Raport Badawczy
Research Report

RB/12/2004

An approximate optimal economic inspection interval for a process with finite production runs

A.M. Deeb, O. Hryniewicz

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 8373578

fax: (+48) (22) 8372772

Kierownik Pracowni zgłaszający pracę:
Prof. dr hab. inż. Olgierd Hryniewicz

Warszawa 2004

An approximate optimal economic inspection interval for a process with finite production runs

Ali Mashat Deeb
Department of Statistics
Al-fatih University
Tripoli, Libya

Olgierd Hryniewicz
Systems Research Institute
Polish Academy of Science
Warsaw, Poland

Abstract

Increased competition on a global market forces producers to follow policies leading to finite production runs. This situation requires the implementation of a new type of inspection procedures with the aim to improve or sustain production quality levels. One of the most important aspects in the design of inspection processes is the specification of inspection intervals. This paper provides a simple procedure to determine an approximate optimal inspection interval h for a given inspection plan characterized by its probability of type-I error α and probability of type-II error β for processes with finite runs.

Key words: inspection plan, inspection interval, just-in-time production, finite runs.

1. Introduction

Quality is the most important decision factor, however, the occurrence of assignable or random causes results in variation of a different degree in the quality characteristic of interest. Thus, it is desirable to inspect the output at different stages of a production process in order to correct it and/or to assure its quality. The inspection is usually done on the basis of random samples drawn periodically from the process. The process of the design of an inspection procedure consists, mainly, of two stages:

1) specification of the inspection (sampling) plan to be preformed at the end of a given inspection interval, and
2) determination of the inspection (sampling) interval for a given inspection plan.

In this paper we focus on the second stage of this process. There many approaches to determine the inspection interval. However, the economical approach has attracted many researchers, who proposed many models and algorithms to determine optimal inspection intervals. Further information on this subject can be found in Bather (1963), Chiu and Wetherill (1975), Duncan (1956, 1978), Gibra (1970), Ladany (1973), Lorenzen and Vance (1986), Montgomery et al (1975), Montgomery (1980, 1982), Panagos et al (1985), Saniga (1989). The general economic model for the optimization of statistical process control can be found in Keats et al (1997). The review of the problems of the optimal design of control charts is given in Ho and Case (1994).

Nowadays industry faces rapid changes in user's requirements, which force firms to follow the "just-in-time" policy that allows them to produce a smaller number of items in response to customer's immediate request. This environment leads to frequent setups of the process, causing shorter (finite) production runs. The new circumstances require methods to determine inspection interval different from those for infinite production runs. Some research has been done to solve this problem, and the most interesting results have been presented in the paper by Del Castillo et al (1996). The problem was also considered in the papers by Quesenberry (1991), Crowder (1992), Del Castillo and Montgomery (1994, 1995) who discussed methods for the calculation of a sampling interval in the context of the design of control charts. Unfortunately, most of the proposed algorithms are too complicated to be used at a production line. In this paper we present an easy to compute procedure that solves the problem of the determination of the optimal economic inspection interval, h, for a process of a finite length. Our approach is based on the results presented in a seminal work of von Collani (1986, 1989). As the objective function we propose the loss per unit produced. The calculation of this characteristics requires a detailed information about the process behavior, the knowledge of statistical properties of the inspection procedure. Moreover, we assume that some economic quantities like the gain from the correctly operating process and some other cost parameters are also known.

The paper is organized as follows. In the second section we introduce the mathematical model of the inspection process when the production period is finite, i.e. in the situation of finite production runs. In the third section we present an economic model that describes the consequences of the implementation of the inspection procedures. This model is used in the fourth section for the optimization of the inspection interval. Finally, in section 5, we discuss the obtained results and present their possible generalizations.

2. The mathematical model of the process

The process under investigation is assumed to have a finite length of t items, and constant production rate of ν items per hour. The process starts in a stable state of control (in-control State I) centered at the target value μ_0. We also assume that its variability is known, and described by the standard deviation σ. Now, let us assume that the process can go out of control, and its deterioration which takes the form of a shift of a known magnitude ($\pm\delta\sigma$) in the process mean. The deterioration shifts the process from the in-control State I to the out-of-control State II characterized by the its mean value, either $\mu_1 = \mu_0 - \delta\sigma$ with probability $P(\mu = \mu_1)$ or $\mu_2 = \mu_0 + \delta\sigma$ with probability $P(\mu = \mu_2)$, where $\delta > 0$ is the shift size, while the variance remains unchanged. Let us formulate some further assumptions.

1) The states of the process are recognized by inspection only.
2) The process is not self-correcting, that is, once a transition to State II has occurred, the process remains there until some corrective actions are taken to return the process to State I.
3) The duration of State I is a random variable, T^*, which is exponentially distributed with a known parameter λ.

The inspection procedure consists of a periodical drawing of samples of known size at interval of h hours. According to the inspection results, appropriate actions should be taken in order to bring the process into State I, however, there are chances for two erroneous signals:

a) a false alarm which occurs with known probability α when the inspection gives a signal that the process is in state II when it is not (type- I error);
b) a non-detection of an existed shift which occurs with known probability β, i.e. indication that the process is in state I whereas it is in state II (type -II error).

Now, let us introduce several random variables that will be used for the formulation of the objective function. Let U_j be the number of samples drawn during state I of the jth cycle. A random event $\{U_j = i\}$ means that during the jth cycle the transition from the in-control State I to the out-of-control State II occurs in the time interval $(ih, (i+1)h)$, and is equivalent to a random event $\{ih < T^* < (i+1)h\}$. The probability mass function of U_j for the exponentially distributed time T^* is given by

$$P(U_j = i) = (1 - e^{-\lambda h})(e^{-\lambda h})^i, i = 0,1,2,\ldots \quad j = 1,2,\ldots \tag{1}$$

It is easy to show that U_j has the expectation

$$E(U_j) = \mu_U = \frac{1-(1-e^{-\lambda h})}{1-e^{-\lambda h}} = \frac{e^{-\lambda h}}{1-e^{-\lambda h}} = \frac{1}{e^{\lambda h}-1} \tag{2}$$

and the variance

$$\sigma_U^2 = \frac{1-(1-e^{-\lambda h})}{(1-e^{-\lambda h})^2} = \frac{e^{-\lambda h}}{(1-e^{-\lambda h})^2} = \frac{e^{\lambda h}}{(e^{\lambda h}-1)^2} \ . \tag{3}$$

Now, let us introduce a random variable V_j that describes the number of samples in the j^{th} cycle drawn during the out-of-control State II. The random event $\{V_j = k\}$ means that the shift is detected after the inspection of the k^{th} sample, i.e. that k samples are required to detect an existed shift during the j^{th} cycle. Since the probability of not detecting the existed shift is equal to β, the probability of its detection is equal to 1- β. Thus, the probability mass function of V_j is given by the following formula

$$P(V_j = k)(1-\beta)\beta^{k-1}, k = 1,2,\ldots; \quad j = 1,2,\ldots \tag{4}$$

Hence, the expected number of samples drawn in State II of each cycle is given by

$$E(V_j) = \mu_V = \frac{1}{1-\beta}, \tag{5}$$

and its variance is given by

$$\sigma_V^2 = \frac{\beta}{(1-\beta)^2} \ . \tag{6}$$

Now, let's denote by F_j the number of false alarms during the j[th] cycle. Since false alarms occur only from sampling in state I, and since each sample in this state triggers a false alarm with probability α, the number of false alarms is described by the binomial distribution with parameters (U_j, α), where U_j is a random variable described previously. Since U_j's are i.i.d. we can easily find the expectation of the conditional random variable $F_j \mid U_j$ that is equal to E ($F_j \mid U_j$) = E ($F \mid U$) = αU, and its variance that is equal to $V(F_j \mid U_j) = V(F \mid U) = \alpha(1-\alpha)U$. Consequently, the unconditional expected number of false alarms observed in each cycle is given by

$$E(F_j) = \mu_F = \frac{\alpha}{e^{\lambda h} - 1}, \; j = 1,2,\dots , \tag{7}$$

and its variance is given by

$$\begin{aligned} \sigma_F^2 &= V(F_j) = V(F) = V[E(F \mid U)] + E[V(F \mid U)] = V(\alpha U) + E[\alpha(1-\alpha)U] = \\ &= \alpha^2 \sigma_U^2 + \alpha(1-\alpha)\mu_U = \frac{\alpha^2 e^{\lambda h}}{(e^{\lambda h} - 1)^2} + \frac{\alpha(1-\alpha)}{e^{\lambda h} - 1} \end{aligned} \tag{8}$$

The next random variable W_j, j=1,2,... represents the number of samples drawn from the process during its j^{th}. It is easy to notice that $W_j = U_j + V_j$ for all j = 1, 2, ... Assuming the independence of U_j and V_j for all j = 1, 2... the sequence of random variables $\{W_j\}_{j=1}^{\infty}$ are i.i.d. with the expectation

$$\mu_W = \mu_U + \mu_V = \frac{1}{e^{\lambda h} - 1} + \frac{1}{1-\beta} = \frac{e^{\lambda h} - \beta}{(1-\beta)(e^{\lambda h} - 1)} = \frac{1 + B(e^{\lambda h} - 1)}{e^{\lambda h} - 1}, \tag{9}$$

and variance

$$\sigma_W^2 = \sigma_U^2 + \sigma_V^2 = \frac{e^{\lambda h}}{(e^{\lambda h} - 1)^2} + \frac{\beta}{(1-\beta)^2} = \frac{e^{\lambda h}}{(e^{\lambda h} - 1)^2} + B(B-1), \tag{10}$$

where

$$B = \frac{1}{1-\beta} . \tag{11}$$

Let S_k be the number of samples taken from the process up to its k^{th} renewal, i.e., S_k gives the time of the k^{th} renewal in terms of number of samples. Hence,

$$S_k = \sum_{j=1}^{k} W_j,\ k = 1,2,\dots. \tag{12}$$

Since $\{W_j\}_{j=1}^{\infty}$ are i.i.d., then the above sequence of random variables defines an ordinary renewal process.

Further, as we have assumed that inspections are performed every h hours, then for a process with a run of t consecutive items at production rate of ν items per hour, the expected number of samples completed during the run is $t/\upsilon h$ samples. Now, let N_t, denotes to the number of renewal cycles completed within a production run of t items (or h/υ samples). To analyze this random variable let us use the approach proposed by Blackwell (1977) and Yang (1983) who utilized the basic results of Cox (1962). Cox (1962) has shown that the approximate expected value of the number of renewals $N_{t'}$ in the time interval $(0,t')$ can be found from the following expression

$$E(N_{t'}) = \frac{t'}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + o(1). \tag{13}$$

Using (13) along with the previous results we arrive at the following approximation for the expected number of the renewal cycles completed during the production run of length t:

$$E(N_t) \approx \frac{t}{\upsilon h \mu_W} + \frac{\sigma_W^2 - \mu_W^2}{2\mu_W^2} \tag{14}$$

Let F_t be a random variable representing the number of false alarms observed during the production run of length t. It is defined by

$$F_t = \sum_{j=1}^{N_t} F_j,. \tag{15}$$

Thus from the well known Wald's equation we find the expected number of false alarms for the whole production run from the following formula

$$E(F_t) = E\left(\sum_{j=1}^{N_t} F_j\right) = E(F_j)E(N_t) \approx \mu_F\left(\frac{t}{\upsilon h \mu_W} + \frac{\sigma_W^2 - \mu_W^2}{2\mu_W^2}\right) \tag{16}$$

This result will be used for the evaluation of economic consequences of the inspection procedure.

3. The economic consequences (costs and profits) of the inspection procedure

This is rather obvious that an item produced during the in-control State I of a production is on average more profitable than that produced in the out-of-control State II. Thus, the process should be kept running in state I as frequently as possible, and whenever an alarm is observed some investigations should be conducted and upon their results the appropriate corrective actions must be taken to put the process in the state of control again. The production process can be considered as a series of renewal cycles, each cycle consisting of the in-control State I period, the out-of-control State II period, and the idle time period necessary for taking renewal actions. In any renewal cycle there are two types of actions associated with the application of an inspection procedure, namely, the inspection actions and the renewal actions.

The inspection actions consist of all actions that are responsible for detecting a shift. They consist of the periodical inspection and testing, and the investigations of the false alarms. The economic consequences of these actions are represented by their respective costs. Let a_1^* be the cost of inspection per sample. Thus, the expected cost of inspection for the whole run is

$$S_t = a_1^* \frac{t}{\upsilon h}. \tag{17}$$

Let a_2^* be the cost of investigating a false alarm (which might include the cost of stopping the process during its investigation). Hence, the expected cost for false alarms for the whole run is given by

$$A_t = a_2^* E(F_t) = a_2^* \mu_F \left(\frac{t}{\upsilon h \mu_W} + \frac{\sigma_W^2 - \mu_W^2}{2\mu_W^2} \right) \tag{18}$$

The renewal actions consist of all duties responsible for the transition of the process from State II to State I. The economic consequences of these actions are of two fold:

- negative ones, that are represented by the costs of the renewal actions, a_3^*, which might include the cost of the possible shutdown of the process while repairing, and

- positive ones, that are represented by the benefit from the transition to the in-control State I .

Suppose that g_1 and g_2 are the expected profits from an item produced in State I and State II, respectively. Thus, $g_1 - g_2$ (≥ 0) is the gain per unit from the transition from the out-of-control State II to the in-control State I. Since the expected duration of State I is $1/\lambda$ hours, and the production rate is υ items per hour, then the expected gain per cycle due to inspection is equal to $\upsilon(g_1 - g_2)/\lambda$. Let b^* be the expected net benefit per renewal, i.e., the difference between the expected gain and the expected renewal cost per cycle. Thus, we have

$$b^* = \frac{g_1 - g_2}{\lambda}\upsilon - a_3^*, \tag{19}$$

and the expected gain from the transitions to state I is given by

$$G_t = b^* E(N_t) \approx b^* \left(\frac{t}{\upsilon h \mu_W} + \frac{\sigma_W^2 - \mu_W^2}{2\mu_W^2} \right). \tag{20}$$

However, if do not use any inspection procedure the process remains in State II until the end of the production run. Thus, the expected gain from producing in State II for the whole run is $t g_2$. The economic consequences of the inspection procedures are used in the next section for finding an optimal inspection interval.

4. Optimization of the inspection interval

Let $L(t)$ be the expected loss incurred in the run. From the considerations presented in the previous sections we can find that $L(t)$ is given by the following formula

$$L(t) = a_1^* \frac{t}{\upsilon h} + a_2^* \mu_F E(N_t) - b^* E(N_t) - t g_2 \ . \tag{21}$$

The expected loss per unit produced expressed as a function of h for a given production run t, is given now by

$$\begin{aligned} L(h \mid t) &= \frac{L(t)}{t} = \frac{1}{t}\left\{ a_1^* \frac{t}{\upsilon h} + a_2^* \mu_F E N_t - b^* E N_t - t g_2 \right\} = \\ &= \frac{a_1^*}{\upsilon h} - \frac{b^* - a_2^* \mu_F}{\upsilon h \mu_W} - \frac{1}{2t}\left(\frac{b^* - a_2^* \mu_F}{\mu_W} \right)\left(\frac{\sigma_W^2 - \mu_W^2}{\mu_W} \right) - g_2 \end{aligned} \tag{22}$$

This function has to be minimized in order to determine the optimal inspection interval h for a process with a finite run t.

To reduce the number of the input parameters of the objective function we can follow von Collani (1986, 1989). The following relation gives the time-standardized loss function

$$S(y \mid r)=\left(L\left(\frac{y}{\lambda} \mid t=\frac{r\upsilon}{\lambda}\right)+g_2\right)\frac{\upsilon}{a_2^{*}\lambda} \tag{23}$$

Let $a_1=\frac{a_1^{*}}{a_2^{*}}$, $b=\frac{b^{*}}{a_2^{*}}$, $y=\lambda h$, $r=\frac{\lambda t}{\upsilon}$, $A=\frac{1}{\alpha}$, and $B=\frac{1}{1-\beta}$.

Hence,

$$S(y \mid r)=\frac{1}{y}\left\{a_1-\frac{b\left(e^{y}-1\right)-\alpha}{e^{y}-b}(1-\beta)\right\}+\frac{1}{2r}\left[\frac{b(e^{y}-1)-\alpha}{e^{y}-\beta}(1-\beta)\right]\left[\frac{e^{y}+\beta}{e^{y}-\beta}\right], \tag{24}$$

or, equivalently.

$$S(y \mid r)=\frac{1}{y}\left\{a_1-\frac{b\left(e^{y}-1\right)-\frac{1}{A}}{1+B\left(e^{y}-1\right)}\right\}+\frac{1}{2r}\left[\frac{b(e^{y}-1)-\frac{1}{A}}{1+B(e^{y}-1)}\right]\left[\frac{1-B(e^{y}+1)}{1-B(e^{y}-1)}\right]. \tag{25}$$

The transformation of the objective function (22) to form of (23) has reduced the complexity of the optimization problem because of the following reasons:

(1) the transformed objective function $S(y \mid r)$ depends only on two cost parameters, namely, a_1 and b instead of four parameters in the original objective function $L(h \mid t)$ given by (22);

(2) the time-standardized objective function $S(y \mid r)$ depends on the process parameters $\frac{1}{\lambda}$, υ, and t only through a new variable r.

The loss function $L(h \mid t)$ attains its minimum at h^{*} iff the time-standardized loss function $S(y \mid r)$attains its minimum at $y^{*}=\lambda h^{*}$. Thus, it is sufficient to optimize the time-standardized loss function $S(y \mid r)$ given by (25) in order to determine the optimal standardized inspection interval y^{*}, and thus the optimal inspection interval h^{*}.

The optimal standardized inspection interval y^* is can be found by solving the following equation

$$\frac{d}{dy}S(y \mid r)=0 \tag{26}$$

After some calculations we present (26) in the following form

$$-\left\{a_1-\frac{b(e^y-1)-\frac{1}{A}}{1+B(e^y-1)}\right\}\frac{1}{y^2}-\frac{\left(b+\frac{B}{A}\right)e^y}{[1+B(e^y-1)]^2}\frac{1}{y}-$$

$$-\frac{1}{2r}\left\{\frac{2b(B-1)e^y}{[1+B(e^y-1)]^2}-\frac{2(B-1)\left(b+\frac{B}{A}\right)e^y}{\left[1+B(e^y-1)\right]^3}+\left(b+\frac{B}{A}\right)\frac{\left[1-B(e^y+1)\right]e^y}{\left[1+B(e^y-1)\right]^3}\right\}=0 \tag{27}$$

Let us introduce the following notation

$$C=\frac{b-Ba_1}{b+\frac{B}{A}} \tag{28}$$

$$D=\frac{1}{r}\left[\frac{bB(B-1)}{b+\frac{B}{A}}\right] \tag{29}$$

$$E=\frac{B}{2r} \tag{30}$$

After some mathematical transformations we obtain the following compact version of (27):

$$\left\{1+B\left[e^y(1+y)-1\right]\right\}\left[1+B(e^y-1)\right]-C\left[1+B(e^y-1)\right]^3+$$
$$+D\left[1+B(e^y-1)\right]y^2e^y+E\left(3-3B-Be^y\right)y^2e^y=0 \tag{31}$$

or, equivalently,

$$\frac{1+B\left[e^{y}(1+y)-1\right]}{\left[1+B(e^{y}-1)\right]^{2}}+\frac{Dy^{2}e^{y}}{\left[1+B(e^{y}-1)\right]^{2}}+\frac{E(3-3B-Be^{y})y^{2}e^{y}}{\left[1+B(e^{y}-1)\right]^{3}}=C \qquad (32)$$

The solution of any of the above equations determines the optimal standardized inspection interval y^{*} for monitoring a process with a finite production run. The solution of these equations requires a numerical procedure. Moreover, the impact of the input parameters on the optimal length of the sampling interval is not visible. Therefore, there is a practical need to obtain an approximate close formula for the optimal inspection interval. To find the approximately optimal inspection interval $\hat{y}$ we expand the left hand side of the equation (32) around y=0, and neglect all terms of order higher than two. This expansion seems to be reasonable if the length of the inspection interval h is small in comparison to the expected time to deterioration $1/\lambda$. After some transformations we arrive at the following equation:

$$1+\frac{1}{2}\{B(1-2B)+2[D+(3-4B)E]\}y^{2}\approx C, \qquad (33)$$

Hence, the approximately optimal standardized inspection interval is given by the following simple formula

$$\hat{y}\approx\sqrt{\frac{2(C-1)}{(1-2B)B+2D+2(3-4B)E)}}=\sqrt{\frac{2r(1-\beta)^{2}(a_{1}+\alpha)}{(1+r)(1+\beta)[b(1-\beta)+\alpha]+2\alpha\beta}} \qquad (34)$$

Once the approximately optimal standardized inspection interval $\hat{y}$ is obtained, the approximately optimal inspection interval $\hat{h}$ is given by

$$\hat{h}=\frac{\hat{y}}{\lambda}=\frac{1}{\lambda}\sqrt{\frac{2r(1-\beta)^{2}(a_{1}+\alpha)}{(1+r)(1+\beta)[b(1-\beta)+\alpha]+2\alpha\beta}}\,. \qquad (35)$$

The determination of the approximately optimal sampling interval ($\hat{h}$) will be summarized in following algorithm.

5. Discussion

The approximate inspection interval $\hat{y}$ depends on a ,β, a_1,b, and r. Thus, it is desirable to investigate the effects of these parameters on $\hat{y}$ and on its accuracy as well. The exact optimal standardized inspection interval y^* has been computed by the minimization of the objective function $S(y|r)$ given by (25) with respect to y using a standard minimization routine. The approximately optimal standardized inspection interval $\hat{y}$ has been computed from (32). The following nine tables contain the comparison of the exact and approximate solutions for various values of α, β, a_1 and r. To compare the exact and the approximate solutions we present not only their values, but the values of the objective function of the respective cases as well.

Table 1

α	β	a_1	b	r	y^*	$\hat{y}$	$S(y^* \mid r)$	$S(\hat{y} \mid r)$
0.01	0.01	0.1	10	10	0.1475	0.1399	-8.479	-8.477
				50	0.1533	0.1453	-8.5354	-8.5334
				500	0.1546	0.1466	-8.5483	-8.5464
			50	10	0.0642	0.0626	-46.537	-46.5361
				50	0.0667	0.0650	-46.6653	-46.6644
				500	0.0672	0.0656	-46.6948	-46.694
			500	10	0.02	0.0198	-488.94	-488.94
				50	0.0208	0.0206	-489.35	-489.35
				500	0.0209	0.0207	-489.444	-489.444
		1	10	10	0.5071	0.4240	-5.6284	-5.5741
				50	0.5268	0.4404	-5.7909	-5.7384
				500	0.5314	0.4443	-5.8281	-5.7761
			50	10	0.2041	0.1897	-39.7362	-39.7114
				50	0.212	0.1970	-40.1169	-40.0929
				500	0.2138	0.1988	-40.2043	-40.1806
			500	10	0.0613	0.0600	-466.713	-466.705
				50	0.0638	0.0623	-467.946	-467.939
				500	0.0643	0.0629	-468.23	-468.223
		5	10	10	1	0.9444	-0.9613	-0.8267
				50	1	0.9808	-1.2142	-1.168
				500	1	0.9895	-1.2711	-1.246
			50	10	0.5048	0.4225	-28.1884	-27.9199
				50	0.5244	0.4388	-28.9989	-28.7396
				500	0.5292	0.4427	-29.1845	-28.9277
			500	10	0.1407	0.1336	-426.869	-426.782
				50	0.146	0.1388	-429.58	-429.496
				500	0.1472	0.1400	-430.204	-430.12

Table 2

α	β	a_1	b	r	y^*	$\hat{y}$	$S(y^* \mid r)$	$S(\hat{y} \mid r)$
0.01	0.05	0.1	10	10	0.1426	0.1344	-8.4218	-8.4193
				50	0.1481	0.1396	-8.4801	-8.4776
				500	0.1494	0.1409	-8.4934	-8.491
			50	10	0.0617	0.0602	-46.4006	-46.3994
				50	0.0641	0.0625	-46.5337	-46.5325
				500	0.0647	0.0630	-46.5643	-46.5632
			500	10	0.0192	0.0190	-488.493	-488.493
				50	0.0199	0.0198	-488.919	-488.919
				500	0.0201	0.0199	-489.018	-489.017
		1	10	10	0.4972	0.4074	-5.4934	-5.4254
				50	0.5158	0.4231	-5.6585	-5.5935
				500	0.52	0.4269	-5.6963	-5.632
			50	10	0.1979	0.1823	-39.3623	-39.3305
				50	0.2053	0.1893	-39.7543	-39.724
				500	0.2072	0.1910	-39.8444	-39.8144
			500	10	0.0591	0.0576	-465.399	-465.389
				50	0.0614	0.0599	-466.679	-466.669
				500	0.0619	0.0604	-466.973	-466.964
		5	10	10	1	0.9073	-0.797	-0.5573
				50	1	0.9423	-1.0507	-0.9031
				500	1	0.9507	-1.1078	-0.9822
			50	10	0.495	0.4060	-27.5139	-27.1777
				50	0.5135	0.4216	-28.3378	-28.0164
				500	0.5182	0.4253	-28.5262	-28.2087
			500	10	0.1359	0.1284	-424.109	-423.996
				50	0.1411	0.1333	-426.91	-426.803
				500	0.1424	0.1345	-427.554	-427.448

Table 3

α	β	a_1	b	r	y^*	$\hat{y}$	$S(y^* \mid r)$	$S(\hat{y} \mid r)$
0.01	0.1	0.1	10	10	0.1367	0.1278	-8.3468	-8.3434
				50	0.142	0.1328	-8.4075	-8.4042
				500	0.1431	0.134	-8.4214	-8.4182
			50	10	0.0589	0.0572	-46.2214	-46.2198
				50	0.0612	0.0594	-46.3607	-46.3593
				500	0.0616	0.0599	-46.3928	-46.3913
			500	10	0.0183	0.0181	-487.906	-487.906
				50	0.019	0.0188	-488.354	-488.353
				500	0.0191	0.019	-488.457	-488.456
		1	10	10	0.4851	0.3874	-5.3171	-5.2295
				50	0.5027	0.4023	-5.4857	-5.4027
				500	0.5068	0.4059	-5.5242	-5.4424
			50	10	0.1902	0.1733	-38.8717	-38.8298
				50	0.1973	0.18	-39.2787	-39.2391
				500	0.1991	0.1816	-39.3721	-39.3331
			500	10	0.0564	0.0548	-463.673	-463.66
				50	0.0586	0.0569	-465.014	-465.001
				500	0.0591	0.0574	-465.322	-465.309
		5	10	10	1	0.8628	-0.5852	-0.2002
				50	1	0.896	-0.8394	-0.5526
				500	1	0.904	-0.8966	-0.6331
			50	10	0.4829	0.386	-26.6331	-26.1998
				50	0.5005	0.4009	-27.4744	-27.0643
				500	0.5047	0.4045	-27.6665	-27.2622
			500	10	0.1302	0.1221	-420.485	-420.335
				50	0.1351	0.1268	-423.404	-423.263
				500	0.1362	0.1279	-424.075	-423.936

Table 4

α	β	a_1	b	r	y^*	$\hat{y}$	$S(y^* \mid r)$	$S(\hat{y} \mid r)$
0.05	0.01	0.1	10	10	0.1736	0.1631	-8.2688	-8.2657
				50	0.1803	0.1694	-8.333	-8.3299
				500	0.1819	0.1709	-8.3477	-8.3447
			50	10	0.0751	0.0731	-46.003	-46.0015
				50	0.0779	0.0759	-46.1511	-46.1497
				500	0.0787	0.0766	-46.1851	-46.1838
			500	10	0.0233	0.0231	-487.133	-487.132
				50	0.0242	0.024	-487.609	-487.609
				500	0.0245	0.0242	-487.719	-487.719
		1	10	10	0.5179	0.4315	-5.5831	-5.5256
				50	0.5379	0.4481	-5.7472	-5.6917
				500	0.5428	0.4521	-5.7847	-5.7298
			50	10	0.2084	0.1934	-39.5803	-39.5541
				50	0.2163	0.2008	-39.9667	-39.9414
				500	0.2183	0.2026	-40.0555	-40.0304
			500	10	0.0626	0.0612	-466.108	-466.1
				50	0.065	0.0635	-467.364	-467.356
				500	0.0655	0.0641	-467.653	-467.645
		5	10	10	1	0.9463	-0.9474	-0.8162
				50	1	0.9827	-1.1997	-1.1578
				500	1	0.9915	-1.2565	-1.2359
			50	10	0.507	0.424	-28.1422	-27.8705
				50	0.5266	0.4404	-28.9543	-28.6921
				500	0.5314	0.4443	-29.1403	-28.8805
			500	10	0.1412	0.1342	-426.624	-426.536
				50	0.1465	0.1393	-429.345	-429.26
				500	0.1479	0.1406	-429.97	-429.886

Table 5

α	β	a_1	b	r	y^*	$\hat{y}$	$S(y^* \mid r)$	$S(\hat{y} \mid r)$
0.05	0.05	.1	10	10	0.168	0.1567	-8.2048	-8.2007
				50	0.1744	0.1627	-8.2709	-8.267
				500	0.176	0.1642	-8.2861	-8.2822
			50	10	0.0724	0.0702	-45.8466	-45.8448
				50	0.0751	0.0729	-46	-45.9983
				500	0.0758	0.0736	-46.0354	-46.0336
			500	10	0.0224	0.0222	-486.614	-486.613
				50	0.0233	0.0231	-487.11	-487.109
				500	0.0235	0.0233	-487.224	-487.223
		1	10	10	0.5077	0.4145	-5.4475	-5.3757
				50	0.527	0.4305	-5.6142	-5.5455
				500	0.5313	0.4343	-5.6523	-5.5844
			50	10	0.202	0.1858	-39.2018	-39.1681
				50	0.2096	0.1929	-39.5996	-39.5675
				500	0.2115	0.1946	-39.6909	-39.6592
			500	10	0.0603	0.0588	-464.772	-464.761
				50	0.0626	0.061	-466.075	-466.064
				500	0.0632	0.0616	-466.374	-466.364
		5	10	10	1	0.909	-0.7834	-0.5461
				50	1	0.944	-1.0366	-0.8923
				500	1	0.9525	-1.0935	-0.9714
			50	10	0.4972	0.4074	-27.4671	-27.1271
				50	0.5159	0.4231	-28.2925	-27.9676
				500	0.5204	0.4269	-28.4814	-28.1602
			500	10	0.1363	0.1289	-423.856	-423.742
				50	0.1416	0.1339	-426.666	-426.558
				500	0.1429	0.135	-427.312	-427.205

Table 6

α	β	a_1	b	r	y^*	$\hat{y}$	$S(y^* \mid r)$	$S(\hat{y} \mid r)$
0.05	0.1	0.1	10	10	0.1612	0.149	-8.1209	-8.1155
				50	0.1674	0.1547	-8.1895	-8.1844
				500	0.1687	0.1561	-8.2053	-8.2003
			50	10	0.0691	0.0668	-45.6414	-45.6389
				50	0.0717	0.0693	-45.8018	-45.7995
				500	0.0724	0.07	-45.8387	-45.8364
			500	10	0.0213	0.0211	-485.933	-485.932
				50	0.0222	0.0219	-486.453	-486.452
				500	0.0223	0.0221	-486.573	-486.572
		1	10	10	0.4959	0.3941	-5.2706	-5.1781
				50	0.5136	0.4093	-5.4406	-5.353
				500	0.5179	0.4129	-5.4794	-5.3931
			50	10	0.1942	0.1766	-38.7051	-38.6607
				50	0.2013	0.1834	-39.118	-39.0761
				500	0.2032	0.1851	-39.2127	-39.1714
			500	10	0.0575	0.0559	-463.016	-463.002
				50	0.0597	0.058	-464.381	-464.367
				500	0.0602	0.0586	-464.695	-464.681
		5	10	10	1	0.8643	-0.5722	-0.188
				50	1	0.8976	-0.8258	-0.5409
				500	1	0.9056	-0.8828	-0.6215
			50	10	0.4852	0.3874	-26.5856	-26.1475
				50	0.5027	0.4023	-27.4284	-27.0137
				500	0.5068	0.4059	-27.6209	-27.212
			500	10	0.1306	0.1226	-420.221	-420.07
				50	0.1357	0.1273	-423.15	-423.008
				500	0.1369	0.1284	-423.823	-423.683

Table 7

α	β	a_1	b	r	y^*	$\hat{y}$	$S(y^* \mid r)$	$S(\hat{y} \mid r)$
0.1	0.01	0.1	10	10	0.2019	0.1878	-8.0506	-8.0457
				50	0.2098	0.1951	-8.1227	-8.118
				500	0.2116	0.1968	-8.1393	-8.1347
			50	10	0.0871	0.0843	-45.437	-45.4347
				50	0.0905	0.0876	-45.6059	-45.6038
				500	0.0911	0.0884	-45.6449	-45.6427
			500	10	0.0269	0.0267	-485.196	-485.195
				50	0.028	0.0277	-485.744	-485.744
				500	0.0282	0.028	-485.871	-485.87
		1	10	10	0.5311	0.4405	-5.5284	-5.467
				50	0.5518	0.4575	-5.6944	-5.6352
				500	0.5566	0.4616	-5.7325	-5.6738
			50	10	0.2136	0.1978	-39.3908	-39.3627
				50	0.2218	0.2054	-39.7842	-39.7571
				500	0.2237	0.2073	-39.8746	-39.8477
			500	10	0.0641	0.0626	-465.37	-465.361
				50	0.0667	0.065	-466.653	-466.644
				500	0.0672	0.0656	-466.949	-466.94
		5	10	10	1	0.9486	-0.93	-0.803
				50	1	0.9851	-1.1816	-1.1451
				500	1	0.9939	-1.2383	-1.2233
			50	10	0.5098	0.4259	-28.0848	-27.8092
				50	0.5295	0.4423	-28.899	-28.633
				500	0.5342	0.4463	-29.0855	-28.822
			500	10	0.1419	0.1348	-426.32	-426.231
				50	0.1473	0.14	-429.052	-428.966
				500	0.1486	0.1413	-429.68	-429.595

Table 8

α	β	a_1	b	r	y^*	$\hat{y}$	$S(y^* \mid r)$	$S(\hat{y} \mid r)$
0.1	0.05	0.1	10	10	0.1956	0.1804	-7.9797	-7.9735
				50	0.2032	0.1874	-8.0539	-8.048
				500	0.2049	0.1891	-8.071	-8.0651
			50	10	0.0839	0.081	-45.2598	-45.2569
				50	0.0871	0.0842	-45.4347	-45.432
				500	0.0881	0.0849	-45.475	-45.4723
			500	10	0.0259	0.0256	-484.601	-484.6
				50	0.0269	0.0266	-485.171	-485.17
				500	0.0272	0.0269	-485.302	-485.301
		1	10	10	0.5211	0.4231	-5.3922	-5.3154
				50	0.5407	0.4394	-5.5607	-5.4874
				500	0.5451	0.4434	-5.5993	-5.5268
			50	10	0.2069	0.19	-39.0066	-38.9706
				50	0.2149	0.1973	-39.4114	-39.377
				500	0.2168	0.1991	-39.5044	-39.4704
			500	10	0.0617	0.0602	-464.006	-463.994
				50	0.0641	0.0625	-465.337	-465.325
				500	0.0648	0.063	-465.643	-465.632
		5	10	10	1	0.9111	-0.7666	-0.5321
				50	1	0.9462	-1.019	-0.8788
				500	1	0.9547	-1.0758	-0.958
			50	10	0.5	0.4092	-27.409	-27.0642
				50	0.5189	0.4249	-28.2364	-27.9069
				500	0.523	0.4287	-28.4257	-28.1
			500	10	0.1371	0.1295	-423.542	-423.426
				50	0.1423	0.1345	-426.363	-426.253
				500	0.1436	0.1357	-427.012	-426.903

Table 9

α	β	a_1	b	r	y^*	$\hat{y}$	$S(y^* \mid r)$	$S(\hat{y} \mid r)$
	0.1	0.1	10	10	0.188	0.1715	-7.887	-7.8787
				50	0.195	0.1781	-7.9638	-7.9561
				500	0.1969	0.1797	-7.9815	-7.9738
			50	10	0.0802	0.0771	-45.0273	-45.0235
				50	0.0832	0.08	-45.21	-45.2064
				500	0.0839	0.0807	-45.2521	-45.2485
			500	10	0.0247	0.0244	-483.819	-483.818
				50	0.0256	0.0253	-484.417	-484.416
				500	0.0259	0.0256	-484.554	-484.553
		1	10	10	0.5089	0.4023	-5.2145	-5.1158
				50	0.5273	0.4178	-5.3863	-5.2929
				500	0.5316	0.4215	-5.4255	-5.3334
			50	10	0.1991	0.1807	-38.5027	-38.4552
				50	0.2066	0.1877	-38.9226	-38.8777
				500	0.2083	0.1893	-39.0189	-38.9748
			500	10	0.0589	0.0572	-462.214	-462.198
				50	0.0611	0.0594	-463.608	-463.593
				500	0.0618	0.0599	-463.928	-463.914
		5	10	10	1	0.8661	-0.556	-0.1728
				50	1	0.8995	-0.8088	-0.5263
				500	1	0.9076	-0.8657	-0.6071
			50	10	0.4876	0.3891	-26.5267	-26.0825
				50	0.5054	0.4041	-27.3713	-26.9509
				500	0.5097	0.4077	-27.5642	-27.1497
			500	10	0.1314	0.1232	-419.894	-419.74
				50	0.1363	0.1279	-422.834	-422.69
				500	0.1376	0.129	-423.51	-423.367

From the analysis of Tables 1 – 9 we arrive at the following conclusions

a) The approximately optimal inspection interval $\hat{y}$ is always shorter than the optimal inspection interval y^* for all considered values of α, β, a_1, b, and r.
b) The smaller is the values of the inspection costs a_1 the better is the approximation.
c) The value of r almost has no effect on the accuracy of the approximation.
d) The benefit per cycle b has a dominant effect on the accuracy of the approximation procedure. The larger is the value of b the better is the approximation.
e) Probabilities of false decisions α and β have minor effect on the accuracy of the approximation. However, by increasing their values we observe a slight improvement of the approximation.

Further analysis of Tables 1 – 9 reveals that from a practical point of view there is no difference between the approximate and the exact values of standardized inspection interval. Moreover, even if such a difference exists then the difference between the corresponding losses is negligible.

The existence of the closed formula for the approximately optimal inspection interval let us formulate some practical observations.

a) Longer inspection intervals correspond to smaller expected shifts of the process mean.
b) Any change in the inspection cost produces a change in the same direction for the optimal inspection interval.
c) The benefit from the inspection affects the interval between inspections in such a way that any change of the benefit b results in the change of the optimal inspection interval in the opposite direction.
d) Any change in the probability of false alarms α produces the change of the optimal inspection interval in the same direction.

e) Changes in β produce changes in the optimal inspection interval in the opposite direction.

f) Increase of the cost of a false alarm results in a decrease of the inspection interval.

g) Changing the renewal cost changes the interval between inspections in the same direction.

h) Small values of the ratio of the production run length to the length of the in-control period, *r*, have minor effect on the optimal inspection interval.

i) Changes of the mean number of occurrences of the assignable cause in a time unit changes the inspection interval in the same direction. The same conclusion holds for the production run length.

j) Changing the production rate changes the inspection interval in the opposite direction.

In the considered model we have assumed that the time between consecutive disorders of the process is described by the exponentially distributed random variable. A possible generalization of the model can be obtained using the approach proposed by Hryniewicz (1992). Another generalization can be obtained when we assume that the search for the assignable cause may not be perfect, as it was proposed in Hryniewicz (1996). When the inspection procedures, e.g. particular control charts, are specified there is also a possibility to look for the optimal values of their parameters.

References

Bather, J. A. (1963) "Control Charts and the Minimization of Costs", JRSS, Ser.B, vol.25, pp. 49-80.

Blackwell, M.T.R. (1977) "The Effect of Short Production Runs on CSP-1", Technometrics, vol. 19, no. 3 , pp. 259-263.

Chiu, W.K., and Wetherill, G.B. (1974) " A Simplified Scheme for the Economic Design of $\overline{\mathrm{X}}$-Charts ", Journal of Quality Technology, vol.6, no.2, pp.63-69.

von Collani, E. (1986) "A Simple Procedure to Determine the Economic Design of an $\overline{\mathrm{X}}$ Control Chart " Journal of Quality Technology, vol.18, no.3, pp.145-151.

von Collani, E. (1989) "The Economic Design of Control Charts", B. G. Teubner Stuttgart.

Cox, D. R. (1962) " Renewal Theory", John Wiley N.Y.

Crowder, S. V. (1992) " An SPC Model for Short Production Runs: Minimizing Expected Cost ", Technometrics, vol. 34, no. 1, pp. 63-73.

Del Castillo, E., and Montgomery, D. C. (1992) " Optimal Design of Control Charts of Finite Length Production Runs" Technical paper, Publication No. 92-10.Arizona State University.

Del Castillo, E., and Montgomery, D. C. (1994) "A general model for the optimal Design of $\overline{X}$-charts used to control short or long run processes", IT Transactions.

Del Castillo, E., and Montgomery, D. C. (1995) "A Kalman filtering process control with an application in semiconductor short run manufacturing", Quality and Reliability Engineering International, vol. 11, pp.101-105.

Del Castillo, E. et al (1996) "A Review of the Statistical Process Control Techniques for Short Run Manufacturing Systems", Communications in Statistics, vol.25, no.11, pp.2723-2737.

Duncan, A. J. (1956) "The Economic Design of $\overline{X}$-Charts Used to Maintain Current Control of a Process", JASA, vol. 51, pp. 228-242.

Duncan, A. J (1978) "The Economic Design of p-Chart to Maintain Current Control of a Process: Some Numerical Results", Technometrics, vol. 20, no. 3, pp. 235-243.

Gibra, I.N. (1970) " Economically Optimal Determination of the Parameters of $\overline{X}$-Control Chart ", Management Science, vol.17, no.9, pp. 635-646.

Ho, Chuanching, and Case K. E. (1994) " Economic Design of Control Charts: A Literature Review for 1981-1991", Journal of Quality Technology, vol.26, no.1, pp. 39-53.

Hryniewicz, O. (1992) " Approximately Optimal Economic Process Control for a General Class of Control Procedures", Frontiers in Statistical Quality Control IV, pp. 201-215.

Hryniewicz, O. (1996) "Optimal procedures of statistical process control in the presence of erroneous process inspections", Control and Cybernetics, vol.26, No.6, pp.1217-1231.

Keats, J.B., Castillo, E.D., von Collani, E., and Saniga, E.M.(1997) " Economic Modeling for Statistical process Control", Journal of Quality Technology, vol.29, no.2, pp.144-162.

Ladany, S. P. (1973) " Optimal Use of Control Charts for Controlling Current Production", Management Science, vol. 19, no. 7, pp. 763-772.

Lorenzen, T. J. and Vance, L. C. (1986) "The Economic Design of Control Charts: a Unified Approach ", Technometrics, Vol. 28, no. 1, pp. 3-10.

Montgomery, D. C., Heikes, R. G. , and Mance, J. F. (1975) " Economic Design of Fraction Defective Control Charts ", Management Science, vol. 21, no. 11, pp. 1272-1283.

Montgomery, D.C. (1980) " The Economic Design of Control Charts: A Review and Literature Survey", Journal of Quality Technology, Vol.12, no.2, pp.75-87.

Montgomery, D.C. (1982) " Economic Design of an $\overline{X}$ Control Chart", Journal of Quality Technology, vol.14, no.1, pp. 40-43.

Panagos, M. R., Heikes , R. G., and Montgomery , D. C. (1985) "Economic Design of $\overline{X}$ Control Charts for Two Manufacturing Process Models", Naval Research Logistic Quarterly, vol. 32, pp. 631-646.

Quesenberry, C.P. (1991) "SPC Q Charts for Start-up Processes and Short or Long Runs", Journal of Quality Technology, vol.23, no.3, pp. 213-224.

Saniga, E. M. (1989) "Economic Statistical Control Charts Designs With an Application to $\overline{X}$ and R Charts", Technometrics, vol. 31, no.3 , pp. 313-320.

Yang, G. L. (1983) "A Renewal Process Approach to Continuous Sampling Plans", Technometrics, vol. 25, no.1, pp. 59-67.