ON CERTAIN SERIES AND DEFINITE INTEGRALS.

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THE series for K in ascending powers of $h'-h$ which was given on p. 147 of vol. xix., and is referred to on p. 73 of the present volume suggests that it would be interesting to express as a definite integral a more general series which includes it as a particular case. It will be seen that we are thus led to several curious results involving definite integrals and series.

The general theorem, §§ 1–5.

§ 1. Since

$$\int_0^\infty e^{-u^2} u^n du = \tfrac{1}{2}\Gamma\left(\frac{n+1}{2}\right),$$

we find, by expanding e^{-xuv} in ascending powers of xuv, and integrating term by term with respect to u and v,

$$\int_0^\infty \int_0^\infty e^{-u^2-v^2-xuv} u^\alpha v^\beta du\, dv$$

$$= \frac{1}{4}\left\{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right) - x\,\Gamma\left(\frac{\alpha+2}{2}\right)\Gamma\left(\frac{\beta+2}{2}\right) + \frac{x^2}{2\,!}\Gamma\left(\frac{\alpha+3}{2}\right)\Gamma\left(\frac{\beta+3}{2}\right) - \&c.\right\}$$

$$= \frac{1}{4}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)\left\{1 + \frac{(\alpha+1)(\beta+1)}{2\,!\,2^2}x^2 + \frac{(\alpha+1)(\alpha+3)(\beta+1)(\beta+3)}{4\,!\,2^4}x^4 + \&c.\right\}$$

$$- \frac{1}{4}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\left\{1 + \frac{\alpha\beta}{2^2}x + \frac{\alpha(\alpha+2)\beta(\beta+2)}{3\,!\,2^4}x^3 + \frac{\alpha(\alpha+2)(\alpha+4)\beta(\beta+2)(\beta+4)}{5\,!\,2^6}x^5 + \&c.\right\}.$$

In order that the integral may be finite α and β must both be >-1. The series is always convergent if x lies between -1 and $+1$.

Transforming the above integral by putting $u=s^2$, $v=t^2$, we see that it may also be written in the form

$$4\int_0^\infty \int_0^\infty e^{-s^4-t^4-xs^2t^2} s^{2\alpha+1} t^{2\beta+1} ds\, dt.$$

§ 2. Replacing x by $2x$ in these results, we have therefore

$$4\int_0^\infty\int_0^\infty e^{-u^2-v^2-2xuv}\,u^\alpha v^\beta\,du\,dv$$

$$=16\int_0^\infty\int_0^\infty e^{-s^4-t^4-2xs^2t^2}\,s^{2\alpha+1}t^{2\beta+1}\,ds\,dt$$

$$=\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)\left\{1+\frac{(\alpha+1)(\beta+1)}{2\,!}x^2+\frac{(\alpha+1)(\alpha+3)(\beta+1)(\beta+3)}{4\,!}x^4+\&c.\right\}$$

$$-\frac{1}{2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\left\{\alpha\beta x+\frac{\alpha(\alpha+2)\beta(\beta+2)}{3\,!}x^3+\frac{\alpha(\alpha+2)(\alpha+4)\beta(\beta+2)(\beta+4)}{5\,!}x^5+\&c.\right\},$$

in which α and β are >-1, and $x>-1$ and <1.

§ 3. Putting $u=r\sin\theta$, $v=r\cos\theta$, we find

$$\int_0^\infty\int_0^\infty e^{-u^2-v^2-2xuv}u^\alpha v^\beta\,du\,dv$$

$$=\int_0^\infty\int_0^{\frac{1}{2}\pi} e^{-(1+x\sin 2\theta)r^2}\,r^{\alpha+\beta}(\sin\theta)^\alpha(\cos\theta)^\beta\,r\,d\theta\,dr$$

$$=\frac{1}{2}\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{\frac{1}{2}\pi}\frac{(\sin\theta)^\alpha(\cos\theta)^\beta}{(1+x\sin 2\theta)^{\frac{1}{2}(\alpha+\beta+2)}}\,d\theta$$

$$=\frac{1}{4}\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{\pi}\frac{(\sin\frac{1}{2}\theta)^\alpha(\cos\frac{1}{2}\theta)^\beta}{(1+x\sin\theta)^{\frac{1}{2}(\alpha+\beta+2)}}\,d\theta.$$

Similarly, putting $s=r\sin\theta$, $t=r\cos\theta$, we find

$$\int_0^\infty\int_0^\infty e^{-s^4-t^4-2xs^2t^2}s^{2\alpha+1}t^{2\beta+1}\,ds\,dt$$

$$=\int_0^\infty\int_0^{\frac{1}{2}\pi} e^{-(1-h\sin^2 2\theta)r^4}\,r^{2\alpha+2\beta+2}(\sin\theta)^{2\alpha+1}(\cos\theta)^{2\beta+1}\,r\,d\theta\,dr,$$

where $h=\frac{1}{2}-\frac{1}{2}x$,

$$=\frac{1}{4}\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{\frac{1}{2}\pi}\frac{(\sin\theta)^{2\alpha+1}(\cos\theta)^{2\beta+1}}{(1-h\sin^2 2\theta)^{\frac{1}{2}(\alpha+\beta+2)}}\,d\theta$$

$$=\frac{1}{8}\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{\pi}\frac{(\sin\frac{1}{2}\theta)^{2\alpha+1}(\cos\frac{1}{2}\theta)^{2\beta+1}}{(1-h\sin^2\theta)^{\frac{1}{2}(\alpha+\beta+2)}}\,d\theta.$$

In this last integral let $\theta = \operatorname{am} u$: it thus becomes

$$= \frac{1}{2^{\alpha+\beta+4}} \Gamma\left(\frac{\alpha+\beta+2}{2}\right) \int_0^{2K} \frac{\operatorname{sn} u\,(1-\operatorname{cn} u)^{\alpha}(1+\operatorname{cn} u)^{\beta}}{(\operatorname{dn} u)^{\alpha+\beta+2}}\,du,$$

where the square of the modulus in the elliptic functions $= \frac{1}{2} - \frac{1}{2}x$.

§ 4. It has thus been shown that the value of the series

$$\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)\left\{1 + \frac{(\alpha+1)(\beta+1)}{2!}x^2 + \frac{(\alpha+1)(\alpha+3)(\beta+1)(\beta+3)}{4!}x^4 + \&\text{c.}\right\}$$

$$-\frac{1}{2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\left\{\alpha\beta x + \frac{\alpha(\alpha+2)\beta(\beta+2)}{3!}x^3 + \frac{\alpha(\alpha+2)(\alpha+4)\beta(\beta+2)(\beta+4)}{5!}x^5 + \&\text{c.}\right\}$$

is represented by the five definite integrals

$$4\int_0^{\infty}\int_0^{\infty} e^{-u^2-v^2-2xuv} u^{\alpha} v^{\beta}\,du\,dv \quad \text{(i)}$$

$$= 16\int_0^{\infty}\int_0^{\infty} e^{-s^4-t^4-2xs^2t^2} s^{2\alpha+1} t^{2\beta+1}\,ds\,dt \quad \text{(ii)}$$

$$= \Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{\pi} \frac{(\sin\frac{1}{2}\theta)^{\alpha}(\cos\frac{1}{2}\theta)^{\beta}}{(1+x\sin\theta)^{\frac{1}{2}(\alpha+\beta+2)}}\,d\theta \quad \text{(iii)}$$

$$= 2\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{\pi} \frac{(\sin\frac{1}{2}\theta)^{2\alpha+1}(\cos\frac{1}{2}\theta)^{2\beta+1}}{(1-h\sin^2\theta)^{\frac{1}{2}(\alpha+\beta+2)}}\,d\theta \quad \text{(iv)}$$

$$= \frac{1}{2^{\alpha+\beta}}\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\int_0^{2K} \frac{\operatorname{sn} u\,(1-\operatorname{cn} u)^{\alpha}(1+\operatorname{cn} u)^{\beta}}{(\operatorname{dn} u)^{\alpha+\beta+1}}\,du \quad \text{(v)},$$

where, as before, α and β are each > -1, x lies between -1 and $+1$, and h, the square of the modulus in the last integral, $= \frac{1}{2} - \frac{1}{2}x$.

§ 5. The formula obtained by equating (iii) and (iv) is perhaps deserving of being noticed separately. It may be written

$$\int_0^{\pi} \frac{(\sin\frac{1}{2}\theta)^{\alpha}(\cos\frac{1}{2}\theta)^{\beta}}{(1+x\sin\theta)^{\frac{1}{2}(\alpha+\beta+2)}}\,d\theta = \int_0^{\pi} \frac{\sin\theta\,(\sin\frac{1}{2}\theta)^{2\alpha}(\cos\frac{1}{2}\theta)^{2\beta}}{(1-h\sin^2\theta)^{\frac{1}{2}(\alpha+\beta+2)}}\,d\theta,$$

where $h = \frac{1}{2} - \frac{1}{2}x$.

The special case $\alpha=-\frac{1}{2}$, $\beta=-\frac{1}{2}$, § 6.

§ 6. If $\alpha=-\frac{1}{2}$ and $\beta=-\frac{1}{2}$, then

$$\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)=\Gamma^2\left(\tfrac{1}{4}\right),$$

and $$\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)=\Gamma^2\left(-\frac{1}{4}\right)=16\Gamma^2\left(\tfrac{3}{4}\right).$$

The series in § 4 therefore becomes

$$\Gamma^2\left(\tfrac{1}{4}\right)\left\{1+\frac{1^2}{2.4}x^2+\frac{1^2.5^2}{2.4.6.8}x^4+\&c.\right\}$$

$$-4\Gamma^2\left(\tfrac{3}{4}\right)\left\{\frac{1}{2}x+\frac{3^2}{2.4.6}x^3+\frac{3^2.7^2}{2.4.6.8.10}x^5+\&c.\right\},$$

and the integral (v)

$$=2\Gamma\left(\tfrac{1}{2}\right)\int_0^{2K}du=4\pi^{\frac{1}{2}}K.$$

If h be the square of the modulus, then $x=1-2h=h'-h$, which is the quantity denoted by λ in the paper in vol. xix. Thus the above result is the same as the expansion of K given on p. 147 of that volume.

The case $\alpha=i-\frac{1}{2}$, $\beta=-i-\frac{1}{2}$, §§ 7, 8.

§ 7. Let $\alpha=i-\frac{1}{2}$, $\beta=-i-\frac{1}{2}$, then the series in § 4

$$=\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}i\right)\Gamma\left(\tfrac{1}{4}-\tfrac{1}{2}i\right)\left\{1-\frac{i^2-\frac{1}{4}}{2\,!}x^2+\frac{(i^2-\frac{1}{4})(i^2-\frac{25}{4})}{4\,!}x^4-\&c.\right\}$$

$$+\tfrac{1}{2}\Gamma\left(-\tfrac{1}{4}+\tfrac{1}{2}i\right)\Gamma\left(-\tfrac{1}{4}-\tfrac{1}{2}i\right)\left\{(i^2-\tfrac{1}{4})x\right.$$

$$\left.-\frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})}{3\,!}x^3+\frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})(i^2-\frac{25}{4})}{5\,!}x^5-\&c.\right\},$$

and the integrals (iii), (iv) and (v) become

$$16\int_0^\infty\int_0^\infty e^{-s^4-t^4-2xs^2t^2}\left(\frac{s}{t}\right)^{2i}ds\,dt$$

$$=2\pi^{\frac{1}{2}}\int_0^\pi\frac{(\tan\frac{1}{2}\theta)^{2i}}{(1-h\sin^2\theta)^{\frac{1}{2}}}d\theta$$

$$=2\pi^{\frac{1}{2}}\int_0^{2K}\left(\frac{1-\operatorname{cn}u}{1+\operatorname{cn}u}\right)^i du,$$

the modulus being as before $=\sqrt{(\frac{1}{2}-\frac{1}{2}x)}$.

In these formulæ i must lie between $-\frac{1}{2}$ and $+\frac{1}{2}$.

§ 8. The two series which occur in the series in the preceding section are the series A_i and B_i of p. 72 of the present volume. It follows therefore that the above three integrals satisfy Legendre's equation.

For example, taking the third integral, we see that if

$$w = \int_0^{2K} \left(\frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}\right)^i du,$$

then w satisfies Legendre's equation

$$(1 - \lambda^2)\frac{d^2 w}{d\lambda^2} - 2\lambda\frac{dw}{d\lambda} + (i^2 - \tfrac{1}{4})\, w = 0,$$

where $\lambda = h' - h$, h being the squared modulus of the elliptic functions.

The case $\alpha = \beta$, §§ 9–11.

§ 9. If $\beta = \alpha$, the series in § 4 becomes

$$\Gamma^2\left(\frac{\alpha+1}{2}\right)\left\{1 + \frac{(\alpha+1)^2}{2\,!}x^2 + \frac{(\alpha+1)^2(\alpha+3)^2}{4\,!}x^4 + \&c.\right\}$$

$$- \tfrac{1}{2}\Gamma^2\left(\frac{\alpha}{2}\right)\left\{\alpha^2 x + \frac{\alpha^2(\alpha+2)^2}{3\,!}x^3 + \frac{\alpha^2(\alpha+2)^2(\alpha+4)^2}{5\,!}x^5 + \&c.\right\},$$

and the five integrals are

$$4\int_0^\infty\int_0^\infty e^{-u^2-v^2-2xuv}(uv)^\alpha\, du\, dv$$

$$= 16\int_0^\infty\int_0^\infty e^{-s^4-t^4-2xs^2t^2}(st)^{2\alpha+1}\, ds\, dt$$

$$= \frac{\Gamma(\alpha+1)}{2^\alpha}\int_0^\pi \frac{(\sin\theta)^\alpha}{(1 + x\sin\theta)^{\alpha+1}}\, d\theta$$

$$= \frac{\Gamma(\alpha+1)}{2^{2\alpha}}\int_0^\pi \frac{(\sin\theta)^{2\alpha+1}}{(1 - h\sin^2\theta)^{\alpha+1}}\, d\theta$$

$$= \frac{\Gamma(\alpha+1)}{2^{2\alpha}}\int_0^{2K}\frac{(\operatorname{sn} u)^{2\alpha+1}}{(\operatorname{dn} u)^{2\alpha+1}}\, du = \frac{\Gamma(\alpha+1)}{2^{2\alpha-1}}\int_0^K (\operatorname{sd} u)^{2\alpha+1}\, du.$$

It is supposed that $\alpha > -1$ and that x lies between -1 and $+1$.

§ 10. The last result shows that, α being > -1,

$$\frac{\Gamma(\alpha+1)}{2^{2\alpha-1}} \int_0^K (\mathrm{sd}\, u)^{2\alpha+1} du$$

$$= \Gamma^2\left(\frac{\alpha+1}{2}\right)\left\{1 + \frac{(\alpha+1)^2}{2!}x^2 + \frac{(\alpha+1)^2(\alpha+3)^2}{4!}x^4 + \&c.\right\}$$

$$- \tfrac{1}{2}\Gamma^2\left(\frac{\alpha}{2}\right)\left\{\alpha^2 x + \frac{\alpha^2(\alpha+2)^2}{3!}x^3 + \frac{\alpha^2(\alpha+2)^2(\alpha+4)^2}{5!}x^5 + \&c.\right\},$$

where the square of the modulus $= \frac{1}{2} - \frac{1}{2}x$.

Putting $\alpha = \frac{1}{2}n - \frac{1}{2}$, so that n may have any value > -1, this formula may be written

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{2^{n-3}} \int_0^K \mathrm{sd}^n u\, du$$

$$= \Gamma^2\left(\frac{n+1}{4}\right)\left\{1 + \frac{(n+1)^2}{2.4}x^2 + \frac{(n+1)^2(n+5)^2}{2.4.6.8}x^4 + \&c.\right\}$$

$$- 4\Gamma^2\left(\frac{n+3}{4}\right)\left\{\frac{1}{2}x + \frac{(n+3)^2}{2.4.6}x^3 + \frac{(n+3)^2(n+7)^2}{2.4\ldots10}x^5 + \&c.\right\}.$$

This is a generalisation of the theorem in § 6, which corresponds to the case $n = 0$.

If h be the squared modulus of the elliptic functions, then $x = \lambda = h' - h$. The theorem under this form will be considered more in detail in a separate paper.

§ 11. By putting $2x$ for x, and changing the sign of x, we can thus express as the sum or difference of two definite integrals the series

$$1 + \frac{(n+1)^2}{2!}x^2 + \frac{(n+1)^2(n+5)^2}{4!}x^4 + \&c.,$$

and $$\frac{1}{2}x + \frac{(n+3)^2}{3!}x^3 + \frac{(n+3)^2(n+7)^2}{5!}x^5 + \&c.$$

The case $\alpha = -\beta$, §§ 12–15.

§ 12. By putting $\beta = -\alpha$ in the general theorem, the series becomes

$$\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha)\left\{1 - \frac{(\alpha^2-1)}{2!}x^2 + \frac{(\alpha^2-1^2)(\alpha^2-3^2)}{4!}x^4 - \&c.\right\}$$

$$+ \tfrac{1}{2}\Gamma(\tfrac{1}{2}\alpha)\,\Gamma(-\tfrac{1}{2}\alpha)\left\{\alpha^2 x - \frac{\alpha^2(\alpha^2-2^2)}{3!}x^3 + \frac{\alpha^2(\alpha^2-2^2)(\alpha^2-4^2)}{5!}x^5 - \&c.\right\},$$

and the integrals become

$$4\int_0^\infty\int_0^\infty e^{-u^2-v^2-2xuv}\left(\frac{u}{v}\right)^\alpha du\,dv$$

$$=16\int_0^\infty\int_0^\infty e^{-s^4-t^4-2xs^2t^2}s^{2\alpha+1}t^{-2\alpha+1}ds\,dt$$

$$=\int_0^\pi \frac{(\tan\frac{1}{2}\theta)^\alpha}{1+x\sin\theta}d\theta$$

$$=\int_0^\pi \frac{\sin\theta\,(\tan\frac{1}{2}\theta)^{2\alpha}}{1-h\sin^2\theta}d\theta$$

$$=\int_0^{2K}\mathrm{sd}\,u\left(\frac{1-\mathrm{cn}\,u}{1+\mathrm{cn}\,u}\right)^\alpha du,$$

the squared modulus being $=\frac{1}{2}-\frac{1}{2}x$.

The quantity α is restricted to values intermediate to -1 and $+1$.

§ 13. The series in the preceding section admits of summation as follows:

It is known that the series

$$\alpha x-\frac{\alpha(\alpha^2-2^2)}{3!}x^3+\frac{\alpha(\alpha^2-2^2)(\alpha^2-4^2)}{5!}x^5-\&c.$$

is equal to $$\frac{\sin(\alpha\sin^{-1}x)}{\sqrt{(1-x^2)}},$$

and that the series

$$1-\frac{\alpha^2-1^2}{2!}x^2+\frac{(\alpha^2-1^2)(\alpha^2-3^2)}{3!}x^4-\&c.$$

is equal to $$\frac{\cos(\alpha\sin^{-1}x)}{\sqrt{(1-x^2)}}.$$

Also

$$\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha)=\frac{\pi}{\sin(\frac{1}{2}+\frac{1}{2}\alpha)\,\pi}=\frac{\pi}{\cos\frac{1}{2}\alpha\pi},$$

and $$\Gamma(\tfrac{1}{2}\alpha)\,\Gamma(1-\tfrac{1}{2}\alpha)=\frac{\pi}{\sin\frac{1}{2}\alpha\pi},$$

so that $$\Gamma(\tfrac{1}{2}\alpha)\,\Gamma(-\tfrac{1}{2}\alpha)=-\frac{2\pi}{\alpha\sin\frac{1}{2}\alpha\pi}.$$

The series in question therefore

$$=\frac{\pi}{\cos\frac{1}{2}\alpha\pi}\frac{\cos(\alpha\sin^{-1}x)}{\sqrt{(1-x^2)}}-\frac{\pi}{\sin\frac{1}{2}\alpha\pi}\frac{\sin(\alpha\sin^{-1}x)}{\sqrt{(1-x^2)}}$$

$$=\frac{2\pi}{\sin\alpha\pi}\frac{\sin\alpha(\frac{1}{2}\pi-\sin^{-1}x)}{\sqrt{(1-x^2)}}=\frac{2\pi}{\sin\alpha\pi}\frac{\sin(\alpha\cos^{-1}x)}{\sqrt{(1-x^2)}}.$$

§ 14. We thus find that

$$4\int_0^\infty\int_0^\infty e^{-u^2-v^2-2xuv}\left(\frac{u}{v}\right)^\alpha du\,dv$$

$$=\int_0^\infty \frac{(\tan\frac{1}{2}\theta)}{1+x\sin\theta}\,d\theta$$

$$=\int_0^\infty \frac{\sin\theta\,(\tan\frac{1}{2}\theta)^{2\alpha}}{1-h\sin^2\theta}\,d\theta, \qquad (h=\tfrac{1}{2}-\tfrac{1}{2}x)$$

$$=\int_0^{2K}\operatorname{sd}u\left(\frac{1-\operatorname{cn}u}{1+\operatorname{cn}u}\right)^\alpha du, \qquad (\text{mod})^2=\tfrac{1}{2}-\tfrac{1}{2}x$$

$$=\frac{2\pi}{\sin\alpha\pi}\,\frac{\sin(\alpha\cos^{-1}x)}{\sqrt{(1-x^2)}},$$

where α and x must both lie between -1 and $+1$.

§ 15. As a particular case, putting $\alpha=0$, we have

$$2\int_0^\infty\int_0^\infty e^{-u^2-v^2-2xuv}\,du\,dv=\int_0^{\frac{1}{2}\pi}\frac{d\theta}{1+x\sin\theta}$$

$$=\int_0^{\frac{1}{2}\pi}\frac{\sin\theta\,d\theta}{1-h\sin^2\theta}=\int_0^K \operatorname{sd}u\,du=\frac{\cos^{-1}x}{\sqrt{(1-x^2)}}.$$

It is easy to verify that

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{1+x\sin\theta}=\frac{\cos^{-1}x}{\sqrt{(1-x^2)}},$$

for $\dfrac{1}{\sqrt{(1-x^2)}}=1+\dfrac{1}{2}x^2+\dfrac{1.3}{2.4}x^4+\dfrac{1.3.5}{2.4.6}x^6+\&c.$

$$=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}(1+x^2\sin^2\theta+x^4\sin^4\theta+x^6\sin^6\theta+\&c.)\,d\theta,$$

and, by differentiating the known expansion of $(\sin^{-1}x)^2$,

$$\frac{\sin^{-1}x}{\sqrt{(1-x^2)}}=x+\frac{2}{3}x^3+\frac{2.4}{3.5}x^5+\frac{2.4.6}{3.5.7}x^7+\&c.$$

$$=\int_0^{\frac{1}{2}\pi}(x\sin\theta+x^3\sin^3\theta+x^5\sin^5\theta+\&c.)\,d\theta,$$

so that

$$\frac{\frac{1}{2}\pi-\sin^{-1}x}{\sqrt{(1-x^2)}}=\int_0^{\frac{1}{2}\pi}\{1-x\sin\theta+x^2\sin^2\theta-x^3\sin^3\theta+\&c.\}\,d\theta$$

$$=\int^{\frac{1}{2}\pi}\frac{d\theta}{1+x\sin\theta}.$$

It is also easy to verify that

$$\int_0^{\frac{1}{2}\pi} \frac{\sin\theta d\theta}{1-h\sin^2\theta} = \frac{\cos^{-1}x}{\sqrt{(1-x^2)}},$$

for

$$\frac{\cos^{-1}x}{\sqrt{(1-x^2)}} = \frac{\cos^{-1}(1-2k^2)}{2k\sqrt{(1-k^2)}} = \frac{\sin^{-1}k}{k\sqrt{(1-k^2)}}$$

$$= 1 + \frac{2}{3}k^2 + \frac{2.4}{3.5}k^4 + \frac{2.4.6}{3.5.7}k^6 + \&c.$$

$$= \int_0^{\frac{1}{2}\pi} (\sin\theta + k^2\sin^3\theta + k^4\sin^5\theta + \&c.)\, d\theta$$

$$= \int_0^{\frac{1}{2}\pi} \frac{\sin\theta d\theta}{1-k^2\sin^2\theta}.$$

The case $\alpha = 1$, $\beta = 2$, §§ 16, 17.

§ 16. When $\alpha = 1$ and $\beta = 2$, the series in § 4

$$= \Gamma(\tfrac{3}{2})\left\{1 + \frac{2.3}{2\,!}x^2 + \frac{2.4.3.5}{4\,!}x^4 + \frac{2.4.6.3.5.7}{6\,!}x^6 + \&c.\right\}$$

$$- \tfrac{1}{2}\Gamma(\tfrac{1}{2})\left\{2x + \frac{1.3.2.4}{3\,!}x^3 + \frac{1.3.5.2.4.6}{5\,!}x^5 + \&c.\right\}$$

$$= \frac{\sqrt{\pi}}{2}\frac{1}{(1+x)^2}.$$

Thus we see that

$$\int_0^\infty\int_0^\infty e^{-u^2-v^2-2xuv}\,uv^2\,du\,dv$$

$$= 4\int_0^\infty\int_0^\infty e^{-s^4-t^4-2xs^2t^2}\,s^3t^5\,ds\,dt = \frac{\sqrt{\pi}}{8}\frac{1}{(1+x)^2},$$

and also

$$\int_0^\pi \frac{\sin\frac{1}{2}\theta\cos^2\frac{1}{2}\theta}{(1+x\sin\theta)^{\frac{5}{2}}}\,d\theta = 2\int_0^\pi \frac{\sin^3\frac{1}{2}\theta\cos^5\frac{1}{2}\theta}{(1-h\sin^2\theta)^{\frac{5}{2}}}\,d\theta$$

$$= \tfrac{1}{8}\int_0^{2K} \frac{\operatorname{sn}u\,(1-\operatorname{cn}u)\,(1+\operatorname{cn}u)^2}{(\operatorname{dn}u)^4}\,du = \tfrac{2}{3}\frac{1}{(1+x)^2}.$$

The last two equations give rise immediately to the following results:

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^3\theta}{(1-h\sin^2\theta)^{\frac{5}{2}}}\,d\theta = \int_0^K \frac{\operatorname{sn}^3 u}{\operatorname{dn}^4 u}\,du = \tfrac{8}{3}\frac{1}{(1+x)^2} = \frac{2}{3h'^2}.$$

The first equation is equivalent to the particular case $n=2$ of the known theorem

$$\int_0^\infty \int_0^\infty \int_0^\infty \dots e^{-x-y-\dots-t-p n \sqrt[n]{(xy\dots t)}} y^{\frac{1}{n}} z^{\frac{2}{n}} \dots t^{\frac{n-1}{n}} \, dx\, dy \dots dt$$
$$= n^{\frac{1}{2}} (2\pi)^{\frac{1}{2}(n-1)} \frac{1.2.3\dots(n-1)}{(p+n)^n}.$$

§ 17. It is evident that we may obtain more general results by differentiating the above theorems with respect to x. For example, n being any positive integer, we have

$$\int_0^\infty \int_0^\infty e^{-u^2-v^2-2xuv} u^n v^{n+1} \, du\, dv = \frac{\sqrt{\pi}}{2^{n+2}} \frac{n!}{(1+x)^{n+1}},$$

giving
$$\int_0^K \frac{\operatorname{sn}^{2n-1} u}{\operatorname{dn}^{2n} u} du = \frac{2.4.6\dots(2n-2)}{1.3.5\dots(2n-1)} \frac{1}{h'^n},$$

where n is any positive integer greater than unity.

The corresponding formula when n is not restricted to integral values is given in § 20 of the following paper.

Formulæ derivable by repeated differentiation, §§ 18, 19.

§ 18. We may also introduce into the double integrals two letters a and b with respect to which, as well as to x, they may be differentiated.

Thus from §§ 15, 16, and the case $\alpha = -\frac{1}{2}$ of § 9, we find

$$\int_0^\infty \int_0^\infty e^{-au^2-bv^2-2xuv} \, du\, dv = \frac{1}{2} \frac{\cos^{-1} \frac{x}{a^{\frac{1}{2}} b^{\frac{1}{2}}}}{\sqrt{(ab-x^2)}},$$

$$\int_0^\infty \int_0^\infty e^{-au^2-bv^2-2xuv} uv^3 \, du\, dv = \frac{\sqrt{\pi}}{8b^{\frac{3}{2}}} \frac{1}{(x+a^{\frac{1}{2}}b^{\frac{1}{2}})^2},$$

$$\int_0^\infty \int_0^\infty e^{-au^2-bv^2-2xuv} u^{-\frac{1}{2}} v^{-\frac{1}{2}} \, du\, dv = \frac{\sqrt{\pi}}{a^{\frac{1}{4}} b^{\frac{1}{4}}} K\left(\frac{1}{2} - \frac{x}{2a^{\frac{1}{2}}b^{\frac{1}{2}}}\right),$$

where $K(h)$ denote the complete elliptic integral corresponding to the squared modulus h.

By differentiating these results m times with respect to a, n times with respect to b, and r times with respect to x, we obtain the values of the integrals in the case in which the powers of u and v which occur are respectively

$$u^{2m+r} v^{2n+r}, \quad u^{2m+r+1} v^{2n+r+3}, \quad u^{2m+r-\frac{1}{2}} v^{2n+r-\frac{1}{2}}.$$

In the first case the exponents are any positive integral numbers whose difference is even, in the second any positive integral numbers whose difference is uneven, and in the third any fractions of the forms $\frac{2s+1}{2}$ and $\frac{2t+1}{2}$, which are not $< -\frac{1}{2}$ and differ by an even number. Zero is of course to be counted as an even number in both cases.

§ 19. For example, taking the last of the three equations and differentiating with respect to a, we find

$$\int_0^\infty\int_0^\infty e^{-au^2-bv^2-2xuv}\, u^{\frac{3}{2}} v^{-\frac{1}{2}}\, du\, dv = \frac{\sqrt{\pi}}{4a^{\frac{5}{4}}b^{\frac{1}{4}}} K - \frac{\sqrt{\pi}}{4a^{\frac{7}{4}}b^{\frac{3}{4}}}\, x \frac{dK}{dh},$$

and, differentiating with respect to x,

$$\int_0^\infty\int_0^\infty e^{-au^2-bv^2-2xuv} u^{\frac{1}{2}} v^{\frac{1}{2}}\, du\, dv = \frac{\sqrt{\pi}}{4a^{\frac{3}{4}}b^{\frac{3}{4}}} \frac{dK}{dh},$$

when, putting a and b equal to unity,

$$\int_0^\infty\int_0^\infty e^{-u^2-v^2-2xuv} u^{\frac{3}{2}} v^{-\frac{1}{2}}\, du\, dv = \frac{\sqrt{\pi}}{4}\left(K - \frac{(h'-h)}{2hh'} G\right),$$

and

$$\int_0^\infty\int_0^\infty e^{-u^2-v^2-2xuv}\, u^{\frac{1}{2}} v^{\frac{1}{2}}\, du\, dv = \frac{\sqrt{\pi}}{8} \frac{G}{hh'}.$$

The differential equation for K derived from the double definite integral, § 20.

§ 20. By introducing a and b into the definite integral in the equation

$$\int_0^\infty\int_0^\infty e^{-s^4-t^4-2xs^2t^2}\, ds\, dt = \frac{\sqrt{\pi}}{4} K\left(\tfrac{1}{2} - \tfrac{1}{2}x\right),$$

we have

$$\int_0^\infty\int_0^\infty e^{-as^4-bt^4-2xs^2t^2}\, ds\, dt = \frac{\sqrt{\pi}}{4a^{\frac{1}{4}}b^{\frac{1}{4}}} K\left(\tfrac{1}{2} - \frac{x}{2a^{\frac{1}{2}}b^{\frac{1}{2}}}\right).$$

It is evident that the effect of differentiating the definite integral twice with respect to x is the same as differentiating it once with respect to a, and once with respect to b, and multiplying by 4; so that,

if
$$u = \frac{1}{a^{\frac{1}{4}}b^{\frac{1}{4}}} K\left(\tfrac{1}{2} - \frac{x}{2a^{\frac{1}{2}}b^{\frac{1}{2}}}\right),$$

then
$$\frac{d^2u}{dx^2} = 4\, \frac{d^2u}{da\, db}.$$

Now $$\frac{d^2u}{dx^2}=\frac{1}{4a^{\frac{5}{4}}b^{\frac{5}{4}}}\frac{d^2K}{dh^2};$$

and $$\frac{du}{da}=-\frac{K}{4a^{\frac{5}{4}}b^{\frac{1}{4}}}+\frac{x}{4a^{\frac{7}{4}}b^{\frac{3}{4}}}\frac{dK}{dh},$$

whence $$\frac{d^2u}{da\,db}=\frac{K}{16a^{\frac{5}{4}}b^{\frac{5}{4}}}-\frac{x}{4a^{\frac{7}{4}}b^{\frac{7}{4}}}\frac{dK}{dh}+\frac{x^2}{16a^{\frac{9}{4}}b^{\frac{9}{4}}}\frac{d^2K}{dh^2}.$$

Substituting in the differential relation, we thus find

$$\frac{d^2K}{dh^2}=K-\frac{4x}{a^{\frac{1}{2}}b^{\frac{1}{2}}}\frac{dK}{dh}+\frac{x^2}{ab}\frac{d^2K}{dh^2},$$

Now $$\tfrac{1}{2}-\frac{x}{2a^{\frac{1}{2}}b^{\frac{1}{2}}}=h,\quad \tfrac{1}{2}+\frac{x}{2a^{\frac{1}{2}}b^{\frac{1}{2}}}=h',$$

whence $$1-\frac{x^2}{ab}=4hh',$$

and $$\frac{x}{a^{\frac{1}{2}}b^{\frac{1}{2}}}=h'-h,$$

so that the result becomes

$$4hh'\frac{d^2K}{dh^2}+4\,(h'-h)\frac{dK}{dh}-K=0,$$

which is the well-known differential equation satisfied by K.

Generalised form of the double integrals, § 21.

§ 21. If in place of the integrals in § 3 we consider the more general integrals in which the exponent is raised to the power n, we find that

$$\int_0^\infty\int_0^\infty e^{-(u^2+v^2+2xuv)^n}u^\alpha v^\beta du\,dv$$

$$=4\int_0^\infty\int_0^\infty e^{-(s^4+t^4+2xs^2t^2)^n}s^{2\alpha+1}t^{2\beta+1}ds\,dt$$

$$=\frac{1}{4n}\Gamma\left(\frac{\alpha+\beta+2}{2n}\right)\int_0^\pi\frac{(\sin\frac{1}{2}\theta)^\alpha(\cos\frac{1}{2}\theta)^\beta}{(1+x\sin\theta)^{\frac{1}{2}(\alpha+\beta+2)}}d\theta$$

$$=\frac{1}{4n}\Gamma\left(\frac{\alpha+\beta+2}{2n}\right)\int_0^\pi\frac{\sin\theta\,(\sin\frac{1}{2}\theta)^\alpha(\cos\frac{1}{2}\theta)^\beta}{(1-h\sin^2\theta)^{\frac{1}{2}(\alpha+\beta+2)}}d\theta$$

$$=\frac{1}{2^{\alpha+\beta+2}n}\Gamma\left(\frac{\alpha+\beta+2}{2n}\right)\int_0^{2K}\frac{\operatorname{sn}u\,(1+\operatorname{cn}u)^\alpha(1+\operatorname{cn}u)^\beta}{(\operatorname{dn}u)^{\alpha+\beta+1}}du.$$

These last three expressions, in which all positive values of n are admissible differ from the corresponding formulæ in §§ 3 and 4 only by the factor $\frac{1}{n}$, which occurs outside the whole expression, and also in the argument of the Gamma function.

Thus, taking the first double integral, we see that its value when the exponent u^2+v^2+2xuv is raised to the power n bears to its value when the exponent is raised to the power m the ratio of

$$\frac{1}{n}\Gamma\left(\frac{\alpha+\beta+2}{2n}\right) \text{ to } \frac{1}{m}\Gamma\left(\frac{\alpha+\beta+2}{2m}\right).$$

DEVELOPMENTS IN POWERS OF $k'^2 - k^2$.

By *J. W. L. Glaisher.*

The general theorem, § 1.

§ 1. In §§ 9 and 10 of the preceding paper it was shown that

$$16\int_0^\infty\int_0^\infty e^{-s^4-t^4-2\lambda s^2t^2}s^n t^n ds\,dt = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2^{n-2}}\int_0^K \text{sd}^n u\,du$$

$$= \Gamma^2\left(\frac{n+1}{4}\right)\left\{1+\frac{(n+1)^2}{2.4}\lambda^2+\frac{(n+1)^2(n+5)^2}{2.4.6.8}\lambda^4+\&\text{c.}\right\}$$

$$-4\Gamma^2\left(\frac{n+3}{4}\right)\left\{\frac{1}{2}\lambda+\frac{(n+3)^2}{2.4.6}\lambda^3+\frac{(n+3)^2(n+7)^2}{2.4\ldots10}\lambda^5+\&\text{c.}\right\},$$

where $\lambda = h' - h$, h and h' denoting k^2 and k'^2 respectively.

The letter n is not restricted to integral values: it may have any value > -1.

Differential relation between P_n and P_{n+2}, § 2.

§ 2. By differentiating the first relation

$$\int_0^\infty\int_0^\infty e^{-s^4-t^4-2\lambda s^2t^2}s^n t^n ds\,dt = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2^{n-6}}\int_0^K \text{sd}^n u\,du,$$