Then $z - z_\alpha = \dfrac{(x-\alpha)(p-q)}{(x-q)(\alpha-q)}$, $z - z_\gamma = \dfrac{(x-\gamma)(p-q)}{(x-q)(\gamma-q)}$,

$$z - z_\beta = z + z_\alpha = \frac{(x-\beta)(p-q)}{(x-q)(\beta-q)}, \quad z + z_\gamma = \frac{(x-\delta)(p-q)}{(x-q)(\delta-q)};$$

and therefore

$$I = \frac{p-q}{\sqrt{(\alpha-q\,.\,\beta-q\,.\,\gamma-q\,.\,\delta-q)}} \int \frac{zdz}{\sqrt{(z^2 - z_\alpha^{\ 2}\,.\,z^2 - z_\gamma^{\ 2})}},$$

which is pseudo-elliptic.

This case clearly corresponds to the two obviously pseudo-elliptic integrals

$$\int \frac{xdx}{\sqrt{(1-x^2\,.\,1-k^2x^2)}} \text{ and } \int \frac{dx}{x\sqrt{(1-x^2\,.\,1-k^2x^2)}},$$

which immediately offer themselves when Jacobi's normal form is used. The other four forms in this case are

$$\int \left[\frac{1-x\sqrt{k}}{1+x\sqrt{k}} \text{ or } \frac{1+x\sqrt{k}}{1-x\sqrt{k}}\right] \frac{dx}{\sqrt{(1-x^2\,.\,1-k^2x^2)}},$$

and $$\int \left[\frac{1-x\sqrt{(-k)}}{1+x\sqrt{(-k)}} \text{ or } \frac{1+x\sqrt{(-k)}}{1-x\sqrt{(-k)}}\right] \frac{dx}{\sqrt{(1-x^2\,.\,1-k^2x^2)}}.$$

ON THE DIVISION OF THE PERIODS OF ELLIPTIC FUNCTIONS BY 9.

By *W. Burnside, M.A.*

The equation which determines $P(3u)$ in terms of $P(u)$ is

$$(3P^4 - \tfrac{3}{2}g_2P^2 - 3g_3P - \tfrac{1}{16}g_2^{\ 2})^2\,P(3u)$$
$$= P^9 + 3g_2P^7 + 24g_3P^6 + \tfrac{15}{8}g_2^{\ 2}P^5 - \tfrac{2}{3}g_2g_3P^4 + (3g_3^{\ 2} - \tfrac{9}{16}g_2^{\ 3})\,P^3$$
$$+ (\tfrac{9}{256}g_2^{\ 4} - \tfrac{3}{2}g_2g_3^{\ 2})\,P + \tfrac{1}{32}g_2^{\ 3}g_3 - g_3^{\ 3} \;\ldots\ldots\ldots\ldots\text{(i)},$$

where for brevity P is written for $P(u)$.

If $$P(3u) = P(3u_0),$$

the nine roots of this equation are

$$\begin{array}{lll} P(u_0)\,, & P(u_0 + \tfrac{2}{3}\omega)\,, & P(u_0 + \tfrac{4}{3}\omega)\,, \\ P(u_0 + \tfrac{2}{3}\omega'), & P(u_0 + \tfrac{2}{3}\omega' + \tfrac{2}{3}\omega), & P(u_0 + \tfrac{2}{3}\omega' + \tfrac{4}{3}\omega), \\ P(u_0 + \tfrac{4}{3}\omega'), & P(u_0 + \tfrac{4}{3}\omega' + \tfrac{2}{3}\omega), & P(u_0 + \tfrac{4}{3}\omega' + \tfrac{4}{3}\omega), \end{array}$$

where 2ω, $2\omega'$ is any pair of primitive periods of the elliptic function; and hence, if $P(3u_0)$ and $P(\frac{2}{3}\omega)$ are both regarded as known quantities,

$$P(u_0) + P(u_0 + \tfrac{2}{3}\omega) + P(u_0 + \tfrac{4}{3}\omega)$$

can only take three different values; or, in other words, the equation (i) of the 9th degree has a resolvent of the 3rd degree, whose coefficients are rational in g_2, g_3, $P(3u_0)$ and $P(\frac{2}{3}\omega)$.

The direct calculation of this resolvent involves rather laborious eliminations; but it may be determined very simply indirectly by finding the two cubic transformations, the successive performance of which leads to the equation (i) for multiplication by 3. The general equation for transformation of order n (odd), viz.

$$P\left(u, \frac{2\omega}{n}, 2\omega'\right) = P(u) + \sum_1^{n-1}\left[\left(P\left(u - \frac{2r\omega}{n}\right) - P\left(\frac{2r\omega}{n}\right)\right]\right.,$$

gives for $n = 3$,

$$P(u, \tfrac{2}{3}\omega, 2\omega') = P(u) + P(u + \tfrac{2}{3}\omega) + P(u - \tfrac{2}{3}\omega) - 2P(\tfrac{2}{3}\omega) \ldots\text{(ii)}$$

The invariants g'_2, g'_3 of the new elliptic function on the left-hand side of this equation are easily determined by expanding each side in powers of u. This gives

$$\frac{1}{u^2} + \frac{g_2'}{20}u^2 + \frac{g_3'}{28}u^4 + \ldots = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + u^2P''(\tfrac{2}{3}\omega)$$
$$+ \tfrac{1}{12}u^4P^{\text{iv}}(u) + \&\text{c.},$$

and therefore
$$g_2' = g_2 + 20P''(\tfrac{2}{3}\omega),$$
$$g_3' = g_3 + 28P^{\text{iv}}(\tfrac{2}{3}\omega):$$

or, if $P(\frac{2}{3}\omega) = a$,

$$g_2' = 120a^2 - 9g_2, \qquad g_3' = 280a^3 - 42ag_2 - 27g_3;$$

a being given by the equation

$$a^4 - \tfrac{1}{2}g_2a^2 - g_3a - \tfrac{1}{48}g_2^2 = 0.$$

The addition equation gives

$$P(u + \tfrac{2}{3}\omega) + P(u - \tfrac{2}{3}\omega) = -2[P(u) + P(\tfrac{2}{3}\omega)] + \tfrac{1}{2}\frac{P'^2(u) + P'^2(\frac{2}{3}\omega)}{[P(u) - P(\frac{2}{3}\omega)]^2};$$

and therefore, on writing

$$P(u) = x, \quad P(u, \tfrac{2}{3}\omega, 2\omega) = y,$$

the equation (ii) for the cubic transformation becomes

$$y = -(x+4a) + \frac{4x^3 - g_2x - g_3 + 4a^3 - g_2a - g_3}{2(x-a)^2}$$

$$= \frac{x^3 - 2x^2a + (7a^2 - \frac{1}{2}g_2)x - 2a^3 - \frac{1}{2}g_2a - g_3}{(x-a)^2} \ldots\text{(iii)}.$$

The equation for giving $P(u, \frac{2}{3}\omega, \frac{2}{3}\omega')$ or z in terms of $P(u, \frac{2}{3}\omega, 2\omega')$ will be of the same form with g'_2 and g'_3 for g_2 and g_3, and b for a, where b is the properly chosen root of the equation

$$b^4 - \tfrac{1}{2}g'_2b^2 - g'_3b - \tfrac{1}{48}g'^2_2 = 0.$$

Hence

$$z = \frac{y^3 - 2y^2b + (7b^2 - \frac{1}{2}g'_2)y - 2b^3 - \frac{1}{2}g'_2b - g'_3}{(y-b)^2} \ldots\text{(iv)},$$

but $z = P(u, \frac{2}{3}\omega, \frac{2}{3}\omega') = 9P(3u, 2\omega, 2\omega')$;

and therefore the elimination of y between (iii) and (iv) must lead to equation (i).

On making the substitution for y in (iv) and calculating the first few terms, there results

$$z = \frac{x^9 - (6a+2b)x^8 + (33a^2 + 12ab + 7b^2 - \frac{3}{2}g_2 - \frac{1}{2}g'_2)x^7 + \ldots}{(x-a)^4[x^3 - 2x^2a + (7a^2 - \frac{1}{2}g_2)x - 2a^3 - \frac{1}{2}g_2a - g_3 - b(x-a)^2]^2}.$$

Comparing this with the equation (i), the coefficient of x^8 in the numerator must vanish and therefore

$$b = -3a.$$

On substituting this value in the coefficient of x^7 in the numerator and equating the result to $3g_2$,

$$g'_2 = 66a^2 + 24ab + 14b^2 - 9g_2$$

$$= 120a^2 - 9g_2,$$

verifying the previously found value, while the coefficient of x^6, which I have not written down as it is rather long, verifies the value of g'_3. The denominator also, on taking account of the equation giving the value of a, will be found at once to reduce to

$$(x^4 - \tfrac{1}{2}g_2x^2 - g_3x - \tfrac{1}{48}g_2^2)^2;$$

moreover, on writing in the equation for b the values found for b, g'_2 and g'_3 having regard to the equation for a, it will be found to be satisfied. These verifications are sufficient to ensure the accuracy of the analysis, the possibility of the process being known independently.

When the values of b, g'_2, g'_3 are substituted in (iv), it becomes

$$z = \frac{y^3 + 6ay^2 + (3a^2 + \frac{9}{2}g_2)\,y - 46a^3 + \frac{57}{2}ag_2 + 27g_3}{(y+3a)^2} \quad \text{...(iv}'\text{)},$$

and now equation (i) may be replaced by equations (iii) and (iv′), and, regarding a as a known quantity, depends for solution on these two cubics.

The process by which (iv′) has been derived from (ii) shews that if

$$z = 9P(3u_0),$$

then $\quad y = P(u_0) + P(u_0 + \frac{2}{3}\omega) + P(u_0 + \frac{4}{3}\omega) - 2a,$

and that therefore, if in (iv′) y' be written for $y + 2a$, the resulting equation will be the resolvent of (i) originally referred to. If for z the four values of $9P(\frac{2}{3}\bar{\omega})$ be substituted successively, the 36 resulting values of x will be the values of $P\left(\frac{2n\omega + 2n'\omega'}{9}\right)$ corresponding to all pairs of values of n, n' except (0, 0), (0, 3), (3, 0), (3, 3), and (3, 6).

To illustrate the application of these equations the case $g_2 = 0$ may be taken. There is no real loss of generality in putting $g_3 = 4$; and if then, following Halphen's notation,

$$\omega_2 = \int_1^\infty \frac{dx}{\sqrt{(4x^3 - 4)}},$$

$2\omega_2$ is the smallest positive real period; while, if

$$\omega'_2 = i\omega_2\sqrt{3},$$

$2\omega'_2$ is the smallest positive pure imaginary period.

Also
$$2\omega = \omega_2 + \omega'_2,$$
$$2\omega' = \omega_2 - \omega'_2$$

are a pair of primitive periods.

The values of a in this case are

$$0,\ 2^{\frac{2}{3}},\ 2^{\frac{2}{3}}\alpha,\ 2^{\frac{2}{3}}\alpha^2, \text{ where } \alpha = -\tfrac{1}{2} - i(\tfrac{1}{2}\sqrt{3}),$$

whence it immediately follows that

$$P(\tfrac{2}{3}\omega_2) = 2^{\frac{2}{3}}, \quad P(\tfrac{2}{3}\omega'_2) = 0.$$

To discriminate between the other two values the equation

$$P(\alpha u) = \alpha P u$$

may be used.

Now
$$\alpha = \frac{-\omega_2 - \omega'_2}{2\omega_2} = -\frac{\omega}{\omega_2},$$

therefore
$$P(\tfrac{2}{3}\omega) = P\left(\frac{\omega_2 + \omega'_2}{3}\right) = 2^{\frac{2}{3}}\alpha,$$

$$P(\tfrac{2}{3}\omega') = P\left(\frac{\omega_2 - \omega'_2}{3}\right) = 2^{\frac{2}{3}}\alpha^2.$$

Taking now in the two cubics (iii) and (iv′)

$$a = 0, \quad z = 0,$$

they reduce to

$$y^3 + 108 = 0,$$
$$x^3 - yx^2 - 4 = 0;$$

one set of values of x is

$P(\frac{2}{9}\omega'_2)$, $P(\frac{2}{9}\omega'_2 + \frac{2}{3}\omega'_2) = P(\frac{8}{9}\omega'_2)$, $P(\frac{2}{9}\omega'_2 + \frac{4}{3}\omega'_2) = P(\frac{4}{9}\omega'_2)$,
and these being real correspond to the real value of y, viz.

$$y = -3 \,.\, 2^{\frac{2}{3}}.$$

Hence these three quantities are the roots of

$$x^3 + 3 \,.\, 2^{\frac{2}{3}} x^2 - 4 = 0,$$

which may be at once found in the form

$$-2^{-\frac{1}{3}} \sec \tfrac{2}{9}\pi, \quad -2^{-\frac{1}{3}} \sec \tfrac{4}{9}\pi, \quad -2^{-\frac{1}{3}} \sec \tfrac{8}{9}\pi,$$

the cube root being the real one.

Now, since $P(\frac{2}{3}\omega'_2) = 0$, $P(\frac{2}{9}\omega'_2)$, $P(\frac{4}{9}\omega'_2)$, and $P(\frac{8}{9}\omega'_2)$ are in ascending order of magnitude, the last being positive. Hence

$$P(\tfrac{2}{9}\omega'_2) = -2^{-\frac{1}{3}} \sec \tfrac{4}{9}\pi,$$

$$P(\tfrac{4}{9}\omega'_2) = -2^{-\frac{1}{3}} \sec \tfrac{2}{9}\pi,$$

$$P(\tfrac{8}{9}\omega'_2) = -2^{-\frac{1}{3}} \sec \tfrac{8}{9}\pi.$$

Since
$$\alpha\omega'_2 = \tfrac{1}{2}(3\omega_2 - \omega'_2) = \omega + 2\omega',$$

and
$$-\alpha^2\omega'_2 = \tfrac{1}{2}(3\omega_2 + \omega'_2) = 2\omega + \omega'.$$

Six more values may be obtained from the given three by applications of the equation

$$P(\alpha u) = \alpha P(u).$$

The sets of values of the argument which correspond to the two imaginary values of y are $\frac{2}{9}$, $\frac{4}{9}$, and $\frac{8}{9}$ of $4\omega - \omega'$ and of $2\omega + \omega'$. There seems to be no ready way of discriminating between these; so that the numerical calculation of the various values of $P(\frac{2}{9}\bar{\omega})$ will probably be most easily performed by finding $P(\frac{2}{9}\omega_2)$, $P(\frac{4}{9}\omega_2)$, $P(\frac{8}{9}\omega_2)$ and then using the addition equation for the rest.

The direct calculation of these three quantities is made by using the values

$$a = 2^{\frac{2}{3}}, \quad z = 9 \,.\, 2^{\frac{2}{3}}.$$

When the real value of y so obtained from (iv′) is used in (iii), the resulting equation has the three quantities in question for its roots.

On making the substitutions, the equation for y is

$$y^3 - 3 \,.\, 2^{\frac{2}{3}} y^2 - 51 \,.\, 2^{\frac{4}{3}} y - 400 = 0\,;$$

and by applying Cardan's method the three roots of this equation are found to be

$$2^{\frac{2}{3}}(1 + 3^{\frac{2}{3}})^2, \quad 2^{\frac{2}{3}}(1 + \alpha 3^{\frac{2}{3}})^2, \quad 2^{\frac{2}{3}}(1 + \alpha^2 3^{\frac{2}{3}})^2.$$

Entering the first of these in equation (iii), and writing

$$x = 2^{\frac{2}{3}} x', \quad \beta = (1 + 3^{\frac{2}{3}})^2,$$

the equation for x' is

$$x'^3 - (2 + \beta)\, x'^2 + (7 + 2\beta)\, x' - (3 + \beta) = 0 \quad \text{...(v)}.$$

The application of Cardan's method of solution to this equation, which has three real roots, leads to forms which are useless for numerical calculation; but the values of x may be obtained in a form explicitly free from imaginaries in the following way.

Suppose the values

$$a = 0, \; z = 9.2^{\frac{2}{3}}$$

be substituted in the two cubics.

The three resulting cubics for x' will be found to be (writing again $x = 2^{\frac{2}{3}} x'$),

$$\left.\begin{aligned} x'^3 - \tfrac{3}{2} \sec \tfrac{2}{9}\pi\, x'^2 - 1 &= 0 \\ x'^3 - \tfrac{3}{2} \sec \tfrac{4}{9}\pi\, x'^2 - 1 &= 0 \\ x'^3 - \tfrac{3}{2} \sec \tfrac{8}{9}\pi\, x'^2 - 1 &= 0 \end{aligned}\right\} \quad \text{............(vi)},$$

and the three sets of values of x will be

$$P(\tfrac{2}{9}\omega_2),\quad P(\tfrac{2}{9}\omega_2 \pm \tfrac{2}{3}\omega'_2),$$
$$P(\tfrac{4}{9}\omega_2),\quad P(\tfrac{4}{9}\omega_2 \pm \tfrac{2}{3}\omega'_2),$$
$$P(\tfrac{8}{9}\omega_2),\quad P(\tfrac{8}{9}\omega_2 \pm \tfrac{2}{3}\omega_2'),$$

the order in which the equations and sets of roots are written not necessarily agreeing. Hence, it follows that the equation (v) has one root in common with each of the equations (vi).

The three values of x may therefore by ordinary elimination be expressed rationally in terms of β and the secants.

The result of the elimination and subsequent simplification is to give the three following values for x', viz.

$$-1 + 3^{\frac{1}{3}} \frac{2 + 3^{\frac{1}{3}} + 3^{\frac{2}{3}} - 2\cos\frac{2}{9}r\pi}{1 - 2\cos\frac{2}{9}r\pi + 4\cos^2\frac{2}{9}r\pi},\quad r = 1, 2, 4.$$

A very rough numerical calculation shews that these are in ascending order of magnitude for the values 4, 1, 2 of r; hence

$$P(\tfrac{2}{9}\omega_2) = -2^{\frac{2}{3}} + 12^{\frac{1}{3}} \frac{2 + 3^{\frac{1}{3}} + 3^{\frac{2}{3}} - 2\cos\frac{4}{9}\pi}{1 - 2\cos\frac{4}{9}\pi + 4\cos^2\frac{4}{9}\pi},$$

while $P(\frac{4}{9}\omega_2)$ and $P(\frac{8}{9}\omega_2)$ are the same functions of $\cos\frac{2}{9}\pi$ and $\cos\frac{8}{9}\pi$.

I have verified the accuracy of this analysis by the actual elimination of x' between (v) and $x'^3 - 3yx'^2 - 1 = 0$, the result being $y^3 - 3y^2 + 1 = 0$, as it should be.

It is interesting to notice that for the value

$$z = 9a$$

the cubic for y can be formally solved in any case in a form suitable for numerical calculation if a be real.

The equation in question becomes on clearing fractions

$$(y-a)^3 - 9(6a^2 - \tfrac{1}{2}g_2)(y-a) - 3(60a^3 - 11ag_2 - 9g_3) = 0.$$

Now, if $\qquad a'^2 = 4a^3 - g_2a - g_3,$

$$a'' = 6a^2 - \tfrac{1}{2}g_2,$$

so that $\qquad g_2 = 12a^2 - 2a'',$

$$g_3 = -8a^3 + 2aa'' - a'^2,$$

the equation becomes

$$(y-a)^3 - 9a''(y-a) - 3(4aa'' + 9a'^2) = 0 \;\ldots\text{(vii)},$$

while the same substitutions throw the equation for a into the form

$$12aa'^2 - a''^2 = 0.$$

Cardan's solution of equation (vii) now is

$$y - a = \frac{3^{\frac{2}{3}}}{2^{\frac{2}{3}}}\frac{a''^{\frac{2}{3}}}{a^{\frac{1}{3}}}\alpha^2 + 3^{\frac{1}{3}}2^{\frac{2}{3}}a^{\frac{1}{3}}a''^{\frac{1}{3}}\alpha, \text{ where } \alpha^3 = 1;$$

or

$$y = a^{\frac{1}{3}}\left[a^{\frac{1}{3}} + \alpha\frac{3^{\frac{1}{3}}}{2^{\frac{1}{3}}}\frac{a''^{\frac{1}{3}}}{a^{\frac{1}{3}}}\right]^2$$

When this value of y is used in the equation for x, it reduces on taking $x - a - \frac{1}{3}(y - a)$, say x', for the unknown, to

$$x'^3 - mx' - mn = 0,$$

where

$$m = \alpha^2\frac{2^{\frac{4}{3}}}{3^{\frac{1}{3}}}a^{\frac{2}{3}}a''^{\frac{2}{3}}\left(1 + \alpha\frac{3^{\frac{1}{3}}}{2^{\frac{4}{3}}}\frac{a''^{\frac{1}{3}}}{a^{\frac{2}{3}}} + \alpha^2\frac{3^{\frac{2}{3}}}{2^{\frac{8}{3}}}\frac{a''^{\frac{2}{3}}}{a^{\frac{4}{3}}}\right),$$

$$n = \alpha\,\frac{2^{\frac{5}{3}}}{3^{\frac{5}{3}}}a^{\frac{1}{3}}a''^{\frac{1}{3}}\left(1 + \alpha\frac{3^{\frac{1}{3}}}{2^{\frac{7}{3}}}\frac{a''^{\frac{1}{3}}}{a^{\frac{2}{3}}}\right),$$

and the only new irrationals involved in the values of x are cube-roots of

$$n + \frac{\gamma}{2^{\frac{2}{3}}.3^{\frac{5}{6}}}\frac{a''^{\frac{2}{3}}}{a^{\frac{1}{3}}},$$

where γ is a primitive twelfth root of unity.

Though not directly connected with the subject of this paper, it may be pointed out that the relation

$$a = -3b$$

found between the constants employed, which may be expressed in the form

$$P(\tfrac{2}{3}\omega,\ 2\omega,\ 2\omega') = -3P(\tfrac{2}{3}\omega',\ \tfrac{2}{3}\omega,\ \tfrac{2}{3}\omega'),$$

is only the first of a series which may be easily verified by obtaining the leading terms of two transformations of odd degree n, which lead to multiplication. The general form of the relation is

$$\sum_{1}^{\frac{1}{2}(n-1)} P\left(\frac{2r\omega}{n},\ 2\omega,\ 2\omega'\right) = -n\sum_{1}^{\frac{1}{2}(n-1)} P\left(\frac{2r\omega'}{n},\ \frac{2\omega}{n},\ 2\omega'\right).$$