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E. Bednarczuk, A. Jezierska, K. Rutkowski

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 3810100

fax: (+48) (22) 3810105

Kierownik Zakładu zgłaszający pracę:
Prof. dr hab. inż. Antoni Żochowski

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Ewa Bednarczuk, Anna Jezierska, Krzysztof Rutkowski

Abstract

In this paper we provide a deeper insight into our recent primal-dual proximal algorithm with memory for solving operator inclusion problem with maximally monotone operators. We propose block-asynchronous version of the best approximation primal-dual proximal algorithm with memory. We concentrate on particular instances which cover the practical problems arising e.g. in image processing. We consider the standard minimization problem of the a sum of lower semicontinuous convex functions, some of them being composed with linear bounded operators. This problems is known to be solved effectively provided that the proximal operators related to the functions involved take closed form expressions. In particular, we formulate optimality conditions for solving the considered problem under less restrictive regularity conditions. The proposed method is illustrated with image reconstruction problem.

I. Introduction

Optimization problems arising in image processing often take the form of minimization of a finite sum of convex proper lower semicontinuous functions.

Let H_i, G_k, $i = 1, \ldots, M$, $k = 1, \ldots, K$ be real Hilbert spaces. Let $f_i : \ H_i \to \mathbb{R} \cup \{+\infty\}$ and $g_k : \ G_i \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex not necessarily differentiable functions and let $L_{ik} : \ H_i \to G_k$ be bounded linear operators, $i = 1, \ldots, M$, $k = 1, \ldots, K$.

In this article we are interested in solving the following optimization problem

$$\min_{p_1 \in H_1, \ldots, p_M \in H_M} \quad \sum_{i=1}^{M} f_i(p_i) + \sum_{k=1}^{K} g_k \left(\sum_{i=1}^{M} L_{ik} p_i \right). \tag{1}$$

In particular when $M = K = 1$ we get

$$\min_{p \in H} F_P(p) := f(p) + g(Lp) \tag{2}$$

and for $M = 1$, $K = 2$ we get

$$\min_{p \in H} F_P(p) := f(p) + g_1(L_1 p) + g_2(L_2 p). \tag{3}$$

The problem of providing solution procedures for problems (1)-(3) has been addressed in numerous papers, e.g. [9], [10], [11], [12], [13], [14], [15], [21].

In [9] an inertial ADMM algorithm has been proposed for the minimization of the sum of two convex proper l.s.c functions, one of which being composed with a linear operator as well as an algorithm for minimization of the sum of finite number of convex proper l.s.c functions. In [10] an inertial forward-backward-forward primal-dual splitting algorithm was proposed for minimizing a finite number of infimal convolution functions composed with linear operators and a convex differentiable function with lipschitzian gradient. In [11] an inertial Douglas-Rachford algorithm has been proposed for minimizing a finite number of infimal convolution functions composed with linear operators. In [14] and [15] generalization of the algorithm of [9] by incorporating more general linear constraints. In [21] an inertial forward-backward algorithm has been proposed to solve saddle point problem corresponding to the minimization of two functions composed with linear operators. The importance of preconditioning procedures applied to saddle point problems and minimization of the sum of two convex functions has been elucidated in [12], [13] (see also [21]).

The present paper is related to our recent paper [4] in which we proposed a primal-dual proximal best approximation algorithm with memory to solve (1). Now we aim at adapting the idea proposed in [16] into our algorithm with memory. This idea relies on asynchronous and block-wise realization of the Fejérian step.

The organization of the paper is as follows. In section 2 we provide necessary theoretical backgrounds and we define the primal-dual approach to solve (1). In section III we recall projection schemes to solve (1). In particular, Iterative Scheme 3 under Assumption 2 is a best approximation algorithm with memory investigated in [4]. To make Iterative Scheme 3 operational, in section IV we provide explicit formulas for projections onto three halfspaces. In section V we provide an asynchronous version of Iterative Scheme 3. In section VI we discuss the results of numerical experiments.

II. Theoretical background

To construct conjugate dual problems we apply the standard approach via perturbation functions as described e.g. in [5], [7] [8].

Let H_i, G_k, $i = 1, \dots, M$, $k = 1, \dots, K$ be real Hilbert spaces. Let $p = (p_1, \dots, p_M) \in \bigoplus_{i=1}^M H_i$. Let $E := \bigoplus_{i=1}^M H_i \times \bigoplus_{k=1}^K G_k$ and let $A: \bigoplus_{i=1}^M H_i \to \bigoplus_{k=1}^K G_k$ be a bounded linear operator defined as

$$A(p) := \left(\sum_{i=1}^M L_{i1} p_i, \dots, \sum_{i=1}^M L_{iK} p_i \right), \tag{4}$$

and let $\tilde{f}: \bigoplus_{i=1}^M H_i \to \mathbb{R} \cup \{+\infty\}$, $\tilde{g}: \bigoplus_{k=1}^K G_k \to \mathbb{R} \cup \{+\infty\}$ be defined as

$$\tilde{f}(p) := \sum_{i=1}^M f_i(p_i), \quad \tilde{g}(v_1, \dots, v_K) := \sum_{k=1}^K g_k(v_k),$$

where $v_k \in G_k$ for $k = 1, \dots, K$.

With this notation the problem (1) is equivalent to

$$\min_{p \in \bigoplus_{i=1}^M H_i} \tilde{f}(p) + (\tilde{g} \circ A)(p). \tag{5}$$

The perturbation function $\Phi: E \to \mathbb{R} \cup \{+\infty\}$ related to problem (5) is

$$\Phi(p, v) := \tilde{f}(p) + \tilde{g}(Ap + v),$$

where $v = (v_1, \dots, v_K) \in \bigoplus_{k=1}^K G_k$. The conjugate $\Phi^*: E \to \mathbb{R} \cup \{+\infty\}$ is

$$\begin{aligned} \Phi^*(p^*, v^*) &= \sup_{(p,v) \in E} (\langle p^* \mid p \rangle + \langle v^* \mid v \rangle - \tilde{f}(p) - \tilde{g}(Ap + v)) \\ &= \tilde{f}^*(p^* - A^* v^*) + \tilde{g}^*(v^*), \end{aligned}$$

where $p^* = (p_1^*, \dots, p_M^*)$, $v^* = (v_1^*, \dots, v_K^*)$, where $v_k^* \in G_k$ for $k = 1, \dots, K$. Then the dual to (5) is

$$\max_{v^* \in \bigoplus_{k=1}^K G_k} -\Phi^*(0, v^*), \tag{6}$$

and the dual to (1) is

$$\min_{v \in \bigoplus_{k=1}^K G_k} \sum_{i=1}^M f_i^* \left(-\sum_{k=1}^K L_{ki}^* v_k^* \right) + \sum_{k=1}^K g_k^*(v_k^*), \tag{7}$$

Where $L_{ki}^*: G_k \to H_i$ is the Hermitian conjugate of L_{ik}. To get basic duality relations for problems (1) and (7) a number of different regularity conditions can be used (see Chapter I of [7]). As shown in [[7], page 14] one of the weakest regularity conditions is based on the strong quasi-relative interior of a set and has the form

$$0 \in \text{sqri}(\text{dom}\tilde{g} - A\text{dom}\tilde{f}) = \text{sqri}(\prod_{k=1}^K \text{dom}\, g_k - A \prod_{i=1}^M \text{dom}\, f_i), \tag{8}$$

where for any function $h: X \to \mathbb{R} \cup \{+\infty\}$ $\text{dom}h := \{x \in X \mid f(x) < +\infty\}$ and the strong quasi-relative interior of a set S is defined as

$$\text{sqri}\, S := \{x \in S \mid \bigcup_{\lambda > 0} \lambda(S - x) \text{ is a closed linear subspace of } H\}.$$

The concept of strong quasi-relative interior was introduced independently by [6], [19], [27], [28].

With the regularity condition (8) we get the following duality relation.

Theorem II.1 *([28, Corollary 2.8.5], [8]) Suppose that the regularity condition* (8) *holds. Then the following are equivalent:*

1) (1) *and* (7) *is solvable*

2) set

$$Z := \Big\{ (p_1, \ldots, p_M, v_1^*, \ldots, v_K^*) \in E \mid -\sum_{k=1}^{K} L_{ki}^* v_k^* \in \partial f_i(p_i), \sum_{i=1}^{M} L_{ik} p_i \in \partial g_k^*(v_k^*), \quad i = 1 \ldots, M, \ k = 1, \ldots, K \Big\} \tag{9}$$

is nonempty.

In the sequel we concentrate on finding an element from the set Z. For convenience of the reader we close this section by recalling the concept of proximity operator which is one of our main tools.

Definition II.2 *For any $x \in H$ and any proper convex and lower semi-continuous function $f : H \to \mathbb{R} \cup \{+\infty\}$ the proximity operator $prox_f(x)$ is defined as the unique solution to the optimization problem*

$$\min_{y \in H} \left(f(y) + \frac{1}{2} \|x - y\|^2 \right).$$

Theorem II.3 *[3, Example 23.3] Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semi-continuous function, $x \in H$ and $\gamma > 0$. Then*

$$J_{\gamma \partial f}(x) = prox_{\gamma f}(x),$$

where $J_A(x) = (Id - A)^{-1}(x)$.

III. Best approximation primal-dual splitting algorithm with memory

In this section we formulate best approximation primal-dual splitting algorithm with memory for problem (1). Let $p = (p_1, \ldots, p_M) \in \prod_{i=1}^{M} H_i$, $v^* = (v_1^*, \ldots, v_K^*) \in \prod_{k=1}^{K} G_k$. Let $A : \ H_1 \times \cdots \times H_M \to G_1 \times \cdots \times G_K$ be the bounded linear operator defined by (4).

For any $a_n = (a_{1,n}, \dots, a_{M,n}) \in \prod_{i=1}^{M} H_i$, $b_n = (b_{1,n}, \dots, b_{K,n}) \in \prod_{k=1}^{K} G_k$ let $\mathcal{H}_n := \{(p, v^*) \in E \mid \varphi_n(p, v^*) \leq 0\}$, where for any $n \in \mathbb{N}$

$$\varphi_n(p, v^*) := \langle p - a_n \mid a_n^* - A^* v^* \rangle + \langle Ap - b_n \mid b_n^* - v^* \rangle,$$

$$a_n^* = (a_{1,n}^*, \dots, a_{M,n}^*) \in \prod_{i=1}^{M} \partial f_i(a_{i,n}),$$

$$b_n^* = (b_{1,n}^*, \dots, b_{K,n}^*) \in \prod_{k=1}^{K} \partial g_k(b_{k,n}).$$

Let $x_n := (p_{1,n}, \dots, p_{M,n}, v_{1,n}^*, \dots, v_K^*)$, $p_n := (p_{1,n}, \dots, p_{M,n})$ and $v_n^* := (v_{1,n}^*, \dots, v_K^*)$ for any $n \in \mathbb{N}$. The starting point of our consideration is the classical projective splitting scheme.

Algorithm 1 Generic primal-dual projection splitting Iterative Scheme

Choose an initial point $x_0 \in E$

Choose a sequence of parameters $\{\lambda_n\}_{n \geq 0} \in (0, 2)$

for $n = 0, 1 \dots$ **do**

  $x_{n+1} = x_n + \lambda_n (P_{\mathcal{H}_n}(x_n) - x_n)$

end for

It was shown in [18], that in case when $M = 1$, $f_1 = 0$ and $L_{1k} = Id$, $k = 1, \dots, K$, Iterative Scheme 1 has the following properties.

Proposition III.1 *([18, Proposition 3.1]) Any sequence $\{x_n\}_{n \in \mathbb{N}}$ generated by Iterative Scheme 1 behaves as follows.*

1) *For any $\bar{x} \in Z$, the sequence $\{\|x_n - \bar{x}\|\}_{n \in \mathbb{N}}$ is nonincreasing — that is $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone with respect to Z.*
2) *If $x_{n_0} \in Z$ for some $n_0 \geq 0$, then $x_n = x_{n_0}$ for all $n \geq n_0$.*
3) *If $\{x_n\}_{n \in \mathbb{N}}$ has a strong accumulation point in S, then the whole sequence converges to that point.*
4) *If Z is nonempty, then $\{x_n\}$ is bounded. Moreover, if there exists $\underline{\lambda}, \bar{\lambda}$ such that $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < 2$ for all n, then*

$$\sum_{n=0}^{+\infty} \|x_{n+1} - x_n\|^2 < +\infty$$

In the general case of problem (1) the following properties of Iterative Scheme 1 has been shown in [16] (see also [1]).

Proposition III.2 *([16, Proposition 4], see also [1, Proposition 3.2]) Suppose that $Z \neq \emptyset$. Then sequence x_n generated by Iterative Scheme 1 has the following properties*

1) $\{x_n\}_{n\in\mathbb{N}}$ *is Fejér monotone with respect to* Z.
2) $\sum_{n=1}^{+\infty} \lambda_n(2-\lambda_n)\|x_{n+1}-x_n\|^2 < +\infty$
3) *Suppose that for every* $p \in H$, *every* $v^* \in G$ *and every strictly increasing sequence* $\{k_n\}_{n\in\mathbb{N}} \in \mathbb{N}$,

$$[p_{k_n} \rightharpoonup p, \quad v^*_{k_n} \rightharpoonup v^*] \implies (p, v^*) \in Z$$

Then $x_n \rightharpoonup \bar{x} = (\bar{p}, \bar{v}^*) \in Z$.

4) *Suppose that the sequences* $\{a_n\}_{n\in\mathbb{N}}$, $\{a^*_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, $\{b^*_n\}_{n\in\mathbb{N}}$ *are bounded. Then*

$$\overline{\lim}\, (\langle p_n - a_n \mid a^*_n + A^*v^*_n\rangle + \langle Ap_n - b_n \mid b^*_n - v^*_n\rangle \leq 0.$$

We consider the following choices of $(a_{i,n}, a^*_{i,n}) \in \operatorname{gra}\partial f_i$, $i = 1, \dots, M$, $(b_{k,n}, b^*_{k,n}) \in \operatorname{gra}\partial g_k$, $k = 1, \dots, K$, $n \in \mathbb{N}$, where for any set-valued mapping B, gra$B$ denotes the graph of B.

Assumption 1

1) *In case when* $M = 1$, $f := f_1 = 0$, $L_{1,k} = Id$ *for* $k = 1, \dots, K$

$$b_{k,n} := prox_{\mu_{k,n} g_k}\left((1 - \sum_{j=0}^{k-1} \alpha_{k,j,n})p_n + \alpha_{k,0,n}a_n + \sum_{j=1}^{k-1} \alpha_{k,j,n}b_{j,n} + \mu_{k,n}v^*_{k,n}\right)$$

$$a_n := prox_{\gamma_n f}(p_n - \gamma_n \sum_{k=1}^{K} v^*_{k,n}),$$

where $\mu_{k,n}$, γ_n, $\alpha_{k,j,n} \in \mathbb{R}$, $0 \leq j < k \leq K$, $n \in \mathbb{N}$ *satisfy some additional requirements (see Algorithm 3 of [18]).*

2) *In the general case of problem* (1)

$$a_{i,n} := prox_{\gamma_n f_i}(p_{i,n} - \gamma_n \textstyle\sum_{k=1}^{K} L^*_{ik}v^*_{k,n}) \qquad i = 1, \dots, M$$

$$b_{k,n} := prox_{\mu_n g_k}(\sum_{i=1}^{M} L_{ki}p_{i,n} + \mu_n v^*_{k,n}) \qquad k = 1, \dots, K,$$

where γ_n, $\mu_n \in [\epsilon, 1/\epsilon]$ *for* $\epsilon > 0$, $n \in \mathbb{N}$.

Observe that if $M = 1$, then $a_n := a_{1,n}$ for all $n \in \mathbb{N}$. The following Proposition is proved in [1] (see also [18]).

Proposition III.3 *With* $\{a_{i,n}\}_{n\in\mathbb{N}}$, $\{b_{k,n}\}_{n\in\mathbb{N}}$ *satisfying requirement 1 or 2 of Assumption 1 the sequence* $\{x\}_{n\in\mathbb{N}}$ *generated by Iterative Scheme 1 behaves as follows.*

1) $\sum_{n=1}^{+\infty} \|p_{i,n+1} - p_{i,n}\|^2 < +\infty$ *for all* $i = 1, \dots, M$ *and* $\sum_{n=1}^{+\infty} \|v^*_{k,n+1} - v_{k,n}\|^2 < +\infty$ *for all* $k = 1, \dots, K$.
2) $\sum_{n=1}^{+\infty} \|p_{i,n} - a_{i,n}\|^2 < +\infty$ *for all* $i = 1, \dots, M$ *and* $\sum_{n=1}^{+\infty} \|v^*_{k,n} - b_{k,n}\|^2 < +\infty$ *for all* $k = 1, \dots, K$.
3) $p_{i,n} \rightharpoonup \bar{p}_i$ *for all* $i = 1, \dots, M$ *and* $v^*_{k,n} \rightharpoonup \bar{v}^*_k$ *for all* $k = 1, \dots, K$, $\bar{x} := (\bar{p}_1, \dots, \bar{p}_M, \bar{v}^*_1, \dots, \bar{v}^*_K) \in Z$.

For any $x, y \in E$ let us define $H(x, y) := \{h \in E \mid \langle h - y \mid x - y\rangle \leq 0\}$. The following strongly convergent projection method relies on the idea which goes back to Haugazeau [3], see also [20], [25], [26].

Algorithm 2 Abstract Haugazeau Algorithm

Choose an initial point $x_0 = (p_0, v_0^*)$

for $n = 0, 1 \ldots$ **do**

$x_{n+1/2} = x_n + \lambda_n(P_{\mathcal{H}_n}(x_n) - x_n)$

$x_{n+1} = P_{H(x_0,x_n)\cap C_n}(x_0)$, where $Z \subset C_n$

end for

Proposition III.4 *([16, Proposition 6], see also [2, Proposition 2.1]) Suppose $Z \neq \emptyset$. Let $\varepsilon > 0$ and $\lambda_n \in (\varepsilon, 1]$ for $n \in \mathbb{N}$. For any sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Iterative Scheme 2 the following hold.*

1) *$\{p_n\}_{n\in\mathbb{N}}$ and $\{v_n^*\}_{n\in\mathbb{N}}$ are bounded.*
2) *$\sum_{n=0}^{+\infty} \|p_{n+1} - p_n\|^2 < +\infty$ and $\sum_{n=0}^{+\infty} \|v_{n+1}^* - x_n^*\|^2 < +\infty$.*
3) *$\sum_{n=0}^{+\infty} \|p_{n+1/2} - p_n\|^2 < +\infty$ and $\sum_{n=0}^{+\infty} \|v_{n+1/2}^* - x_n^*\|^2 < +\infty$.*
4) *Suppose that the sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{a_n^*\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, $\{b_n^*\}_{n\in\mathbb{N}}$ are bounded. Then*

$$\overline{\lim}\, (\langle p_n - a_n \mid a_n^* + A^* v_n^*\rangle + \langle Ap_n - b_n \mid b_n^* - v_n^*\rangle \leq 0.$$

5) *Suppose that, for every $(p, v^*) \in Z$ and every strictly increasing sequence $\{q_n\}_{n\in\mathbb{N}}$ in $\mathbb{N}$,*

$$[p_{q_n} \rightharpoonup p \quad \text{and} \quad v_{q_n}^* \rightharpoonup v^*] \implies (p, v^*) \in Z.$$

Then $\{p_n\}_{n\in\mathbb{N}}$ converges strongly to $\bar{p}$ and $\{v_n^*\}_{n\in\mathbb{N}}$ converges strongly to $\bar{v}$ and $(\bar{p}, \bar{v}^*) \in Z$

In [2] the following Iterative Scheme 3 has been proposed with $C_n = H(x_n, x_{n+1/2})$.

Algorithm 3 Proximal primal-dual best approximation iterative scheme for finite number of functions

Choose an initial point $x_0 = (p_0, v_0^*) \in E$ and $\varepsilon > 0$,

$p_0 = (p_{1,0}, \dots, p_{M,0})$, $v_0^* = (v_{1,0}^*, \dots, v_{K,0}^*)$

Choose sequences of parameters $\{\lambda_n\}_{n\geq 0} \in (0,1]$ and $\{\gamma_n\}_{n\geq 0}, \{\mu_n\}_{n\geq 0} \in [\varepsilon, 1/\varepsilon]$

for $n = 0, 1 \dots$ **do**

Fejerian step

for $i = 1, \dots, M$ **do**

$a_{i,n} = \text{prox}_{\gamma_n f_i}(p_{i,n} - \gamma_n \sum_{k=1}^K L_{ki}^* v_{k,n}^*)$

$a_{i,n}^* = \gamma_n^{-1}(p_{i,n} - a_{i,n}) - \sum_{k=1}^K L_{ki}^* v_{k,n}^*$

end for

for $k = 1, \dots, K$ **do**

$b_{k,n} = \text{prox}_{\mu_n g_k}(\sum_{i=1}^M L_{ik} p_{i,n} + \mu_n v_{k,n}^*)$

$b_{k,n}^* = \mu_n^{-1}(\sum_{i=1}^M L_{ik} p_{i,n} - b_{k,n}) + v_{k,n}^*$

$s_{M+k,n}^* = b_{k,n} - \sum_{i=1}^M L_{ik} a_{i,n}$

end for

for $i = 1, \dots, M$ **do**

$s_{i,n}^* = a_{i,n}^* + \sum_{k=1}^K L_{ki}^* b_{k,n}^*$

end for

$s_n^* = (s_{1,n}^*, \dots, s_{M,n}^*, s_{M+1,n}^*, \dots, s_{M+K,n}^*)$

$\eta_n = \sum_{i=1}^M \langle a_{i,n} \mid a_{i,n}^* \rangle + \sum_{k=1}^K \langle b_{k,n} \mid b_{k,n}^* \rangle$

$H_n = \{h \in E \mid \langle h \mid s_n^* \rangle \leq \eta_n\}$

if $\|s_n^*\| = 0$ **then**

$\bar{p} = p_n$, $\bar{v}^* = v_n^*$

Terminate

else

$x_{n+1/2} = x_n + \lambda_n(P_{H_n}(x_n) - x_n)$

Haugazeau step

$x_{n+1} = P_{H(x_0; x_n) \cap C_n}(x_0)$

end if

end for

Proposition III.5 *[2, Propoistion 4.2] Let* $x_n = (p_n, v_n^*) = (p_{1,n}, \dots, p_{M,n}, v_{1,n}^*, \dots, v_{K,n}^*) \in E$*, let* $C_n := H(x_n; x_{n+1/2})$ *for all* $n \in \mathbb{N}$*. The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *generated by Iterative Scheme 3 has the* *following properties*

1) $\sum_{n=1}^{+\infty} \|p_{i,n+1} - p_{i,n}\|^2 < +\infty$ *for all* $i = 1, \ldots, M$ *and* $\sum_{n=1}^{+\infty} \|v^*_{k,n+1} - v^*_{k,n}\|^2 < +\infty$ *for all* $k = 1, \ldots, K$.
2) $\sum_{n=1}^{+\infty} \|p_{i,n} - a_{i,n}\|^2 < +\infty$ *for all* $i = 1, \ldots, M$ *and* $\sum_{n=1}^{+\infty} \| \sum_{i=1}^{M} L_{ik} p_{i,n} - b_{k,n}\|^2 < +\infty$ *for all* $k = 1, \ldots, K$.
3) $p_{i,n} \to \bar{p}_i$ *for all* $i = 1, \ldots, M$ *and* $v^*_{k,n} \to \bar{v}^*_k$ *for all* $k = 1, \ldots, K$, $\bar{x} := (\bar{p}_1, \ldots, \bar{p}_M, \bar{v}^*_1, \ldots, \bar{v}^*_K) = P_Z(x_0) \in Z$.

In [4] we proposed the Iterative Scheme 3 with the following choices of C_n.

Assumption 2 *Let* C_n *be given as one of the following*

1) $C_n := H(x_n, x_{n+1/2}) \cap H(x_{n-1}, x_{n-1/2})$ *for* $n \geq 1$ *and* $C_0 = H(x_0, x_{1/2})$,
2) $C_n := H(x_n, x_{n+1/2}) \cap H(x_0, x_{n-1})$ *for* $n \geq 1$ *and* $C_0 = H(x_0, x_{1/2})$,
3) $C_n := H(x_n, x_{n+1/2}) \cap H(x_0, \tau_n x_n + (1 - \tau_n) x_{n-1}))$ *for* $\tau_n \in (0, 1), n \geq 1$ *and* $C_0 = H(x_0, x_{1/2})$.

Under these choices of C_n it was shown in [4] that sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Iterative Scheme 3 has properties given in Proposition III.5. Let us note that the Haugazeau step of Iterative Scheme 3 with C_n defined as in Assumption 2 requires projection onto the intersection of three halfspaces.

IV. Projections onto the intersection of three halfspaces

In this section we consider the problem of finding projection onto the intersection of three halfspaces. As mentioned above this problem is in the core of Haugazeau step of Iterative Scheme 3.

Let us consider the general situation, i.e. let $u_i \in H$, $u_i \neq 0$ be elements of a Hilbert space H, $\eta_i \in \mathbb{R}$ and $C_i = \{h \in H \mid \langle h \mid u_i \rangle \leq \eta_i\}$ for $i \in K := \{1, 2, 3\}$. Finding the projection of element $x \in H$ onto $C := C_1 \cap C_2 \cap C_3$ is equivalent to solving the following optimization problem

$$\min_{\substack{h \in H \\ \langle h \mid u_1 \rangle \leq \eta_1 \\ \langle h \mid u_2 \rangle \leq \eta_2 \\ \langle h \mid u_3 \rangle \leq \eta_3}} \frac{1}{2}\|h - x\|^2. \tag{10}$$

Let

$$G := \begin{bmatrix} \|u_1\|^2 & \langle u_1 \mid u_2 \rangle & \langle u_1 \mid u_3 \rangle \\ \langle u_2 \mid u_1 \rangle & \|u_2\|^2 & \langle u_2 \mid u_3 \rangle \\ \langle u_3 \mid u_1 \rangle & \langle u_3 \mid u_2 \rangle & \|u_3\|^2 \end{bmatrix}$$

and for any $I \subset K$, $I \neq \emptyset$, let $G_{I,I}$ be the submatrix of G composed of rows indexed by I and columns indexed by I only.

By Theorem 6.41 of [17] the solution of (10) is of the form

$$\bar{x} = x - \bar{\nu}_1 u_1 - \bar{\nu}_2 u_2 - \bar{\nu}_3 u_3. \tag{11}$$

To find $\bar{\nu} = [\bar{\nu}_i]_{i \in K}$ we propose the algorithm summarized in Iterative Scheme 4.

Algorithm 4 Algorithm for finding $\bar{\nu}$

Let $\mathcal{K}$ be a set of all nonempty subsets of K
while $\mathcal{K} \neq \emptyset$ **do**
  Choose randomly $I \in \mathcal{K}$
  if $\det G_{I,I} \neq 0$ **then**
    Find $\nu = [\nu_i]_{i \in I}$ such that $G_{I,I}\nu = [\langle x \mid u_i \rangle - \eta_i]_{i \in I}$
    if $\nu > 0$ **then**
      if for all $i \in K \backslash I$, $\langle x - \sum_{k \in I} \nu_k u_k \mid u_i \rangle - \eta_i \leq 0$ **then**
        Terminate, $\bar{\nu}_i = \nu_i$ for $i \in I$ and $\bar{\nu}_i = 0$ for $i \in K \backslash I$
      end if
    end if
  end if
  $\mathcal{K} := \mathcal{K} \backslash I$
end while

It was shown in [24] that there exists at least one subset I of K such that:

- $\det G_{I,I} \neq 0$,
- solution of $G_{I,I}\nu = [\langle x \mid u_i \rangle - \eta_i]_{i \in I}$ is positive,
- for all $i \in K \backslash I$, $\langle x - \sum_{k \in I} \nu_k u_k \mid u_i \rangle - \eta_i \leq 0$.

Since there exists only 7 nonempty subsets of K, Iterative Scheme 4 requires at most 7 iterations. Furthermore the number of iterations can be reduced to 4.

V. Asynchronous block-iterative best approximation scheme

Following the idea proposed in [16] in the present section we modify our Iterative Scheme 3 in order to be able to perform calculations in asynchronous and block-wise way. We adopt the following notations and assumptions as used in [16].

Assumption 3

1) *Let $\tilde{M}$ be a strictly positive integer. Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty subsets of $\{1,2,\ldots,M\}$ and $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty subsets of $\{1,2,\ldots,K\}$ such that*

$$I_0 = I,\ K_0 = K,$$
$$(\forall n \in \mathbb{N})\ \left(\bigcup_{j=n}^{n+\tilde{M}-1} I_j = \{1,2\ldots,M\} \text{ and } \bigcup_{j=n}^{n+\tilde{M}-1} K_j = \{1,2\ldots,K\}\right).$$

2) *Let $\tilde{D}$ be a strictly positive integer. For every $i \in 1\ldots,M$ and $k \in 1\ldots,K$ let $\{c_i(n)\}_{n\in\mathbb{N}}$ and $\{d_k(n)\}_{n\in\mathbb{N}}$ be sequences in $\mathbb{N}$ such that*

$$(\forall n \in \mathbb{N}) \quad \begin{array}{l} (\forall i \in \{1,\ldots,M\})\ n - \tilde{D} \le c_i(n) \le n, \\ (\forall k \in \{1,\ldots,K\})\ n - \tilde{D} \le d_k(n) \le n. \end{array}$$

3) *Let $\varepsilon \in (0,1)$ and, for every $i \in \{1,\ldots,M\}$ and every $k \in \{1,\ldots,K\}$, $\{\gamma_{i,n}\}_{n\in\mathbb{N}}$ and $\{\mu_{k,n}\}_{n\in\mathbb{N}}$ be sequences in $[\varepsilon, 1/\varepsilon]$.*

The following proposition proved in [16] provides the properties of the asynchronous and block-wise variant of the calculations of auxiliary data in the Fejérian step.

Proposition V.1 *([16, Proposition 7]) Suppose that $Z \neq \emptyset$ and that the following are satisfied*

1) *For every $i \in \{1,\ldots,M\}$, $\{p_{i,n}\}_{n\in\mathbb{N}}$ is a bounded sequence in H_i and, for every $k \in \{1,\ldots,K\}$, $\{v^*_{k,n}\}_{n\in\mathbb{N}}$ is a bounded sequence in G_k.*
2) *Assumption 3 is in force.*
3) *For every $n \in \mathbb{N}$, set*

for $i \in I_n$ ***do***

$a_{i,n} = prox_{\gamma_{i,c_i(n)} f_i}(p_{i,c_i(n)} - \gamma_{i,c_i(n)} \sum_{k=1}^K L^*_{ki} v^*_{i,c_i(n)})$

$a^*_{i,n} = \gamma^{-1}_{i,c_i(n)}(p_{i,c_i(n)} - a_{i,n}) - \sum_{k=1}^K L^*_{ki} v^*_{i,c_i(n)}$

end for

for $i \in \{1,\ldots,M\}\backslash I_n$ ***do***

$(a_{i,n}, a^*_{i,n}) = (a_{i,n-1}, a^*_{i,n-1})$

end for

for $k \in K_n$ ***do***

$b_{k,n} = prox_{\mu_{k,d_k(n)} g_k}(\sum_{k=1}^K L_{ki} p_{k,d_k(n)} + \mu_{k,d_k(n)} v^*_{k,d_k(n)})$

$b^*_{k,n} = v^*_{k,d_k(n)} + \mu^{-1}_{k,d_k(n)}(\sum_{k=1}^K L_{ki} p_{k,d_k(n)} - b_{k,n})$

end for

for *For every* $k \in \{1,\ldots,K\}\backslash K_n$ ***do***

$(b_{k,n}, b^*_{k,n}) = (b_{k,n-1}, b^*_{k,n-1})$

end for

and define

$$(\forall_{n\in\mathbb{N}}) \quad \begin{aligned} a_n &= \{a_{i,n}\}_{i\in 1,\ldots,m}, \; a_n^* = \{a_{i,n}^*\}_{i\in 1,\ldots,m}, \\ b_n &= \{b_{k,n}\}_{k\in 1,\ldots,K}, \; b_n^* = \{b_{k,n}^*\}_{k\in 1,\ldots,K}. \end{aligned}$$

Then the following hold

1) $(\forall_{n\in\mathbb{N}}\ \forall_{i\in 1,\ldots,M}\ \forall_{k\in 1,\ldots,K})\ a_{i,n}^* \in \partial f_i(a_{i,n})$ *and* $b_{k,n}^* \in \partial g_k(b_{k,n})$.
2) $\{a_n\}_{n\in\mathbb{N}}$, $\{a_n^*\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, $\{b_n^*\}_{n\in\mathbb{N}}$ *are bounded.*
3) *Suppose that the following are satisfied*
   a) $\sum_{n=0}^{+\infty} \|p_{n+1} - p_n\|^2 < +\infty$ *and* $\sum_{n=0}^{+\infty} \|v_{n+1}^* - x_n^*\|^2 < +\infty$.
   b) $\overline{\lim} \left(\langle p_n - a_n \mid a_n^* + A^* v_n^*\rangle + \langle Ap_n - b_n \mid b_n^* - v_n^*\rangle\right) \leq 0$.
   c) $\{q_n\}_{n\in\mathbb{N}}$ *is a strictly increasing sequence in* $\mathbb{N}$, *for every* $i \in \{1,\ldots,M\}$ $p_{i,q_n} \rightharpoonup \bar{p}_i$ *and, for every* $k \in \{1,\ldots,K\}$ $v_{k,q_n}^* \rightharpoonup \bar{v}_k^*$.

   Then

$$p_n - a_n \to 0, \quad a_n^* + A^* v_n^* \to 0, \quad Ap_n - b_n \to 0, \quad v_n^* - b_n^* \to 0,$$

   and $(\bar{p}, \bar{v}^*) \in Z$.

Iterative Scheme 5 is an asynchronous and block-iterative modification of our Iterative Scheme 3 which takes into account modifications of the Fejérian step proposed in [16, Proposition 7(c)].

Algorithm 5 Asynchronous block-wise proximal primal-dual best approximation iterative scheme for finite number of functions with memory

Choose an initial point $x_0 = (p_0, v_0^*) \in \bigoplus_{i=1}^{M} H_i \times \bigoplus_{k=1}^{K} G_k$ and $\varepsilon > 0$,
$p_0 = (p_{1,0}, \dots, p_{M,0})$, $v_0^* = (v_{1,0}^*, \dots, v_{K,0}^*)$
Let $I_n, K_n, c_i(n), d_i(n), \gamma_{i,n}, \mu_{i,n}$ satisfy Assumption 3
Let C_n be defined as in Assumption 2
for $n = 0, 1 \dots$ **do**
  Asynchronous block-wise Fejérian step
  for $i \in I_n$ **do**
    $a_{i,n} = \text{prox}_{\gamma_{i,c_i(n)} f_i}(p_{i,c_i(n)} - \gamma_{i,c_i(n)} \sum_{k=1}^{K} L_{ki}^* v_{k,c_i(n)}^*)$
    $a_{i,n}^* = \gamma_{i,c_i(n)}^{-1}(p_{i,c_i(n)} - a_{i,n}) - \sum_{k=1}^{K} L_{ki}^* v_{k,c_i(n)}^*$
  end for
  for $i \in \{1, \dots, M\} \backslash I_n$ **do**
    $(a_{i,n}, a_{i,n}^*) = (a_{i,n-1}, a_{i,n-1}^*)$
  end for
  for $k \in K_n$ **do**
    $b_{k,n} = \text{prox}_{\mu_{k,d_k(n)} g_k}(\sum_{i=1}^{M} L_{ik} p_{i,d_k(n)} + \mu_{k,d(n)} v_{k,d_k(n)}^*)$
    $b_{k,n}^* = \mu_{k,d_k(n)}^{-1}(\sum_{i=1}^{M} L_{ik} p_{i,d_k(n)} - b_{k,n}) + v_{k,d_k(n)}^*$
  end for
  for $k \in \{1, \dots, K\} \backslash K_n$ **do**
    $(b_{k,n}, b_{k,n}^*) = (b_{k,n-1}, b_{k,n-1}^*)$
  end for
  for $i = 1, \dots, M$ **do**
    $s_{i,n}^* = a_{i,n}^* + \sum_{k=1}^{K} L_{ki}^* b_{k,n}^*$
  end for
  for $k = 1, \dots, K$ **do**
    $s_{M+k,n}^* = b_{k,n} - \sum_{i=1}^{M} L_{ik} a_{i,n}$
  end for
  $s_n^* = (s_{1,n}^*, \dots, s_{M,n}^*, s_{M+1,n}^*, \dots, s_{M+K,n}^*)$
  $\eta_n = \sum_{i=1}^{M} \langle a_{i,n} \mid a_{i,n}^* \rangle + \sum_{k=1}^{K} \langle b_{k,n} \mid b_{k,n}^* \rangle$
  $H_n = \{h \in E \mid \langle h \mid s_n^* \rangle \leq \eta_n\}$

if $\|s_n^*\| = 0$ **then**
$\bar{p} = p_n$, $\bar{v}^* = v_n^*$
Terminate
else
$x_{n+1/2} = x_n + \lambda_n(P_{H_n}(x_n) - x_n)$
Haugazeau step
$x_{n+1} = P_{H(x_0,x_n)\cap C_n}(x_0)$
end if
end for

Remark V.2 *Let us note that Haugazeau step in Iterative Scheme 5 coincides formally with the Haugazeau step in Iterative Scheme 3. However, practical realizations of the Haugazeau steps differ significantly due to the fact that they depend on Fejérian steps which are asynchronous block-wise and synchronous, respectively.*

Proposition V.3 *Suppose that $Z \neq \emptyset$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by Iterative Scheme 5. Define*

$$(\forall_{n\in\mathbb{N}}) \quad \begin{array}{l} a_n = \{a_{i,n}\}_{i\in\{1,\dots,M\}},\ p_n = \{p_{i,n}\}_{i\in\{1,\dots,M\}}, \\ b_n^* = \{b_{k,n}^*\}_{k\in\{1,\dots,K\}},\ v_n^* = \{v_{k,n}^*\}_{k\in\{1,\dots,K\}}. \end{array}$$

Then $\{a_n\}_{n\in\mathbb{N}}$ and $\{p_n\}_{n\in\mathbb{N}}$ converge strongly to $\bar{p} \in \prod_{i=1}^M H_i$, $\{b_n^*\}_{n\in\mathbb{N}}$ and $\{v_n^*\}_{n\in\mathbb{N}}$ converge strongly to $\bar{v}^* \in \prod_{k=1}^K G_k$, and $(\bar{p}, \bar{v}^*) \in Z$.

The proof of Proposition V.3 follows the idea of the proof of Theorem 7 of [16].

Proof. Iterative Scheme 5 is an instance of Abstract Haugazeau Algorithm from Iterative Scheme 2, hence for all $i \in \{1,\dots,M\}$ and all $k \in \{1,\dots,K\}$ the sequences $\{p_{i,n}\}_{n\in\mathbb{N}}$, $\{v_{k,n}^*\}_{n\in\mathbb{N}}$ are bounded. Thus, for the sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{a_n^*\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, $\{b_n^*\}_{n\in\mathbb{N}}$ we can apply Proposition V.1. By Proposition III.4, statements (3a) and (3b) of Proposition V.1 hold. Let $\{q_n\}_{n\in\mathbb{N}}$ be any strictly increasing sequence in $\mathbb{N}$ and for every $i \in \{1,\dots,M\}$, $p_{i,q_n} \rightharpoonup \bar{p}_i$, and for every $k \in \{1,\dots,K\}$, $v_{k,q_n}^* \rightharpoonup \bar{v}_k^*$. Then, by (3c) of Proposition V.1

$$p_n - a_n \to 0, \quad v_n^* - b_n^* \to 0$$

and $(\bar{p}, \bar{v}^*) \in Z$, where $\bar{p} := \{\bar{p}_i\}_{i\in\{1,\dots,M\}}$ and $\bar{v} := \{\bar{v}_k^*\}_{k\in\{1,\dots,K\}}$. Now since for any strictly increasing sequence $\{q_n\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ we have

$$[p_{q_n} \rightharpoonup \bar{p} \quad \text{and} \quad v_{q_n}^* \rightharpoonup \bar{v}^*] \implies (\bar{p}, \bar{v}^*) \in Z$$

and by statement (5) of Proposition III.4, we have $p_n \to \bar{p}$ and $v_n^* \to \bar{v}^*$. ∎

VI. Numerical results

The evaluation experiments concern the image inpainting problem which corresponds to the recovery of an image $\bar{p} \in \mathbb{R}^d$ from lossy observations $y = L_1\bar{p}$, where $L_1 \in \mathbb{R}^{d\times d}$ is a diagonal matrix such that for $i = 1, \dots, d$ we have $L(i,i) = 0$, if the pixel i in the observation image y is lost and $L(i,i) = 1$, otherwise. The considered optimization problem is of the form

$$\min_{p_1,\dots,p_M \in H} \quad \sum_{k=1}^{M} |p_k| + \omega \sum_{l=1}^{M} TV(Dp_l) + \sum_{r=1}^{M} \iota_{y_r}(Dp_r) + \sum_{t=1}^{M} \iota_{\mathcal{S}}(Dp_t)$$

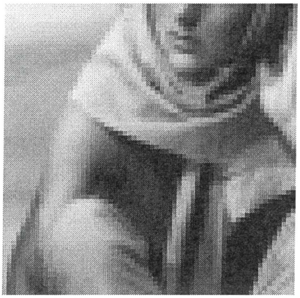
where

1) p_i, $i = 1\dots, M$ are patches of dimension $w^2 \times 1$,
2) ω is a regularization parameter
3) dictionary matrix D contains d normalised vectors (atoms) D_k; stored as columns in $D = (D_1, \dots, D_d) \in \mathbb{R}^{d\times w^2}$,
4) $TV : \mathbb{R}^d \mapsto \mathbb{R}$ is a discrete isotropic total variation functional [23], i.e. for every $p \in \mathbb{R}^d$, $TV(p) = g(L_2 p) := \left(\sum_i^d {}_1([\Delta^{\mathrm{h}} p]_i)^2 + ([\Delta^{\mathrm{v}} p]_i)^2\right)^{1/2}$ with $L_2 \in \mathbb{R}^{2d\times d}, L_2 := \left[(\Delta^{\mathrm{h}})^\top \ (\Delta^{\mathrm{v}})^\top\right]^\top$, where $\Delta^{\mathrm{h}} \in \mathbb{R}^{d\times d}$ (resp. $\Delta^{\mathrm{v}} \in \mathbb{R}^{d\times d}$) corresponds to a horizontal (resp. vertical) gradient operator,
5) ι is the indicator function defined as:
$$\iota_{\mathcal{S}}(p) = \begin{cases} 0 & \text{if } \ p \in \mathcal{S} \\ +\infty & \text{otherwise,} \end{cases}$$
6) the function $\iota_{\mathcal{S}}(p)$ is imposing the solution to belong to the set $\mathcal{S} = [0,1]^d$.

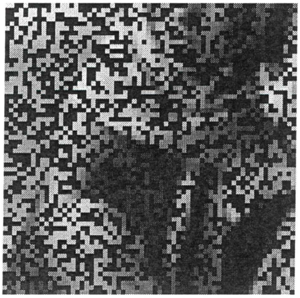
In the following experiments, we consider the cases of lossy observations:

1) with κ randomly chosen pixels which are unknown
2) with unknown pixels given by a structured mask.

Following [22] we use dictionary learning to obtain a dictionary D from a corrupted picture. Then we iterate Algorithm 5 with respect to block-activation of functions $|p_k|$ satisfying Assumption 3 (1).



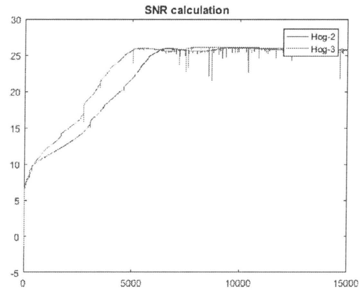
(a) Original



(b) Degraded

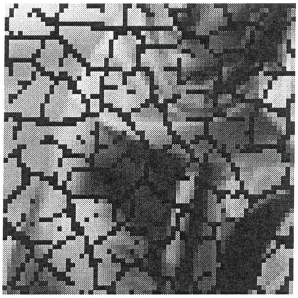
(c) Restored via [16], Alg-2

(d) Restored Ours, Alg-3



(e) SNR

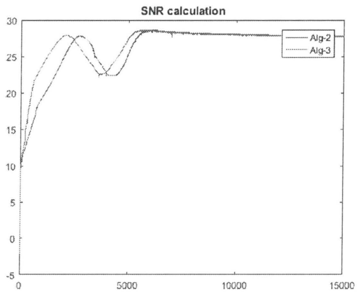
(f) Original



(g) Degraded

(h) Restored via [16], Alg-2

(i) Restored Ours, Alg-3



(j) SNR

VII. Conclusions

In the present paper the numerical experiments are performed only for synchronous version of the algorithm. The future work will concentrate on testing different strategies of block-activations as well as asynchronicity.

References

[1] A. Alotaibi, P. L. Combettes, and N. Shahzad. Solving coupled composite monotone inclusions by successive Fejér approximations of their Kuhn-Tucker set. *SIAM J. Optim.*, 24(4):2076–2095, 2014.

[2] A. Alotaibi, P.L. Combettes, and N. Shahzad. Best approximation from the Kuhn-Tucker set of composite monotone inclusions. *Numerical Functional Analysis and Optimization*, 36(12):1513–1532, 2015.

[3] H.H. Bauschke and P.L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces (CMS Books in Mathematics)*. Springer, 2011.

[4] E. M. Bednarczuk, A. Jezierska, and K. E. Rutkowski. Inertial proximal best approximation primal-dual algorithm. *ArXiv e-prints*, October 2016.

[5] J. Frédéric Bonnans and Alexander Shapiro. *Perturbation analysis of optimization problems*. Springer Series in Operations Research. Springer-Verlag, New York, 2000.

[6] J. Borwein and R. Goebel. Notions of relative interior in Banach spaces. *Journal of Mathematical Sciences*, 115(4):2542–2553, 2003.

[7] R. I. Boţ. *Conjugate duality in convex optimization*, volume 637 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 2010.

[8] R. I. Boţ and E. R. Csetnek. Regularity conditions via generalized interiority notions in convex optimization: New achievements and their relation to some classical statements. *Optimization*, 61(1):35–65, 2012.

[9] R. I. Boţ and E. R. Csetnek. An inertial alternating direction method of multipliers. *Minimax Theory Appl.*, 1(1):29–49, 2016.

[10] R. I. Boţ and E. R. Csetnek. An inertial forward-backward-forward primal-dual splitting algorithm for solving monotone inclusion problems. *Numerical Algorithms*, 71(3):519–540, 2016.

[11] R. I. Boţ, E. R. Csetnek, and C. Hendrich. Inertial Douglas-Rachford splitting for monotone inclusion problems. *Appl. Math. Comput.*, 256:472–487, 2015.

[12] K. Bredies and H. Sun. Preconditioned alternating direction method of multipliers for the minimization of quadratic plus non-smooth convex functionals, 2015.

[13] Kristian Bredies and Hongpeng Sun. Preconditioned Douglas–Rachford splitting methods for convex-concave saddle-point problems. *SIAM Journal on Numerical Analysis*, 53(1):421–444, 2015.

[14] C. Chen, R. H. Chan, S. Ma, and J. Yang. Inertial proximal ADMM for linearly constrained separable convex optimization. *SIAM J. Imaging Sci.*, 8(4):2239–2267, 2015.

[15] C. Chen, S. Ma, and J. Yang. A general inertial proximal point algorithm for mixed variational inequality problem. *SIAM J. Optim.*, 25(4):2120–2142, 2015.

[16] Patrick L. Combettes and Jonathan Eckstein. Asynchronous block-iterative primal-dual decomposition methods for monotone inclusions. *Mathematical Programming*, pages 1–28, 2016.

[17] F. Deutsch. *Best Approximation in Inner Product Spaces*. CMS Books in Mathematics. Springer, 2001.

[18] J. Eckstein and B. F. Svaiter. General projective splitting methods for sums of maximal monotone operators. *SIAM J. Control Optim.*, 48(2):787–811, 2009.

[19] V. Jeyakumar and Henry Wolkowicz. Generalizations of slater's constraint qualification for infinite convex programs. *Mathematical Programming*, 57(1):85–101, 1992.

[20] Guo ji Tang and Nan jing Huang. Strong convergence of a splitting proximal projection method for the sum of two maximal monotone operators. *Operations Research Letters*, 40(5):332 – 336, 2012.

[21] D. A. Lorenz and T. Pock. An inertial forward-backward algorithm for monotone inclusions. *Journal of Mathematical Imaging and Vision*, 51(2):311–325, 2015.

[22] V. Naumova and K. Schnass. Dictionary Learning from Incomplete Data. *ArXiv e-prints*, January 2017.

[23] L I Rudin, S Osher, and E Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena*, 60(1-4):259–268, 1992.

[24] K. E. Rutkowski. Explicit formulas for projections onto intersection of a finite number of halfspaces in Hilbert spaces. *ArXiv e-prints*, June 2016.

[25] M.V. Solodov and B.F. Svaiter. Forcing strong convergence of proximal point iterations in a Hilbert space. *Mathematical Programming*, 87(1):189–202.

[26] Y. J. Wang, N. H. Xiu, and J. Z. Zhang. Modified extragradient method for variational inequalities and verification of solution existence. *Journal of Optimization Theory and Applications*, 119(1):167–183, 2003.

[27] C. Zălinescu. Solvability results for sublinear functions and operators. *Zeitschrift für Operations Research*, 31(3):A79–A101, 1987.

[28] C. Zalinescu. *Convex Analysis in General Vector Spaces.* World Scientific, 2002.