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Pricing forward contracts on the Polish Power Exchange

M. Pawłowski, P. Nowak

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 3810100

fax: (+48) (22) 3810105

Kierownik Zakładu zgłaszający pracę:
Prof. dr hab. inż. Olgierd Hryniewicz

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Michał Pawłowski, Piotr Nowak

Abstract

The aim of this paper is to demonstrate a model of the Polish Power Exchange (POLPX) energy spot prices. The model is a result of a profound analysis dedicated to the Polish market's characteristics.

The proposed dynamics of the spot prices is driven by a mean-reverting jump-diffusion stochastic process with mixed-exponentially distributed jumps. The suggested approach contains several tailor-made ideas which have not been considered in the literature yet.

In the calibration of the model to the forward curve the exact, analytical formula for a forward price is derived and the notions of the market price of jump and diffusion risks are deployed. As a result, practitioners may avoid approximation or simulation methods while valuing over-the-counter forward contracts.

1 Introduction

The liberalization of the Polish energy market was a consequence of an implementation of the Directive 96/92/EC by the European Union, concerning common rules for the internal market in electricity. On this basis, an act on energy law was legislated in April 1997. The basic principles guiding the reforms were to separate electricity from its transmission services and to treat electric energy not like a common good, but as a commodity which may be traded in a similar way to equities or currency. To create and administer a market for: electricity generators, companies involved in energy trading, energy suppliers and industry clients, which could face up to all privatization programmes, the POLPX was established (started to operate in December 1999).

The spot prices time series on the POLPX exhibits all the distinctive (compared to other purpose markets prices trajectories, for example share markets) attributes: daily, weekly, yearly seasonality, mean-reversion to the marginal cost of production level and sudden spikes (negative or positive jumps of prices with almost immediate returns to the seasonal level) caused e.g. by a failure of a transmission network, outage of power plants or a sudden decrease (or increase) in temperature, in conjunction with inelasticity of demand and supply. In the context of pricing derivatives, there is no possibility to build a replicating strategy for any payoff – storage of the underlying is infeasible on a large scale, production and consumption have to be balanced all the time.

In the paper the stochastic process S of electrical energy spot prices on the Warsaw exchange is proposed. The aforementioned unique attributes of the electricity spot prices, which are typical also for the POLPX, are taken into account. Simultaneously, the possibility of pricing forward contracts with the underlying asset is taken into consideration.

The contribution of the paper to the theory of one-factor electricity spot prices models is manifold:

- we propose to model jump size by the mixed-exponential distribution with the separated from zero support (Section 4);
- each holiday (resulting in a "deterministic downward spike" in price) is treated separately as a part of the seasonality (Subsection 6.1);

- the procedure of spikes filtering is aimed at maximizing the p-value of the deseasonalised, and with deleted jumps, log-returns (Section 7);
- for the derived analytical formula for the forward curve, the mathematically elegant method of the calibration of the model to the quoted forward contracts, deploying the notions of the market prices of diffusion and jump risks, is introduced (Section 10).

In 1973 Fischer Black and Myron Scholes obtained the European option pricing formula, applying a geometric Brownian motion to model the dynamics of the underlying asset. An analytically closed form of the arbitrage-free option price is one of the main advantages of the Black-Scholes model. However, it is widely known that the Black-Scholes approach suffers from two main drawbacks. Firstly, contrary to the theoretical assumptions, distribution of the log-returns of the underlying process S in the real market is leptokurtic and skewed to the left. Secondly, the implied volatility as a function of the strike price forms a "U-shape" whereas its constancy is assumed in the model. Such an empirical phenomenon is well documented and is called the volatility smile.

Many alternatives to the model introduced by Black and Scholes have been proposed since the authors published their seminal paper. Among them Levy processes, i.e. stochastic processes with independent and stationary increments, have been adopted to describe log-prices of underlying assets. Two main categories of Levy processes commonly used in mathematical finance are: jump-diffusion and infinite activity models.

Merton in [24] was the first to use a jump-diffusion model to option valuation. He assumed that jump risk is not systematic and the log-price process of the underlying asset is a sum of a Brownian motion with drift and a compound Poisson process with normally distributed jump size. Kou in [19] and Kou and Wang in [20] proposed a jump-diffusion model similar to Merton's, assuming the asymmetric double exponential distribution of jump sizes. For more general ideas we refer the reader to [1, 8, 9, 11, 16], where the phase-type (PHM), hyperexponential (HEM) and mixed-exponential jump-diffusion (MEM) models of underlying assets were applied. Since the mixed-exponential distribution can approximate any distribution in the sense of weak convergence, MEM was an inspiration for our electricity spot prices model. In [26, 28, 29] Levy jump-diffusions with discrete distributions of jump sizes were adhibited and various sources of uncertainty on the financial market were considered. In these approaches semimartingale characteristics were also used, see e.g. [25, 32].

The second category of Levy processes consists of models with infinite number of jumps in every finite interval. Two important processes from this category used in finance are the Normal Inverse Gaussian model and the Variance Gamma model, see e.g. [23] and [3].

Our model is preceded by the selective overview of concepts arising in stochastic analysis. This is in Section 2. The next paragraph includes a presentation of four representative one-factor models of spot prices which are currently widely studied and used by both researchers and practitioners. The rest of the paper is organized as follows. In Section 4 the dynamics of the custom-made model for the Polish market is enunciated. Section 5 familiarizes the reader with historical data chosen for analysis. In Section 6 the method of adjusting seasonality to the historical time series is written up in details. Having read Section 7, one becomes acquainted with the algorithm of detection of spikes in prices. The course of a process of the parameters estimation is comprised in Section 8. In Section 9 the discretization of the continuous-time dynamics, as well as the comparison of the simulated this way trajectories with the historical series (tests for a goodness of fit) are performed. Section 10 demonstrates the method of calibration of the model to the quoted forward contracts using the analytical forward price and the notion of a market price of risk. The last section concludes.

2 Stochastic preliminaries

Let $\mathcal{T} = [0, T^*]$, $T^* > 0$ be a finite time interval. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$ be a filtered probability space.

Definition 1. *We call a random variable $\tau : \Omega \to \mathcal{T} \cup \{\infty\}$ a stopping time, if for each $t \in \mathcal{T}$ $\{\omega \in \Omega : \tau(\omega) \le t\} \in \mathcal{F}_t$.*

Definition 2. *We call a stochastic process $X = (X_t)_{t\in\mathcal{T}}$ a martingale, if random variables X_t are $\mathcal{F}_t$-measurable, $\mathbb{E}|X_t| < \infty$ for $t \in \mathcal{T}$ and*

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s$$

for $s \le t$. If additionally the family $(X_t)_{t\in\mathcal{T}}$ is uniformly integrable, the process X is called a uniformly integrable martingale.

Definition 3. *We call a stochastic process $X = (X_t)_{t\in\mathcal{T}}$ a local martingale, if one can find a nondecreasing family of stopping times $\{\tau_k\}_{k=1}^{\infty}$ such that $\tau_k \uparrow T^*$ ($\mathbb{P}$-a.s.) and the stopped processes $X_t^{\tau_k} = X_{\tau_k\wedge t}\mathbb{1}_{\{\tau_k>0\}}$ are uniformly integrable martingales for each k.*

Definition 4. *We call a stochastic process $X = (X_t)_{t\in\mathcal{T}}$ a semimartingale, if it is representable as a sum*

$$X_t = X_0 + A_t + M_t,\ t \in \mathcal{T},$$

where A is a process of bounded variation (over each finite interval $[0,t]$), M is a local martingale, both defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$ satisfying the usual conditions, i.e. the σ-algebra $\mathcal{F}$ is $\mathbb{P}$-complete, $\mathcal{F}_t$, $t \in \mathcal{T}$, contain all the sets in $\mathcal{F}$ of $\mathbb{P}$-probability zero and is right continuous ($\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t, s\in\mathcal{T}} \mathcal{F}_s$, $t \in \mathcal{T}$).

Definition 5. *For two semimartingales X and Y the quadratic covariance process is the process $[X,Y]$ defined on the same filtered probability space, such that*

$$[X,Y]_t = X_tY_t - \int_0^t X_{s-}dY_s - \int_0^t Y_{s-}dX_s - X_0Y_0, \tag{1}$$

$t \in \mathcal{T}$, where $\int_0^t X_{s-}dY_s$ and $\int_0^t Y_{s-}dX_s$ are stochastic integrals with respect to Y and X, respectively.

Definition 6. *A point process*

$$0 < U_1 < U_2 < \dots,$$

where U_n are nonnegative random variables, represents the appearances over time of some event. Equivalent representation of the point process involves the notion of an associated counting process defined by

$$N_t = \sum_{n\ge1} \mathbb{1}_{\{U_n\le t\}},\ t \in \mathcal{T}. \tag{2}$$

Remark 1. *A special case of the counting (point) process is a Poisson process, when in (2)*

$$U_n = \sum_{i=1}^{n} \tau_i, \tag{3}$$

where $(\tau_i)_{i\ge1}$ is a sequence of independent exponential random variables with intensity parameter λ.

Definition 7. *We call a pair $(U_n, Z_n)_{n\ge0}$ a marked point process, if U_n is a point process and Z_n is a sequence of $\mathbb{R}$-valued random variables.*

Definition 8. *An integer-valued random measure $N(dt, dz)$ corresponding to the marked point process $(U_n, Z_n)_{n\geq 0}$ is defined by*

$$N(t, A) = \sum_{n\geq 1} \mathbb{1}_{\{U_n\leq t\}}\mathbb{1}_{\{Z_n\in A\}}, \ t \in \mathcal{T}.$$

If the point process U_n is in the form (3), then we deal with a Poisson random measure.

Definition 9. *May $N(dt, dz)$ denote the Poisson random measure. May $\nu(dz)$ be the density function of Z_i, $i \geq 1$, under $\mathbb{P}$. We define a compensated Poisson random measure by*

$$\widetilde{N}(dt, dz) = N(dt, dz) - \lambda\nu(dz)dt.$$

Definition 10. *A compound Poisson process with intensity λ is a stochastic process defined as*

$$J_t = \sum_{i=1}^{N_t} Z_i, \ t \in \mathcal{T},$$

where Z_i are i.i.d. jump sizes and N_t is a Poisson process with intensity λ independent from $(Z_i)_{i\geq 1}$.

Let $W = (W_t)_{t\in\mathcal{T}}$ be a $(\mathcal{F}, \mathbb{P})$ Wiener process.

Theorem 1. Girsanov theorem. *Let $\theta_t^{\mathbb{Q}}$ and $q_t^{\mathbb{Q}}(z) \geq 0$ be predictable processes such that the process*

$$Z_t = \exp\Bigg(- \int_0^t \theta_s^{\mathbb{Q}} dW_s - \frac{1}{2}\int_0^t \left(\theta_s^{\mathbb{Q}}\right)^2 ds + \int_0^t\int_{\mathbb{R}} \ln(q_s^{\mathbb{Q}}(z))N(ds, dz) +$$
$$\int_0^t\int_{\mathbb{R}} \left(1 - q_s^{\mathbb{Q}}(z)\right)\lambda\nu(dz)ds\Bigg) \tag{4}$$

exists for $0 \leq t \leq T^*$. Suppose that

$$\mathbb{E}\left[Z_{T^*}\right] = 1. \tag{5}$$

Define the equivalent probability measure $\mathbb{Q}$ on $\mathcal{F}$ by

$$d\mathbb{Q}(\omega) = Z_{T^*}d\mathbb{P}(\omega), \ \ \omega \in \Omega$$

and define the process

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \theta_s^{\mathbb{Q}} ds,$$

as well as the random measure

$$\widetilde{N}^{\mathbb{Q}}(dt, dz) = (1 - q_t^{\mathbb{Q}}(z))\lambda\nu(dz)dt + \widetilde{N}(dt, dz),$$

where $\widetilde{N}(dt, dz)$ is a $(\mathcal{F}, \mathbb{P})$ compensated Poisson random measure.
Then, $W_t^{\mathbb{Q}}$ is a $(\mathcal{F}, \mathbb{Q})$ Wiener process and $\widetilde{N}^{\mathbb{Q}}(dt, dz)$ a $(\mathcal{F}, \mathbb{Q})$ compensated Poisson random measure of $N^{\mathbb{Q}}(dt, dz)$, while $N^{\mathbb{Q}}(dt, dz)$ is the $(\mathcal{F}, \mathbb{Q})$ Poisson random measure with a predictable intensity of the compound Poisson process under $\mathbb{Q}$

$$\lambda^{\mathbb{Q}} = \lambda \int_{\mathbb{R}} q_t^{\mathbb{Q}}(z)\nu(dz) \tag{6}$$

and a new density function under $\mathbb{Q}$ of $Z_i, i \geq 1$,

$$\nu^{\mathbb{Q}}(dz) = \frac{q_t^{\mathbb{Q}}(z)\nu(dz)}{\int\limits_{\mathbb{R}} q_t^{\mathbb{Q}}(z)\nu(dz)}, \tag{7}$$

where $q_t^{\mathbb{Q}}(z)$ binds 6 and 7

$$q_t^{\mathbb{Q}}(z) = \frac{\lambda^{\mathbb{Q}}\nu^{\mathbb{Q}}(dz)}{\lambda\nu(dz)}. \tag{8}$$

In terms of $N^{\mathbb{Q}}(dt, dz)$,

$$\widetilde{N}^{\mathbb{Q}}(dt, dz) = N^{\mathbb{Q}}(dt, dz) - \lambda q_t^{\mathbb{Q}}(z)\nu(dz)dt.$$

Proof. We refer the reader to Theorem 1.33 in [30], to Theorem 11.6.9 in [33] and to Lemma 11.6.8 especially for a proof of 8, also in [33]. □

Theorem 2. Novikov criterion. *May Z_t be defined as in 4. If either*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int\limits_0^{T^*}\left(\theta_s^{\mathbb{Q}}\right)^2 ds + \int\limits_0^{T^*}\int\limits_{\mathbb{R}}\left(q_s^{\mathbb{Q}}(z)\ln(q_s^{\mathbb{Q}}(z)) + 1 - q_s^{\mathbb{Q}}(z)\right)\nu(dz)ds\right)\right] < \infty \tag{9}$$

or

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int\limits_0^{T^*}\left(\theta_s^{\mathbb{Q}}\right)^2 ds + \int\limits_0^{T^*}\int\limits_{\mathbb{R}}\left(q_s^{\mathbb{Q}}(z)\right)^2 N(ds, dz)\right)\right] < \infty, \tag{10}$$

then (5) holds, i.e.

$$\mathbb{E}\left[Z_{T^*}\right] = 1$$

and thus $(Z_t)_{t\in[0,T^*]}$ is a martingale and the measure $\mathbb{Q}$ defined by

$$d\mathbb{Q}(\omega) = Z_{T^*}d\mathbb{P}(\omega), \quad \omega \in \Omega$$

on $\mathcal{F}$ is a probability measure.

Proof. We refer the reader to Theorem 1.36 in [30]. □

3 An overview of existing one-factor models of spot prices

Alongside the discrete-time econometric models such as ARMA, ARIMA or GARCH, one of the most obvious choices in class of continuous-time models, when modelling electrical energy spot prices, are one-factor models. This is because of their good adaptivity to data, existence, not infrequently, of analytical solutions to numerous provided issues (e.g. formulas for forward prices), as well as multiple approximation methods (for pricing options, etc.).

We focus our attention on four representative, currently widely studied one-factor, energy spot prices models.

3.1 A mean reverting diffusion model with sesonality

A prototype one-factor model for electricity prices, which became a milestone in commodity pricing, was introduced in [31] and developed in [22]. However, the model is lumbered with one serious drawback – it does not handle jumps in spot prices. Anyway, it reflects another two fundamental features: mean-reversion and deterministic seasonality.

Let us denote by S_t a spot price at a moment t (time is measured in years, single time step on the modelled markets is one day). The dynamics in a continuous setting is described by

$$S_t = \exp(g(t) + X_t), \tag{11}$$

where $g(t)$ is a deterministic seasonality function and X_t is a mean reverting (to the mean equal 0) process whose dynamics is driven by

$$dX_t = -\alpha X_t dt + \sigma dW_t.$$

α is a speed of mean reversion, σ is a constant volatility. It implies

$$dS_t = \alpha(\rho(t) - \ln S_t)S_t dt + \sigma S_t dW_t,$$

where

$$\rho(t) = \frac{1}{\alpha}\left(\frac{dg(t)}{dt} + \frac{1}{2}\sigma^2\right) + g(t)$$

under suitable conditions for $g(t)$. An explicit solution for $\ln S_t$ might be derived:

$$\ln S_t = g(t) + X_0 e^{-\alpha t} + \sigma \int_0^t e^{\alpha(s-t)} dW_s,$$

thus S_t for fixed t has a log-normal distribution.

Two different forms of seasonality functions are analysed. The first is

$$g(t) = a + bD_t + \sum_{i=2}^{12} c_i M_t^i,$$

where

$$D_t = \begin{cases} 1 & \text{if the moment } t \text{ is a holiday or a weekend} \\ 0 & \text{otherwise,} \end{cases}$$

$$M_t^i = \begin{cases} 1 & \text{if the moment } t \text{ is during the i-th month} \\ 0 & \text{otherwise,} \end{cases}$$

for $i \in \{2, \ldots, 12\}$ and a, b, c_i constant parameters.

Another proposal looks more adequate, as it is not a piecewise constant function:

$$g(t) = a + bD_t + c\cos\left((t + d)\frac{2\pi}{365}\right),$$

where D_t is defined as above, a, b, c and d are constant parameters. The cosine function is responsible for the annual seasonality in a continuous form.

3.2 A mean-reverting jump diffusion model with seasonality

A natural enhancement to the previous model is to add the possibility of jumps to the driving process.

$$S_t = \exp(g(t) + X_t), \tag{12}$$

where $g(t)$ is a deterministic seasonality function and X_t is a mean-reverting (to the mean equal to 0) process, the increases of which are assumed to follow

$$dX_t = -\alpha X_t dt + \sigma(t) dW_t + dJ_t, \tag{13}$$

where $\sigma(t)$ is a deterministic, time dependent volatility, $J_t = \sum_{i=1}^{N_t} Z_i$ is a compound Poisson process with some constant intensity (see Definition (10)), Z_i are i.i.d. jump magnitudes of a normal $N(-\frac{\sigma_J^2}{2}, \sigma_J^2)$ distribution, W_t is a Brownian motion. May Z be a random variable of $N(-\frac{\sigma_J^2}{2}, \sigma_J^2)$ distribution as well. An implementation of the above model was conducted in [17].

The authors in [10] recommend using the seasonality function g which fits the observed monthly averages with the Fourier series of order 5. Another proposal comes from [5], where

$$g(t) = a + \beta t + \gamma \cos(\varepsilon + 2\pi t) + \delta \cos(\eta + 4\pi t). \tag{14}$$

The role of the parameters is as follows: a stands for fixed costs of the production of electricity, β denotes the long run linear trend in the costs. The periodicity is contained in both cosines reflecting the market with two prices maxima per year. Such an approach is adequate for many representative markets, e.g. Nord Pool, German EEX.

One of the results of investigations performed in [5] states also that the usage of the Nomal distribution of spikes has the effect of overestimation of skewness and kurtosis. Nonetheless, the overall grade of the Normal distribution, in the application of the jump size modelling, is quite satisfactory, even though the normality Kolmogorov-Smirnov test at 5% significance level indicates to reject the H_0 for the tested jumps (the same situation repeats for tests for other distributions, e.g. truncated exponential or Pareto).

Another type of the jump-size distribution, which may turn out to better match the empirical data, is an asymmetric double-exponential distribution with density

$$f(z) = q\xi e^{\xi z} \mathbb{1}_{\{z<0\}} + p\eta e^{-\eta z} \mathbb{1}_{\{z \geq 0\}},$$

where p, $q > 0$ are the probabilities of upward and downward jumps, respectively. The restrictions $\eta_1 > 1$, $\eta_2 > 0$ and $p + q = 1$ are imposed. A very detailed description of the model with jumps sampled from this distribution may be found in [6].

3.3 The threshold model

The originators of the model [14] proposed to decompose S_t as in (12), with

$$dX_t = -\alpha X_t dt + \sigma dW_t + h(X_t^-) dJ_t,$$

where $J_t = \sum_{i=1}^{N_t} Z_i$, but this time N_t is a time-inhomogeneous Poisson process with time-dependent jump intensity in the form

$$\lambda(t) = \theta_1 s(t),$$

or in the extended form

$$\lambda(t) = \theta_1 s(t) \left(1 + \max\{0, \ln(S_t^-) - \overline{S}(t)\}\right).$$

θ_1 has an interpretation of the expected number of spikes per time unit. $\overline{S}(t)$ is a specified threshold from which the spikes activity grows. $s(t)$ is a normalized periodic jump intensity shape proposed in the form

$$s(t) = \left(\frac{2}{1 + |\sin(\frac{\pi(t-\tau)}{k})|} - 1\right)^d,$$

where k is the multiple of the peaking levels beginning at time τ, d adjusts the dispersion of jumps around the peaking times (how short spikes last). It is important to mention, that the jump sizes $Z_i, i \geq 1$, in this model possess a truncated exponential distribution with density function

$$f(z) = \frac{\theta_2 \exp(-\theta_2 z)}{1 - \exp(-\theta_2 \psi)}, \quad 0 \leq z \leq \psi.$$

ψ denotes the maximal possible jump size whereas θ_2 is the average jump size. The function h assures the correct direction of the jumps and is defined as

$$h(Y_t) = \begin{cases} 1 & \text{if } Y_t < \phi(t) \\ -1 & \text{if } Y_t \geq \phi(t) \end{cases},$$

where $\phi(t)$ is a threshold that for instance may be set as a constant spread $\Delta > 0$ over the seasonality $g(t)$, i.e.

$$\phi(t) = g(t) + \Delta > \overline{S}(t),$$

where the seasonality is introduced as in (14).

The authors in [5] critisize the choice of the truncated exponential distrbution due to the fact, that it disallows for big jumps exceeding the fixed threshold ψ (determined by historical data). As a remedy, they propose another jump size distributions like something between Pareto and truncated exponential (in the sense of kurtosis), for instance gamma. The authors also claim, that the h function is responsible only for preventing two consecutive price values being above the threshold, whereas the parameter α is responsible for mean reversion. Hopelessly, the estimate of α is higher than expected for a base signal and smaller than required to dampen a spike. They generally call the usage of any thresholds appearing in this model into question – for example the influence of the stochastic spike intensity they find trifling.

However, the model gained popularity as it is very interesting from the theoretical point of view and at the time (2006) it represented the novel idea. Both the threshold and mean-reverting jump diffusion models belong to the canon of energy spot price modelling.

3.4 A supply/demand Barlow's model

In this approach we assume that

$$S_t = \begin{cases} (1 + \alpha X_t)^{\frac{1}{\alpha}}, & 1 + \alpha X_t > \varepsilon_0 \\ \varepsilon_0^{1/\alpha}, & 1 + \alpha X_t \leq \varepsilon_0 \end{cases},$$

$$dX_t = \lambda(a - X_t)dt + \sigma dW_t,$$

for $\alpha \neq 0$ and $S_t = \exp(X_t)$ for $\alpha = 0$. $\varepsilon_0^{1/\alpha}$ is a known constant representing a minimum price for $\alpha > 0$ and a maximum price for $\alpha < 0$, characteristic to the particular market.

The reason the model (often called a nonlinear Ornstein-Uhlenbeck model) has such a structure lies in some assumptions of the demand and supply game, as well as the predefined forms of demand and supply functions which have to be equated in order to assign a correct price to the energy unit.

3.5 Huisman and de Jong 2-regime switching model

Nowadays, the class of regime switching models is a fast developing and popular family of models for electricity prices' repatterning due to the popularity of Markov models and because the energy spot price at every moment may be assigned one out of several, unique in the context of price behaviour, states.

[12] present the model in a discrete-time setting, inasmuch as the number of transitions must be countable. In this model the decomposition (12) does not make the X process mean revert to zero, but to constants μ_M and μ_S:

$$X_{M,t+1} = X_{M,t} + \alpha\left(\mu_M - X_{M,t}\right) + \varepsilon_{M,t},$$

$$\varepsilon_{M,t} \sim N(0, \sigma_M)$$

is an evolution of S in a mean-reverting regime, whereas

$$X_{S,t+1} = \mu_S + \varepsilon_{S,t},$$

$$\varepsilon_{S,t} \sim N(0, \sigma_S)$$

is an equation for S in a spike regime. α, σ_M, $\sigma_S > 0$. The probabilities of switching from one regime to another, or remaining in the given regime, are contained in a 2×2 Markov transition matrix.

3.6 De Jong 3-regime switching model with Poisson jumps

After the decomposition (12), the mean-reverting regime is modelled in the form

$$X_{M,t+1} = X_{M,t} + \alpha(\mu - X_{M,t}) + \sigma\varepsilon_t,$$

whilst the spike regimes $X_{u,t}$ and $X_{d,t}$ are for $i \in \{u, d\}$

$$X_{i,t+1} = X_{i,t} + \alpha(\mu - X_{i,t}) + \sum_{i=1}^{n_t+1} Z_{i,t},$$

where $n_t \sim Poiss(\lambda)$ and jump-up regime

$$Z_{u,t} \sim N(\mu^u, \sigma^u),$$

jump-down regime

$$Z_{d,t} \sim N(\mu^d, \sigma^d),$$

$\varepsilon_t \sim N(0, 1)$, α, σ, σ_u, $\sigma_d > 0$, μ, μ_u, $\mu_d \in \mathbb{R}$. From the base regime it is only possible to stay there or to move upwards, whereas from the up-jump regime the chain with probability 1 moves to the down-jump regime and afterwards surely moves to the base regime.

3.7 Janczura and Weron 3-regime switching model

Nowadays, the class of regime switching models is a fast developing and popular family of models for electricity prices repatterning due to the popularity of Markov models and because the energy spot price at every moment may be assigned one out of several, unique in the context of price behaviour, states.

The latest ideas in the regime switching modelling of energy were shown in [15]. There are three different regimes for a spot price – for each there is a specific dynamics introduced. The two are spikes regimes (for increase and decrease in price, respectively). The latter is a base regime when the price moves in a noisy way and the amplitudes are small.

May $Y_t = S_t - g(t)$, where $g(t)$ is a deterministic seasonality fitted to the price index values.

$$\ln(Y_{t,u} - Y(0.5)) \sim N(\mu_u, \sigma_u^2), \quad Y_{t,u} > Y(0.5),$$

$$\ln(-Y_{t,d} + Y(0.5)) \sim N(\mu_d, \sigma_d^2), \quad Y_{t,d} < Y(0.5),$$

$$dY_{t,b} = (\alpha - \beta Y_{t,b})dt + \sigma_b Y_{t,b}^{\gamma} dW_t,$$

where $Y(q)$ denotes the q-quantile in the given deseasonalised prices (our dataset), γ and other parameters in the formulas are constants. A Markov transition matrix 3×3 contains probabilities of moves between the regimes. Instead of a log-normal law, one may use arbitrary distribution for Y (such an extension is proposed in [21]), e.g. Pareto:

$$Y_t \sim F_{Pareto}(y; \alpha, \lambda) = 1 - \left(\frac{\lambda}{y}\right)^{\alpha}, \quad Y_t > \lambda \geq Y(0.5).$$

3.8 The Markov-switching linear regression model

The spot price is modelled as

$$S_t = X_t^T \beta_{W_t} + \varepsilon_t,$$

$\varepsilon_t \sim N(0, \sigma^2_{W_t})$, $W = \{1, 2, \ldots, n\}$ is a set of n states (regimes), $\forall i, j \in W$ $\mathbb{P}(W_t | W_{t-1} = j) = p_{ij}$. X_t is a vector of exogenous variables at time t, β_{W_t} is a vector of regression coefficients in a state W_t, $\sigma^2_{W_t}$ is an error's variance in the given regime, p_{ij} are transition probabilities between states i and j.

3.9 The potential Levy jump-diffusion model with jumps

In this model the deseasonalised logarithms of spot prices undergo the equations

$$dX_t = -U'(X_t)dt + \sigma dW_t + dL_t,$$

where $\sigma > 0$, L is a pure jump α-stable non-Gaussian Levy process, $U(x)$ is a specially selected potential function with restriction $U : \mathbb{R} \longrightarrow \mathbb{R}$ is twice continuously differentiable, $U(x) \to \infty$ as $|x| \to \infty$,

$$\int_{-\infty}^{\infty} \exp(-2U(x)/\sigma^2)dx < \infty.$$

Without the Levy component, X_t has the Gibbs distribution with density

$$\pi_\sigma(x) = \frac{\exp(-2U(x)/\sigma^2)}{\int_{-\infty}^{\infty} \exp(-2U(y)/\sigma^2)dy}.$$

The potential function can be sufficiently estimated from the historical data. It may be for instance assumed, that U is a polynomial of some even, higher than 2, degree. Then the model's parameters may be estimated simultaneously from the observed deseasonalised logarithms of prices using the maximum likelihood method or the generalized method of moments.
The authors in [7] propose the following form of derivative of U :

$$U'(x) = \begin{cases} A(x-m)^{s-1} + C_0, & x - m > c \\ \gamma(x-m) + C_0, & |x - m| \geq c \\ -A(x-m)^{s-1} - C_0, & x - m < -c \end{cases}.$$

The parameter A is determined explicitly from the continuity of U', m is so called global mean level parameter.

3.10 The model with a normal inversed Gaussian Levy driving process

The choice of infinite activity processes to model the electricity spot price is very advantageous because the single Levy component may be responsible for small, typical daily movements (the Levy jumps are so frequent that they can supersede the motion of the Wiener process), as well as for big jumps in prices (substitution for the Poisson process).

In a model introduced in [4] the deseasonalised logarithms of the spot prices undergo the equation

$$dX_t = a(m - X_t)dt + dL_t,$$

for $a, m \geq 0$. L_t is a Levy process

$$L_t = \nu t + \sigma W_t + \int_{|z|<1} z\widetilde{N}((0,t], dz) + \int_{|z|\geq 1} zN((0,t], dz),$$

where $\sigma > 0$, $\nu \in \mathbb{R}$, N is a homogeneous Poisson random measure with a compensator $l(dz)dt$. The σ-finite measure $l(dz)$ on the Borel sets of $\mathbb{R}$ is a Levy measure which satisfies $l(0) = 0$ and $\int_{\mathbb{R}} \min(1, z^2) l(dz) < \infty$. Using the Ito lemma one obtains

$$X_t = x_0 e^{-at} + \frac{m}{a}(1 - e^{-at}) + \int_0^t e^{-a(t-s)} dL_s.$$

L_1 is assumed to have the normal inverse Gaussian (NIG) distribution belonging to the class of generalized hyperbolic distributions. The density function of the NIG is in the form

$$f(x; \mu, \alpha, \beta, \delta) = \frac{\alpha\delta}{\pi} \exp\left(\delta\sqrt{(\alpha^2 - \beta^2)} + \beta(x - \mu)\right) \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}}$$

with $\mu \in \mathbb{R}$ the location of the density, $\beta \in \mathbb{R}$ the skewness parameter, $\alpha \geq |\beta|$ measures the heaviness of the tails and $\delta > 0$ is the scale parameter. The function K_1 is the modified Bessel function of the second kind and order 1.

4 The model for the Polish Power Exchange spot prices

In this section a model of the spot prices suited to the Polish market data is presented. The idea of the mixed-exponential distribution of jumps is drawn from [9] (where the authors model the assets prices), but connection of such jump distribution with mean-reverting process and exploitation of it to the electricity spot prices modelling has not been explored yet. The propounded distribution has a distinguishing property that it can approximate any distribution with respect to weak convergence as closely as possible. In the light of problems with matching a distribution to a dataset of jumps, such property seems to be a countermeasure. Nonetheless, the price to pay is plenty of parameters to be estimated, appearing in the formulation of the jump-size distribution (16) (however, in practice, taking $n = m = 2$ suffices to ensure very good accuracy).

One of the biggest merit of simplicity of the model formulation (15) is that it enables to derive a closed-form forward price, which is described in Section (10). This fact cannot be overstated – with such a tool a very precise calibration to quoted contracts on the market is feasible. Moreover, having tuned the model's parameters to market data, one may use the analytical formula for the forward price to calculate the risk-neutral price of any tailor-made forward contract with no necessity to apply any simulation or approximation method.

We start with the decomposition of the spot price process S_t:

$$S_t = \exp(g(t) + X_t),$$

$$dX_t = -\alpha X_t dt + \sigma dW_t + dJ_t, \tag{15}$$

where α and σ are constants, $(W_t)_{t \in \mathcal{T}}$ is a Wiener process, $(J_t)_{t \in \mathcal{T}}$ is a compound Poisson process of the form

$$J_t = \sum_{i=1}^{N_t} Z_i, \quad t \in \mathcal{T},$$

with constant intensity λ, Z_i are i.i.d. jump magnitudes of translated mixed-exponential distribution, i.e. with density

$$f(z) = q_d \sum_{i=1}^{m} q_i \xi_i e^{\xi_i (z - m_d)} \mathbb{1}_{\{z < m_d\}} + p_u \sum_{j=1}^{n} p_j \eta_j e^{-\eta_j (z - m_u)} \mathbb{1}_{\{z > m_u\}}, \tag{16}$$

where $q_d, p_u \geq 0$, $q_d + p_u = 1$,
$q_i, p_j \in (-\infty, \infty)$, $\sum_{i=1}^{m} q_i = \sum_{j=1}^{n} p_j = 1$, $\xi_i > 0, \eta_j > 1$. q_d and p_u are the probabilities of negative and positive jumps, respectively. $m_d < 0$ is a minimal (with respect to the absolute value) value of negative jumps, $m_u > 0$ is a minimal value of positive jumps. A necessary condition for $f(z)$ to be a density function is

$$q_1, p_1 > 0, \ \sum_{i=1}^{m} q_i \xi_i \geq 0, \ \sum_{j=1}^{n} p_j \eta_j \geq 0.$$

One of possible sufficient conditions is

$$\sum_{i=1}^{k} q_i \xi_i \geq 0, \ \sum_{j=1}^{l} p_j \eta_j \geq 0$$

for all $k \in \{1, \ldots, m\}$, $l \in \{1, \ldots, n\}$.
A special case of the mixed-exponential distribution is a hyperexponential distribution, when all parameters q_i and p_j are nonnegative.

The separation from zero of the support of the density function is caused by the fact that either positive or negative jumps are extreme events, therefore highly greater than zero with respect to the absolute value.

All the introduced above processes and random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is generated by W and J and augmented to encompass $\mathbb{P}$-null sets from $\mathcal{F} = \mathcal{F}_{T^*}$. The filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$ satisfies usual assumptions: σ-algebra $\mathcal{F}$ is $\mathbb{P}$-complete, the filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is right continuous and each $\mathcal{F}_t$ contains all the $\mathbb{P}$-null sets from $\mathcal{F}$.

Using the Ito lemma, one obtains that S_t follows the stochastic differential equation

$$dS_t = \alpha(\rho(t) - \ln S_t)S_t dt + \sigma S_t dW_t + S_t(e^Z - 1)dN_t, \tag{17}$$

where

$$\rho(t) = \frac{1}{\alpha}\left(\frac{dg(t)}{dt} + \frac{1}{2}\sigma^2\right) + g(t).$$

5 Data description

The data selected for estimation comes from the POLPX's IRDN index with the time range of September 2011 – January 2014 (844 quotations) with an exception for a jump-size distribution's estimation due to the scarcity of jumps (much more data is required for the stable estimation) – from September 2005 to September 2013 (2924 quotations). However, in the longer series there is a substantial quantitative change in the prices behaviour, therefore the shortened series is used for the estimation of the remaining parameters.

It is important to note here that by a spot price we mean a weighted (by volume) average price of daily transactions – a standard day-ahead reference index for contracts with delivery of energy during the whole upcoming day.

Without any deep analysis one can state the fact that prices undergo some yearly seasonal fluctuations, negative trend, but the most conspicuous are: weakly seasonality – indices values on Sundays are unequivocally smaller than on other days, and presence of jumps. Jumps may be categorized into two groups: there are only a few rises or falls that are apparently higher than others. The second group consists of a big number of smaller jumps which absolute values slightly exceed or are at the level of the Sundays drops. Both groups will be detected during estimation. All kinds of movements have their mirror images – the prices come back to the long run mean which is a seasonality.

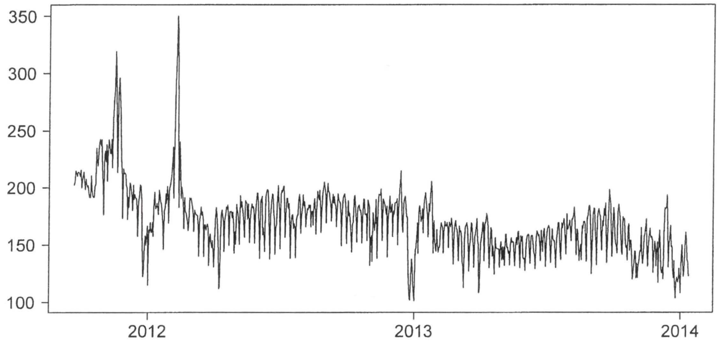

Figure 1: Spot prices in PLN/MWh

6 Seasonality fitting

From the specification of our model we know, that subtracting the seasonality function from the series of prices we are left with a kind of residue which is modelled by the zero-mean-reverting jump-diffusion process (15). In other words, after filtering spikes from this residue, the returns of the remaining noise must have normal distribution. Therefore, the relevant, objective criterion for measuring the goodness of seasonality combined with removal of spikes should be a p-value of the appropriate statistical test for normality of the mentioned returns. The stage of constructing the seasonality is divided into several steps.

6.1 Deterministic downward spikes

The Polish market has a distinctive feature that the prices substantially come down for a one-day period, if this day is a national holiday. Thus, these deterministic downward spikes should be extracted from the time series in the first instance as a first component of the seasonality. An estimator of the value of the jump is a mean of decreases of prices (from a day preceding the holiday, if it is not Sunday or another holiday, to the holiday) during years which are chosen to estimation. There are 12 such deterministic downward jumps each year.

In the process of deseasonalisation the estimators of jumps are added to the holidays prices, but only in case the holiday is not on Sunday. If so, the difference of the estimator of the jump and the average Sunday drop is added – if this difference is positive, or nothing is added otherwise (because then the typical Sunday fall occurs).

6.2 Detrending, fitting of weekly and yearly oscillations

The next step of the deseasonalisation is a removal of a linear trend. The fitted linear model is in the form $y = -0.0004t + 5.3$, where t is time in days – the negative trend is observed.

Thereafter, to eliminate the weekly seasonality, the means of logarithms of prices of all days within a week are subtracted from the log-series (all of them in range 5.28 - 5.35 apart from Sunday's which is 5.15).

Yearly seasonality is matched the other way. The Fourier series of order 12 is fitted to the monthly averages of log-prices and then subtracted from the prices series. The reason the magnitude of the order is 12, is the best properties of matching the seasonality in further steps (i.a. the p-value of the deseasonalised log-returns without jumps). To make the series fluctuate around zero, the average index value is added to the whole range of the deseasonalised prices.

6.3 Annual sinusoidal function

In order to detect any remaining annual movements, other than described above, the periodic (one year period) sinusoidal function of the form

$$a + bt + \sum_{k=1}^{3} c_k \sin\left(\frac{2k\pi t}{365}\right) + d_k \cos\left(\frac{2k\pi t}{365}\right)$$

is estimated by the nonlinear least-squares method and subtracted from the series (Figure 2). The estimated values of parameters are shown in Table 1.

a	b	c_1	d_1
$6.76 \cdot 10^{-3}$	$-1.67 \cdot 10^{-5}$	$1.83 \cdot 10^{-2}$	$3.34 \cdot 10^{-2}$
c_2	d_2	c_3	d_3
$-5.49 \cdot 10^{-3}$	$6.06 \cdot 10^{-3}$	$-3.31 \cdot 10^{-2}$	$-1.04 \cdot 10^{-2}$

Table 1: Estimated parameters by the nonlinear least-squares method

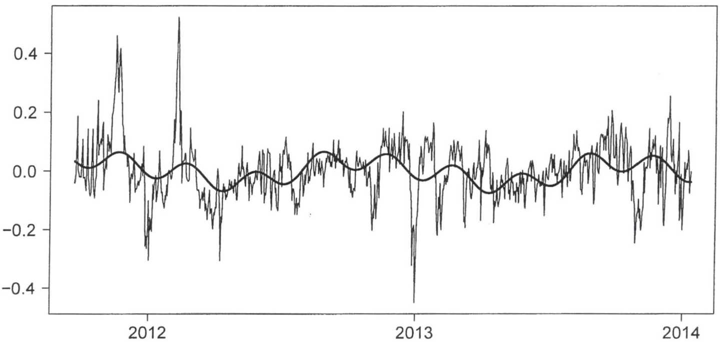

Figure 2: Annual sinusoidal function fitted to the partially deseasonalised log-price series

6.4 The comprehensive form of the seasonality

Applying consecutively the previous steps, one obtains the complete form of the seasonality, which is shown on Figure 3. It is also interesting to observe the seasonality in some magnification, for instance around Christmas (Figure 4).

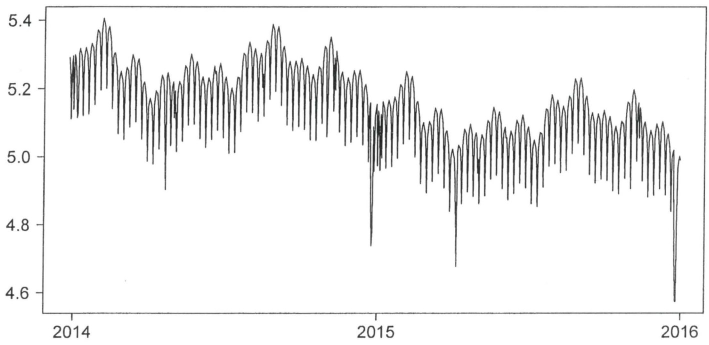

Figure 3: The overall seasonality function in logarithmic scale in PLN/MWh

7 Filtering of spikes (jumps with mean-reversions)

Filtering of spikes is performed by the iterative procedure: in the first step all jumps which absolute value exceeds some predefined threshold, for instance three times the standard deviation of the deseasonalised log-returns, are removed from the series. In the next step the same action is made, but this time the standard deviation is calculated basing on the thinned series of returns. New jumps are filtered and deleted and the process continues until in some iteration no jumps are found.

The most important aspect of this method is to fix the threshold so as to maximize the p-value of the Shapiro-Wilk normality test for the deseasonalised, and with deleted jumps, log-returns – the assumptions of the model must be fulfilled. For our data the threshold turned out to be $2.59s$, where s is the standard deviation of the series obtained in each step of the described procedure. The maximized p-value is equal to 0.053, whereas some other normality tests indicate even better results of p-values for the chosen threshold: Anderson-Darling – 0.38, Kolmogorov-Smirnov – 0.74, Jarque-Bera – 0.69, Shapriro-Francia – 0.14. There is no evidence to reject the null hypothesis of the log-returns normality at the 5% significance level.

8 Estimation of the jump-diffusion's parameters

8.1 Base signal parameters assessment

Having deseasonalised the log-price series and removed jumps with mean-reversions (which happen immediately after the jumps), one may estimate the parameter σ appearing in equation (15). This volatility is estimated as a mean of the rolling standard deviation of the time-scaled increments $\frac{P_i - P_{i-1}}{\sqrt{t_i - t_{i-1}}}$ (see [13], formula 3.10):

$$\sigma(t_k) = \sqrt{\frac{1}{m-1} \sum_{i=k-m+1}^{k} \left(\frac{P_i - P_{i-1}}{\sqrt{t_i - t_{i-1}}} - \sum_{j=k-m+1}^{k} \frac{P_j - P_{j-1}}{\sqrt{t_j - t_{j-1}}} \right)^2},$$

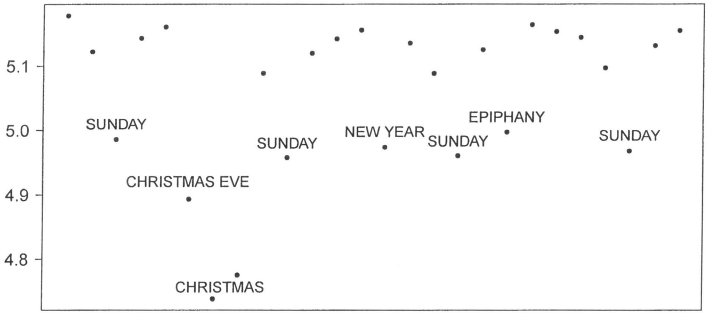

Figure 4: Seasonality function around Christmas 2014

$$\sigma = \frac{\sum_{k=m}^{M} \sigma(t_k)}{M - m + 1},$$

P is the deseasonalised and devoid of spikes log-price index, $m = 30$, $M = 791$, $k \in \{m, \dots, M\}$ (after removing of jumps and mean-reversions there are 791 log-returns). For all $i \in \{1, \dots, 791\}$ $t_i - t_{i-1} = \frac{1}{365}$. The estimated value $\sigma = 0.91$.

Determination of the mean-reversion's velocity α is based on the deseasonalised log-prices, but in the presence of spikes. One has to regress the deseasonalised log-prices series bereft of its first element versus the deseasonalised log-prices series without its last element, which is a direct cause of the discretized form (see details in Subsection 9.1) of the equation (15):

$$X_{t_k} = e^{-\alpha \Delta t} X_{t_{k-1}} + \rho_{t_k},$$

where ρ_{t_k} is the sum of integrals of the Wiener process and the compound Poisson process between times t_{k-1} and t_k. The value of the regression coefficient $e^{-\alpha \Delta t}$ is significantly different from zero – the speed of mean-reversion achieved this way equals $\alpha = 0.25$.

The results of the augmented Dickey-Fuller test applied for the deseasonalised log-prices indicate that there is no unit root in our time series data – the mean-reversion is indeed present.

8.2 Evaluation of the jump-size distribution's parameters

As mentioned in Section 5, much longer time series is used for the estimation of the jump magnitude's distribution to assure that there is enough data for a stable assessment of this kind of a rare event. The algorithm describing how it is done was explained in Section 7, nevertheless a salient modification is prerequisite at this moment. In fact, when using the iterative method written up earlier, numerous unnecessary (for this application) returns, which are simply the mean-reversions of the process, are filtered. It means that the consecutive decreases (increases) after the jumps (of the opposite sign to the jump's sign), which exceed the threshold, cannot be taken into account if one wants to obtain the actual estimators of the jump-size distribution.

In the density function specification (16) $m = n = 2$ is taken. 242 returns are classified as jumps by the filtering algorithm on the series of 2924 observations. Out of these jumps, 56 are the mean-reversions and thus are not considered in the estimation. Accordingly, the yearly frequency of the Poisson process λ is equal to $(242 - 56)/2924 \cdot 365 = 23.22$. Counting upward and downward jumps yields $q_d = 0.65$, $p_u = 0.35$. Extreme values of filtered jumps give $m_d = -0.12$, $m_u = 0.12$.

The remaining parameters are estimated by the maximum likelihood method – see Table 2. The parameters q_1, q_2, p_1, p_2 are all positive, so that the jump distribution turns out to be

q_1	q_2	ξ_1	ξ_2	p_1	p_2	η_1	η_2
0.6	0.4	8.41	38.72	0.13	0.87	3.72	29.71

Table 2: The estimated parameters of the mixed-exponential jump-size distribution

the hyperexponential distribution, a special case of the mixed-exponential distribution. Figure 5 illustrates the adjustment of the density to the histogram of filtered jumps.

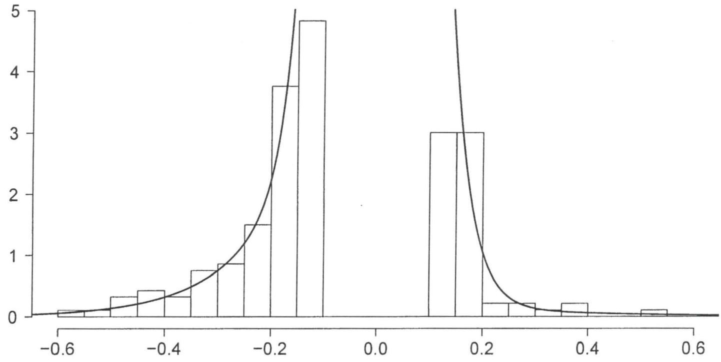

Figure 5: Mixed-exponential distribution fitted to the empirical distribution of jumps

9 Simulation of the spot prices and tests for the trajectories

9.1 Discretization of the process

Lemma 1. *Let X_t follow the equation (15) and may $0 \leq s \leq t$, $t \in \mathcal{T}$. Then*

$$X_t = e^{-\alpha(t-s)} X_s + \int_s^t \sigma e^{-\alpha(t-u)} dW_u + \sum_{s<u\leq t,\ \Delta N_u \neq 0} e^{-\alpha(t-u)} \Delta J_u. \tag{18}$$

Moreover,

$$\int_s^t \sigma e^{-\alpha(t-u)} dW_u \sim N\left(0, \sigma\sqrt{\frac{1-e^{-2\alpha(t-s)}}{2\alpha}}\right). \tag{19}$$

Proof. Let $s, t \in \mathcal{T}$ and $Z_t = X_t Y_t$ for $Y_t = e^{\alpha(t-s)}$. Clearly, Y_t is a continuous, increasing process and therefore (see [32]) $[X, Y] = 0$. From the differential form of (1),

$$\begin{aligned} dZ_t &= \alpha Z_t dt + e^{\alpha(t-s)} dX_t = \alpha Z_{t-} dt + e^{\alpha(t-s)} \left(-\alpha X_t dt + \sigma dW_t + dJ_t\right) = \\ &\alpha Z_t dt + e^{\alpha(t-s)} \left(-\alpha X_t dt + \sigma dW_t + dJ_t\right) = e^{\alpha(t-s)} \left(\sigma dW_t + dJ_t\right). \end{aligned} \tag{20}$$

For $s \leq t$ one obtains

$$Z_t = Z_s + \int_s^t \sigma e^{\alpha(u-s)} dW_u + \sum_{s<u\leq t,\ \Delta N_u \neq 0} e^{\alpha(u-s)} \Delta J_u,$$

which gives the equation (18). Since, for each continuous function φ,

$$\int_s^t \varphi(u)\, dW_u \sim N\left(0, \sqrt{\int_s^t \varphi^2(u)\, du}\right)$$

and

$$\int_s^t \sigma^2 e^{-2\alpha(t-u)} du = \sigma^2 \frac{1 - e^{-2\alpha(t-s)}}{2\alpha},$$

one obtains (19). □

Hence, the discretized dependency between the consecutive "daily" values of the process X_t is of the form

$$X_{t_k} = X_{t_{k-1}} \exp\left(\frac{-\alpha}{365}\right) + \sigma\sqrt{\frac{1 - \exp\left(\frac{-2\alpha}{365}\right)}{2\alpha}} N(0,1) + \sum_{i=1}^{N_{1/365}} Z_i, \tag{21}$$

where $N(0,1)$ is a standard normally distributed variable, $N_{1/365}$ is a Poisson random variable with the intensity parameter $\frac{\lambda}{365}$, Z_i are mixed-exponentially distributed random variables. A sample trajectory with added seasonality and trend is shown on Figure 6 on a background of the historical path.

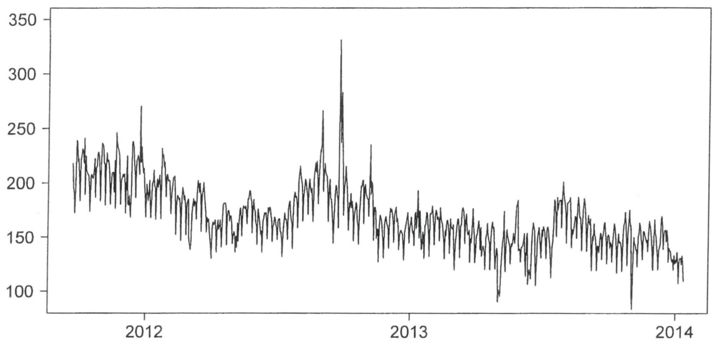

Figure 6: Simulated sample path in PLN/MWh

9.2 Goodness of fit of the sample paths

The comparison of the two moments and 5%, 95% quantiles of the historical spot prices (in PLN/MWh) which are used for estimation and an average of 5000 simulated trajectories is shown in Table 3.

	mean	st. dev.	5% quantile	95% quantile
real data	171.44	29.82	129.55	215.36
simulations	168.52	30.65	122.67	218.96

Table 3: Moments and quantiles of the historical spot prices and 5000 simulated paths

	mean	st. dev.	5% quantile	95% quantile
real data	-0.00059	0.106	-0.164	0.187
simulations	-0.00074	0.118	-0.176	0.199

Table 4: Moments and quantiles of the historical and 5000 simulated log-returns (average)

The comparison of the two moments and 5%, 95% quantiles of the historical log-returns and an average of the log-returns of 5000 simulated trajectories is shown in Table 4.

It must be stated clearly that the deviations of the presented above simulation summaries from the real ones are caused only by the fact, that the jump-size distribution's parameters are estimated on the much more longer time series, and as a result their values diverge from those which may be obtained on the main, shorter series. But this series is too short to allow for a stable estimation of all jump-size distribution's parameters. However, changing merely the probability of the upward jump to the value esitmated on the shorter series (equal to 0.45), one receives much more accurate mean of the simulated trajectories.

Notwithstanding, these differences are not really significant. The Kolmogorov-Smirnov test for the equality of distributions of the real log-increases and the log-increases of the simulated data gives no evidence to reject the null hypothesis of the equality of the mentioned distributions – the averaged p-value (over 5000 samples) is equal to 0.47.

Finally, the reestimation procedure was conducted, i.e. for each simulated path all the parameters were estimated and then were averaged over samples – the resulting parameters' values were very similar to those computed during the estimation described in Section 8.

10 Calibration of the model to the quoted forward contracts

10.1 Market price of risk – introduction

Once the model is entirely composed out of historical data, the application of it to the actual market situation requires calibration to the quoted forward curve because the actual market participants' expectation may not reflect the historical features like the prices trend or the absolute level of the prices. In order to accommodate this necessity, we put into the model some degrees of freedom which allow for calibration, and from the economical point of view illustrate how much the market pays for investing in the risky, tradable asset (electricity spot price in our case). The magnitudes of the changes in parameters are called market prices of risk. As in the model there are two sources of randomness, both market price of diffusion risk as well as jump risk are introduced.

10.2 Derivation of the analytical formula for the forward price

One of advantages of the model is that it enables derivation of the analytical formula

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t] \tag{22}$$

for the forward prices F(t,T), $0 < t \leq T$, $T \in \mathcal{T}$. An equivalent risk-neutral measure $\mathbb{Q}$ appears in this expression. This is a tool which enables to make the model suited to the actual prices.

From mathematical point of view, there are uncountably many equivalent, potentially risk-neutral measures. From financial point of view, the market is incomplete, as there are more sources of randomness (hence risk) than risky assets, thus not every payoff may be replicated (hedged) with this underlying asset and not risky one, for instance a bank account or a bond. The form of dynamics under any equivalent measure allows us to pin down the form of the market prices of risk. Ascribing the concrete numerical values to the parameters which denote the market prices of risk uniquely determines the choice of the appropriate risk-neutral measure.

Suppose now, that we change our physical measure $\mathbb{P}$ to the equivalent, potentially risk-neutral measure $\mathbb{Q}$.

Definition 11. *We call the parameters appearing in the Radon-Nikodym density (4), $\theta_t^{\mathbb{Q}}$ and $q_t^{\mathbb{Q}}(z)$, the market price of diffusion risk and the market price of jump risk, respectively.*

Having defined the notions of the market prices of risk, we may return to the problem of calculating the forward price (22). From now onwards, we will consider only the class of equivalent measures $\mathbb{Q}$, where the market price of diffusion risk is time-independent, i.e. $\theta_t^{\mathbb{Q}} \equiv \theta^{\mathbb{Q}}$, and under which the jump-size distribution's density does not change (which reflects the market realities), i.e. $\nu^{\mathbb{Q}}(dz) = \nu(dz)$, $Z^{\mathbb{Q}} = Z$. From (8), these assumptions imply that $q_t(z) = \frac{\lambda^{\mathbb{Q}}}{\lambda}$ and thus it is a constant. As both market prices of risk are constant, the Novikov criterion (Theorem (2)) holds and the condition (5) is satisfied – the Girsanov theorem (Theorem (1)) may be applied.

One may observe from the Girsanov theorem that after the change of the measure there is a transformation of a drift in the asset's dynamics and a modification of the intensity, as well as the jump-size distribution of the compound Poisson process. As a result, the SDE (15) under the equivalent measures $\mathbb{Q}$ may be written as

$$dX_t = -\alpha(X_t + \frac{\sigma\theta_t^{\mathbb{Q}}}{\alpha})dt + \sigma dW_t^{\mathbb{Q}} + Z^{\mathbb{Q}}dN_t^{\mathbb{Q}} = -\tilde{\alpha}X_t dt + \sigma dW_t^{\mathbb{Q}} + Z^{\mathbb{Q}}dN_t^{\mathbb{Q}} \tag{23}$$

because the new terms appearing in the drift might be appended to the seasonality function, see [18], Chapter 4.2.

Theorem 3. *The analytical formula for the forward price within the model defined in Section 4 by (15) and (16) is equal to*

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t] = G(T)\left(\frac{S_t}{G(t)}\right)^{e^{-\alpha(T-t)}} \exp\left(\int_t^T \sigma e^{-\alpha(T-s)}\left(\frac{1}{2}\sigma e^{-\alpha(T-s)} - \theta^{\mathbb{Q}}\right)ds\right)\cdot$$

$$\exp\left(\int_t^T \left(e^{m_d}q_d\sum_{i=1}^m q_i\frac{\xi_i e^{\alpha(T-s)}}{\xi_i e^{\alpha(T-s)}+1} + e^{m_u}p_u\sum_{j=1}^n p_j\frac{\eta_j e^{\alpha(T-s)}}{\eta_j e^{\alpha(T-s)}-1}\right)\lambda^{\mathbb{Q}}ds - \lambda^{\mathbb{Q}}(T-t)\right). \tag{24}$$

Proof. Let us revise the equation (17). Using the Ito lemma for the process $Y_t = \ln(S_t)$ and changing the physical measure $\mathbb{P}$ to the equivalent risk-neutral measure $\mathbb{Q}$, one obtains

$$dY_t = \alpha(\widetilde{\rho}^{\mathbb{Q}}(t) - Y_t)dt + \sigma dW_t^{\mathbb{Q}} + Z dN_t^{\mathbb{Q}}, \tag{25}$$

where

$$\widetilde{\rho}^{\mathbb{Q}}(t) = \frac{1}{\alpha}\frac{dg(t)}{dt} + g(t) - \frac{\sigma\theta^{\mathbb{Q}}}{\alpha}.$$

After multiplying both sides of (25) by $e^{-\alpha(T-t)}$ and integrating from t to T, the equation converts to

$$\int_t^T e^{-\alpha(T-s)}dY_s = \int_t^T e^{-\alpha(T-s)}dg(s) + \int_t^T \alpha e^{-\alpha(T-s)}g(s)ds - \int_t^T \alpha e^{-\alpha(T-s)}Y_s ds -$$

$$\int_t^T \sigma\theta^{\mathbb{Q}} e^{-\alpha(T-s)}ds + \int_t^T \sigma e^{-\alpha(T-s)}dW_s^{\mathbb{Q}} + \int_t^T e^{-\alpha(T-s)}ZdN_s^{\mathbb{Q}}. \tag{26}$$

Because

$$-\int_t^T \alpha e^{-\alpha(T-s)}Y_s ds = e^{-\alpha(T-t)}Y_t - Y_T + \int_t^T e^{-\alpha(T-s)}dY_s \tag{27}$$

and

$$\int_t^T \alpha e^{-\alpha(T-s)}g(s)ds = g(T) - e^{-\alpha(T-t)}g(t) - \int_t^T e^{-\alpha(T-s)}dg(s), \tag{28}$$

we may write

$$Y_T = g(T) + (Y_t - g(t))e^{-\alpha(T-t)} - \int_t^T \sigma\theta^{\mathbb{Q}}\, e^{-\alpha(T-s)}ds + \int_t^T \sigma e^{-\alpha(T-s)}dW_s^{\mathbb{Q}} + \int_t^T e^{-\alpha(T-s)}ZdN_s^{\mathbb{Q}}. \tag{29}$$

Applying the Dynkin lemma and a technique similar as in [27], we obtain the following equality

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T \sigma e^{-\alpha(T-s)}dW_s^{\mathbb{Q}} + \int_t^T e^{-\alpha(T-s)}ZdN_s^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right] =$$

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T \sigma e^{-\alpha(T-s)}dW_s^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right]\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T e^{-\alpha(T-s)}ZdN_s^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right]. \tag{30}$$

Using (29), (30), the fact that $S_T = e^{Y_T}$ and denoting $G(t) = e^{g(t)}$,

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t] = G(T)\left(\frac{S_t}{G(t)}\right)^{e^{-\alpha(T-t)}} \exp\left(-\int_t^T \sigma\theta^{\mathbb{Q}}\, e^{-\alpha(T-s)}ds\right)\cdot$$

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T \sigma e^{-\alpha(T-s)}dW_s^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right]\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T e^{-\alpha(T-s)}ZdN_s^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right] = G(T)\left(\frac{S_t}{G(t)}\right)^{e^{-\alpha(T-t)}}\cdot$$

$$\exp\left(\int_t^T \sigma e^{-\alpha(T-s)}\left(\frac{1}{2}\sigma e^{-\alpha(T-s)} - \theta^{\mathbb{Q}}\right)ds\right)\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T e^{-\alpha(T-s)}ZdN_s^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right], \tag{31}$$

inasmuch as

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T \sigma e^{-\alpha(T-s)}dW_s^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right] = \exp\left(\frac{1}{2}\int_t^T \sigma^2 e^{-2\alpha(T-s)}ds\right). \tag{32}$$

Following the considerations in [10] (part A of Appendix), we may write

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T e^{-\alpha(T-s)}ZdN_s^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right] = \exp\left(\int_t^T \left(\mathbb{E}^{\mathbb{Q}}\left[e^{e^{-\alpha(T-s)}Z}\right] - 1\right)\lambda^{\mathbb{Q}}ds\right). \tag{33}$$

The latter exponent may be explicitly calculated. For this purpose, first of all we calculate

$$\mathbb{E}^{\mathbb{Q}}\left[e^{e^{-\alpha(T-s)}Z}\right],$$

where Z has a translated mixed-exponential distribution with density (16). The straightforward calculation yields

$$\mathbb{E}^{\mathbb{Q}}\left[e^{e^{-\alpha(T-s)}Z}\right] = e^{m_d}q_d\sum_{i=1}^{m} q_i\frac{\xi_i e^{\alpha(T-s)}}{\xi_i e^{\alpha(T-s)}+1} + e^{m_u}p_u\sum_{j=1}^{n} p_j\frac{\eta_j e^{\alpha(T-s)}}{\eta_j e^{\alpha(T-s)}-1}. \tag{34}$$

Thus,

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T e^{-\alpha(T-s)}ZdN_s^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right] =$$

$$\exp\left(\int_t^T\left(e^{m_d}q_d\sum_{i=1}^{m} q_i\frac{\xi_i e^{\alpha(T-s)}}{\xi_i e^{\alpha(T-s)}+1} + e^{m_u}p_u\sum_{j=1}^{n} p_j\frac{\eta_j e^{\alpha(T-s)}}{\eta_j e^{\alpha(T-s)}-1}\right)\lambda^{\mathbb{Q}}ds - \lambda^{\mathbb{Q}}(T-t)\right), \tag{35}$$

which finishes the proof. □

10.3 Results of the calibration to the real data

The period chosen for calibration starts at the beginning of the 2014 year. Since the first quoted on the POLPX week of 2014 begins on 30th December 2013, t in formula for $F(t,T)$ is Friday, 27th December 2013. The electricity spot price on this day (last known value of the index) was equal to $S_t = 121.6$ PLN/MWh. In order to calibrate our already estimated model to the market participants' expectations, we choose the quoted on 27th December 2013 liquid forward contracts, i.e. only those with some open positions. There are 9 such contracts, forward prices of which are given in Table 5 (supply of the electrical energy during the period indicated in the contracts' names):

M1_14	M2_14	M3_14	Q1_14	Q2_14	Q3_14	Q4_14	Y_14	Y_15
152	154.50	148	151.25	148.65	158.26	150.25	152.86	158.30

Table 5: Forward prices of liquid contracts quoted on 27th December 2013

The process of calibration begins with evaluating the analytical forward prices (24) for t equal to 27th December 2013 and T equal to all dates within the periods of supply of the energy for all selected forward contracts, where the values of $\theta^{\mathbb{Q}}$ and $\lambda^{\mathbb{Q}}$ remain unspecified. Afterwards, the results, as the functions of these parameters, are averaged within the respective contracts delivery periods. Thereafter, the mean square error is calculated for the obtained analytical forward prices and the quoted ones so as to minimize the error with respect to $\theta^{\mathbb{Q}}$ and $\lambda^{\mathbb{Q}}$. The outcomes of this procedure, which uniquely determine the choice of the equivalent risk-neutral measure, are presented in Table 6. Let us note that the calibrated value of the intensity parameter achieves the minimum, boundary value equal to 0. The possible reason justifying this conjuncture is a small volatility of the quoted liquid forward contracts (flat term structure).

The calibrated forward prices, preceded by the series of spot prices, are shown on Figure 7.

11 Conclusions

In the article the authors introduce the new model for the electricity spot prices, which are quoted on the Polish Power Exchange, taking into account all the specificity of the Warsaw

$\theta^{\mathbb{Q}}$	$\lambda^{\mathbb{Q}}$	$q_t^{\mathbb{Q}}(z) = \lambda^{\mathbb{Q}}/\lambda$
19.93	0	0

Table 6: Results of calibration

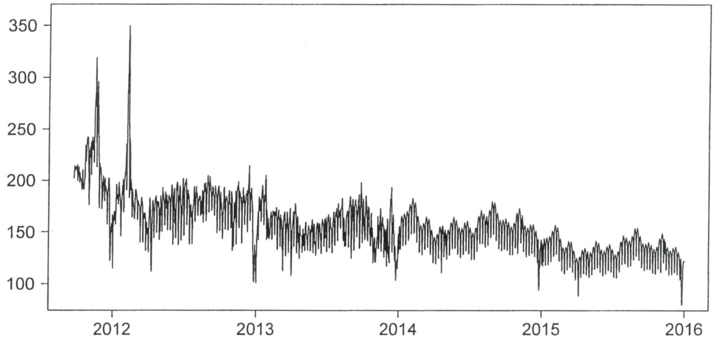

Figure 7: Historical spot prices and calibrated to market forward prices in PLN/MWh, time horizon: 31st December 2015

market, as well as the electrical energy prices specificity in general. Several novel ideas concerning seasonality matching, spikes filtering, jump-size distribution, etc. are put into practice. The parameters are estimated basing on the historical data. The model is validated by performing simulations and tests for the goodness of fit, which legitimize the proposed approach. Finally, the analytical formula for the forward prices is derived allowing for the convenient calibration of the model to the forward contracts quoted on the exchange, making use of the notions of the market prices of diffusion and jump risks.

The future work will concern valuing of options within the model.

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