \frac{\tfrac{1}{2}\Gamma(\tfrac{1}{2})}{\tfrac{1}{16}\Gamma^2(\tfrac{1}{4})}$$
$$= \frac{2}{K^0} = \frac{8 G^0}{\pi},$$

and by putting
$$\alpha = -\tfrac{1}{4}, \ \beta = \tfrac{3}{4}, \ \gamma = 1,$$
we find
$$1 - \frac{3}{4^2} - \frac{3^2 . 7}{4^2 . 8^2} - \frac{3^2 . 7^2 . 11}{4^2 . 8^2 . 12^2} - \&\text{c.} = \frac{\Gamma(\tfrac{1}{2})}{\Gamma(\tfrac{5}{4}) \, \Gamma(\tfrac{1}{4})} = \frac{4\Gamma(\tfrac{1}{2})}{\Gamma^2(\tfrac{1}{4})}$$
$$= \frac{1}{K^0} = \frac{4 G^0}{\pi}.$$

§53. Putting $x = 1$ in the formulæ of §§ 50 and 51, and substituting the above values for the series, we find
$$\int_0^{\frac{1}{2}\pi} G \, d\gamma = 4 \, (G^0)^2,$$
$$\int_0^{\frac{1}{2}\pi} I \, d\gamma = 2 \, (G^0)^2 - \tfrac{1}{2} (K^0)^2,$$
$$\int_0^{\frac{1}{2}\pi} E \, d\gamma = 2 \, (G^0)^2 + \tfrac{1}{2} (K^0)^2,$$

γ being the modular angle.

The last two equations give by subtraction

$$\int_0^{\frac{1}{2}\pi} K d\gamma = (K^0)^2,$$

which is the formula found in § 38.

§ 54. By subtracting the first formula from the second and third, we find

$$\int_0^{\frac{1}{2}\pi} k^2 K d\gamma = 2 (G^0)^2 + \tfrac{1}{2} (K^0)^2 = \int_0^{\frac{1}{2}\pi} E d\gamma,$$

and $$\int_0^{\frac{1}{2}\pi} k'^2 K d\gamma = - 2 (G^0)^2 + \tfrac{1}{2} (K^0)^2 = - \int_0^{\frac{1}{2}\pi} I d\gamma,$$

The case n = 3, §§ 55–57.

§ 55. When $n = 3$, the general formula gives

$$\int_{-x}^{x} \frac{P_3 d\lambda}{\sqrt{(x^2 - \lambda^2)}} = 2^3 \pi \frac{\Gamma^2\left(\frac{3}{2}\right)}{\Gamma(3)} \left\{ 1 + \frac{3^2}{2^2} x^2 + \frac{3^2 \cdot 5^2}{2^2 \cdot 4^2} x^4 + \frac{3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2} x^6 + \&c. \right\}.$$

It can be shown (see § 57) that the series

$$1 + \frac{3^2}{2^2} x^2 + \frac{3^2 \cdot 5^2}{2^2 \cdot 4^2} x^4 + \frac{3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2} x^6 + \&c.$$

is equal to

$$\frac{2 (G + E)}{\pi (1 - x^2)}, \quad (\text{mod.} = x),$$

so that the equation becomes

$$\int_{-x}^{x} \frac{P_3 d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \frac{2\pi (G + E)}{1 - x^2}, \quad (\text{mod.} = x).$$

§ 56. Now, by § 6,

$$P_3 = \frac{3 (h' - h)^2 + 4hh'}{8h^2 h'^2} \frac{\gamma}{kk'} - \frac{3 (h' - h)}{8h^2 h'^2}$$

$$= \frac{2 (1 + 2\lambda^2)}{(1 - \lambda^2)^{\frac{5}{2}}} \cos^{-1}\lambda - \frac{6\lambda}{(1 - \lambda^2)^2}.$$

Thus

$$\int_{-x}^{x} \frac{P_5 d\lambda}{\sqrt{(x^2 - \lambda^2)}} = 4 \int_{0}^{x} \frac{(1 + 2\lambda^2) \cos^{-1}\lambda d\lambda}{(1 - \lambda^2)^{\frac{5}{2}} (x^2 - \lambda^2)^{\frac{1}{2}}}$$

$$= 4 \int_{0}^{x} \frac{(1 + 2\lambda^2) (\frac{1}{2}\pi - \sin^{-1}\lambda) d\lambda}{(1 - \lambda^2)^{\frac{5}{2}} (x^2 - \lambda^2)^{\frac{1}{2}}} = 2\pi \int_{0}^{x} \frac{(1 + 2\lambda^2) d\lambda}{(1 - \lambda^2)^{\frac{5}{2}} (x^2 - \lambda^2)^{\frac{1}{2}}}$$

$$= 2\pi \int_{0}^{\frac{1}{2}\pi} \frac{1 + 2x^2 \sin^2\theta}{(1 - x^2 \sin^2\theta)^{\frac{5}{2}}} d\theta, \text{ by putting } \lambda = x \sin \theta,$$

$$= 2\pi \int_{0}^{K} \frac{1 + 2x^2 \operatorname{sn}^2 u}{\operatorname{dn}^4 u} du, \quad (\text{mod.} = x)$$

$$= 2\pi \int_{0}^{K} (3 \operatorname{nd}^4 u - 2 \operatorname{nd}^2 u) \, du.$$

Now, the squared modulus being h,

$$\int_{0}^{K} \operatorname{nd}^4 u \, du = \frac{2 (1 + h')}{3h'} \int_{0}^{K} \operatorname{nd}^2 u \, du - \frac{K}{3h'},$$

and

$$\int_{0}^{K} \operatorname{nd}^2 u \, du = \frac{E}{h'}, \text{[*]}$$

whence

$$3 \int_{0}^{K} \operatorname{nd}^4 u \, du - 2 \int_{0}^{K} \operatorname{nd}^2 u \, du = \left(\frac{2 (1 + h')}{h'} - 2\right) \frac{E}{h'} - \frac{K}{h'}$$

$$= \frac{2E - h'K}{h'^2} = \frac{G + E}{h'^2}.$$

Thus, when the modulus is x, the value of the integral

$$\int_{-x}^{x} \frac{P_5 d\lambda}{\sqrt{(x^2 - \lambda^2)}}$$

has been shown to be

$$\frac{2\pi (G + E)}{1 - x^2}, \quad (\text{mod.} = x),$$

agreeing with the value of the integral which was derived from the general formula in the preceding section. We have thus obtained a verification of the general formula in the case $n = 3$, but no new result.

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[*] *Messenger*, vol. XI., pp. 129, 134.

§ 57. The value assigned to the series upon the right-hand side of the equation in § 55 may be obtained as follows:

Differentiating, with respect to h, the formula

$$\frac{2K}{\pi} = 1 + \frac{1^2}{2^2} h + \frac{1^2 . 3^2}{2^2 . 4^2} h^2 + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} h^3 + \&c.,$$

we find

$$\frac{2}{\pi} \frac{G}{hh'} = \frac{1^2}{2} + \frac{1^2 . 3^2}{2^2 . 4} h + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6} h^2 + \frac{1^2 . 3^2 . 5^2 . 7^2}{2^2 . 4^2 . 6^2 . 8} h^3 + \&c.,$$

whence, multiplying by h and differentiating again,

$$\frac{2}{\pi} \frac{d}{dh} \left(\frac{G}{h'} \right) = \frac{1}{2} \left\{ 1 + \frac{3^2}{2^2} h + \frac{3^2 . 5^2}{2^2 . 4^2} h^2 + \frac{3^2 . 5^2 . 7^2}{2^2 . 4^2 . 6^2} h^3 + \&c. \right\}.$$

Now

$$\frac{d}{dh} \frac{G}{h'} = \frac{1}{h'} \frac{K}{2} + \frac{G}{h'^2} = \frac{2G + h'K}{2h'^2} = \frac{G + E}{2h'^2},$$

so that the equation becomes

$$\frac{2}{\pi} \frac{G + E}{h'^2} = 1 + \frac{3^2}{2^2} h + \frac{3^2 . 5^2}{2^2 . 4^2} h^2 + \frac{3^2 . 5^2 . 7^2}{2^2 . 4^2 . 6^2} h^3 + \&c.,$$

which, when h is replaced by x, is the result quoted in § 55.

The general theorem when n is a positive integer, §§ 58–63.

§ 58. The method employed in the preceding section suggests a symbolic form in which the theorem of § 41 may be exhibited when n is a positive integer.

Taking only the first of the two integrals, this theorem may be written

$$\int_{-x}^{x} \frac{P_{2n-1} d\lambda}{\sqrt{(x^2 - \lambda^2)}} = 2^{2n-3} \pi \frac{\Gamma^2 (\frac{1}{2}n)}{\Gamma (n)} \left\{ 1 + \frac{n^2}{2^2} x^2 + \frac{n^2 (n+2)^2}{2^2 . 4^2} x^4 \right.$$
$$\left. + \frac{n^2 (n+2)^2 (n+4)^2}{2^2 . 4^2 . 6^2} x^6 + \&c. \right\},$$

where

$$P_{2n-1} = \int_0^K (\operatorname{sd} u)^{2n-1} du.$$

§ 59. Let S_n denote the series

$$1 + \frac{n^2}{2^2} x^2 + \frac{n^2 (n+2)^2}{2^2 . 4^2} x^4 + \frac{n^2 (n+2)^2 (n+4)^2}{2^2 . 4^2 . 6^2} x^6 + \&c.,$$

then by performing the operations we easily find (as in the preceding section) that

$$\frac{1}{x}\frac{d}{dx} x \frac{d}{dx} S_n = n^2 S_{n+2}.$$

If we denote the operation

$$\frac{1}{x}\frac{d}{dx} x \frac{d}{dx} \text{ by } \zeta,$$

this equation may be written

$$S_{n+2} = \frac{1}{n^2} \zeta S_n.$$

§ 60. Thus we find

$$S_3 = \zeta S_1,$$

$$S_5 = \frac{1}{3^2} \zeta^2 S_1,$$

&c.,   &c.,

and in general

$$S_{2r+1} = \frac{1}{1^2.3^2...(2r-1)^2} \zeta^r S_1.$$

Now, when $n = 2r + 1$, the coefficient of S_n in the general theorem

$$= 2^{4r-1}\pi \frac{\Gamma^2(r+\frac{1}{2})}{\Gamma(2r+1)}$$

$$= 2^{4r-1}\pi \frac{(r-\frac{1}{2})^2(r-\frac{3}{2})^2...(\frac{1}{2})^2 \Gamma^2(\frac{1}{2})}{(2r)!}$$

$$= 2^{2r-1}\pi^2 \frac{(2r-1)^2(2r-3)^2...3^2.1^2}{(2r)!}.$$

Thus, in this case, the theorem becomes

$$\int_{-x}^{x} \frac{P_{4r+1}d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \frac{2^{2r-1}\pi^2}{(2r)!} \zeta^r S_1,$$

or, since $S_1 = \dfrac{2K}{\pi}$, (mod. $= x$),

$$\int_{-x}^{x} \frac{P_{4r+1}d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \frac{2^{2r}\pi}{(2r)!} \left(\frac{1}{x}\frac{d}{dx} x \frac{d}{dx}\right)^r K,$$

the modulus of the complete elliptic integral K on the right-hand side of the equation being x.

§ 61. By means of the theorem (§ 2)

$$\frac{dP_n}{d\lambda} = -\frac{n+1}{4}\, P_{n+2},$$

we may express P_{4r+1} as a repeated derivative of P_1. We thus find

$$P_{4r+1} = \frac{2^{2r}}{(2r)\,!}\, \left(\frac{d}{d\lambda}\right)^{2r} P_1,$$

and the formula may be written in the form

$$\int_0^x \frac{d^{2r}P_1}{d\lambda^{2r}}\, \frac{d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \tfrac{1}{2}\pi^2 \left(\frac{1}{x}\frac{d}{dx}\, x\,\frac{d}{dx}\right)^r S_1,$$

or, replacing P_1 and S_1 by their values,

$$\int_{-x}^x \left\{\frac{d^{2r}}{d\lambda^{2r}}\frac{\cos^{-1}\lambda}{\sqrt{(1-\lambda^2)}}\right\} \frac{dx}{\sqrt{(x^2-\lambda^2)}} = \pi \left(\frac{1}{x}\frac{d}{dx}\, x\,\frac{d}{dx}\right)^r K,\ (\text{mod.} = x)$$

for, by § 6,

$$P_1 = \frac{\gamma}{kk'} = \frac{\cos^{-1}\lambda}{\sqrt{(1-\lambda^2)}}\,.$$

The cases $r = 0$ and $r = 1$ have been considered in §§ 45 and 55–57.

§ 62. Proceeding as in § 60, we have

$$S_4 = \frac{1}{2^2}\,\zeta S_2,$$

$$S_6 = \frac{1}{2^2.4^2}\,\zeta^2 S_2,$$

$$\&\text{c.,}\qquad \&\text{c.,}$$

and in general

$$S_{2r+2} = \frac{1}{2^{2r}\,(r\,!)^2}\,\zeta^r S_2.$$

When $n = 2r + 2$, the coefficient of S_n

$$= 2^{4r+1}\pi\,\frac{(r\,!)^2}{(2r+1)\,!},$$

so that the theorem becomes

$$\int_{-x}^{x} \frac{P_{4r+3}\,d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \frac{2^{2r+1}\pi}{(2r + 1)!}\, \zeta^r S_2,$$

that is

$$\int_{-x}^{x} \frac{P_{4r+3}\,d\lambda}{\sqrt{(x^2 - \lambda^2)}} = \frac{2^{2r+1}\pi}{(2r + 1)!} \left(\frac{1}{x}\frac{d}{dx}\, x\,\frac{d}{dx}\right)^r \frac{1}{1 - x^2}.$$

We may also write the theorem in the form

$$\int_{-x}^{x} \frac{d^{2r+1}P_1}{d\lambda^{2r+1}}\, \frac{d\lambda}{\sqrt{(x^2 - \lambda^2)}} = -\pi \left(\frac{1}{x}\frac{d}{dx}\, x\,\frac{d}{dx}\right)^r S_2,$$

or

$$\int_{-x}^{x} \left\{\frac{d^{2r+1}}{d\lambda^{2r+1}}\, \frac{\cos^{-1}\lambda}{\sqrt{(1 - \lambda^2)}}\right\} \frac{dx}{\sqrt{(x^2 - \lambda^2)}} = -\pi \left(\frac{1}{x}\frac{d}{dx}\, x\,\frac{d}{dx}\right)^r \frac{1}{1 - x^2}.$$

The case $r = 0$ was considered in § 48.

§ 63. It may be observed that if we put $x^2 = z$, the operator ζ

$$= \frac{1}{x}\frac{d}{dx}\, x\,\frac{d}{dx} = 4\frac{d}{dz}\, z\,\frac{d}{dz},$$

so that the theorems may be written

$$\int_{-x}^{x} \frac{d^{2r}P_1}{d\lambda^{2r}}\, \frac{d\lambda}{\sqrt{(x^2 - \lambda^2)}} = 2^{2r-1}\pi \left(\frac{d}{dz}\, z\,\frac{d}{dz}\right)^r K,$$

where $z = x^2$ is the square of the modulus of K,

and $$\int_{-x}^{x} \frac{d^{2r+1}P_1}{d\lambda^{2r+1}}\, \frac{d\lambda}{\sqrt{(x^2 - \lambda^2)}} = -2^{2r}\pi \left(\frac{d}{dz}\, z\,\frac{d}{dz}\right)^r \frac{1}{1 - z},$$

where $z = x^2$.