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Optimal Stochastic Model in l_n^1–norm for Option Pricing via Simulations

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Abstract

The paper is devoted to the problem of fitting optimal stochastic process of underlying asset movements in the option pricing. We use martingale theory and Monte Carlo methods to simulate some Levy processes. We argue that presented method may be used for solving the "volatility smile" problem. A real market example of finding an appropriate process is also described.

Keywords: *option pricing, Brownian motion, Poisson process, martingale theory, Monte Carlo methods*

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1 Introduction

The Black-Scholes formula is a very important and one of the best known results in financial mathematics. It enables to calculate the price of a derivative under the assumption that there is not a possibility of arbitrage in the market. General considerations concerning no-arbitrage pricing of derivatives one can find in [2], [4] and [6].

It is assumed in the Black-Scholes model that the stochastic process which describes the price movements of an underlying financial instrument $\mathcal{S}_t$ is the geometrical Brownian motion. The advantage of this assumption is a simplicity of the pricing formula. However, it is commonly known that the Black-Scholes formula does not describe the real "behaviour" of derivatives very well.

There are two main differences between the Black–Scholes model and the real market. The first is non-symmetric distribution of the standardized returns of logarithms of $\mathcal{S}_t$. The second difference, called "volatility smile", consists in the "U" shape of the graph of volatility as the function of striking price, while from the Black–Scholes model it follows that this function should be constant.

Our paper is devoted to improve the pricing model. To simplify our considerations, especially from statistical point of view, we focus our attention on improvement of the model with respect to the second issue. For that purpose we look for a stochastic process fitted to the real prices with respect to the mentioned property of the volatility. As the domain of our explorations we choose the exponential function of a linear combinations of Brownian motion, two independent Poisson processes and the drift. The set of exponents of this type is a natural subclass of Levy processes. It contains non-continuous processes, which may model jumps of the underlying asset. We look for the best model with respect to a suitable l_n^1-norm. The next step is to price the derivative under the assumption of no arbitrage. It requires calculation of the form of the equivalent martingale measure for $\mathcal{S}_t$, which is a complicated problem from mathematical point of view. The counterpart of the Black–Scholes formula is also complicated for computation and therefore in place of analytical calculations we use Monte Carlo methods.

Basic assumptions and theorems are included in Section 2. In particular, this section contains mathematical details of our pricing model. Section 3 is dedicated to description of the fitting method. We also discuss results of an application of our method for the no-arbitrage pricing of a real derivative. Finally we present some conclusions.

2 Stochastic models of underlying asset movements

In this section we present a few models of stochastic processes of underlying asset movements. We also discuss a methodology of using martingale theory and Monte Carlo methods for option pricing.

2.1 Basic assumptions and theorems

The process of movements of the underlying asset (e.g. index, price of a stock) may be modelled by some stochastic process (see e.g. [2], [4], [6]). A classical example of such a process is the geometrical Brownian motion and a well known result is the Black–Scholes formula for the calculation of the price of European–style options (see e.g. [1], [2], [4]).

Geometrical Brownian motion and many other important stochastic processes for modelling movements of the underlying asset are special cases of transformations of Levy processes. In this paper we present an approach suitable for these general processes based on theory of martingales. It uses the local characteristics for Levy processes (see [6]). Let us now introduce some necessary concepts and basic facts, which can be also found in [5] and [6].

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ be a probability space with right–continuous, complete filtration $(\mathcal{F}_t)_{t\in[0,T]}$, where $T < \infty$. The assumption that T is finite is fulfilled in this paper because the financial instruments considered here have only finite life–time intervals.

Def. 1. *A stochastic process $(Y_t)_{t\in[0,T]}$ is called a Levy process if it fulfils the following conditions:*

1. $Y_0 = 0$ *a.s. (almost surely),*
2. $(Y_t)_{t\in[0,T]}$ *has independent increments,*
3. *for all $s \geq 0$ and $t \geq 0$ the random variable $Y_{t+s} - Y_t$ has the same distribution as $Y_s - Y_0$,*
4. *for almost all $\omega \in \Omega$ the trajectories of $(Y_t)_{t\in[0,T]}$ are right–continuous and have left–side limits (i.e. they are* cadlag functions*),*
5. *Y_t is a stochastically continous process.*

We assume that $(Y_t)_{t\in[0,T]}$ is an $\mathcal{F}_t$–adapted process. This means that for each time moment τ we know the whole behaviour of process Y_t till τ.

For Levy processes the local characteristics, called Levy characteristics, are the following functions:

$$B_t : [0,T] \to \mathbb{R}, \; B_t = bt \;, \tag{1}$$

$$C_t : [0,T] \to \mathbb{R}, \; C_t = ct \;, \tag{2}$$

$$\nu : [0,T] \to \mathcal{M}(\mathbb{R}), \; \nu_t = \nu(dx)t, \; \nu(\{0\}) = 0, \; \int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(dx) < \infty \;, \tag{3}$$

where b and c are some constants, and $\mathcal{M}(\mathbb{R})$ is the space of non–negative measures on $\mathbb{R}$.

Let $(Y_t)_{t\in[0,T]}$ be the Levy process.

Def. 2. *The following transformation of probabilistic measure P for each $a \in \mathbb{R}$*

$$dP(a) = \exp\left(aY_T - T\left(ab + \frac{a^2c}{2} + \int_{\mathbb{R}} (e^{ax} - 1 - ag(x))\nu(dx)\right)\right) dP \;,$$

where $g(x) = x\mathbb{1}(|x| \le r)$ and r is a positive constant connected with characteristics, is called the Esscher transformation.

Theorem 1. *The triplet $(b(a), c(a), \nu(a))$ of the Levy characteristics of the process $(Y_t)_{t\in[0,T]}$ with respect to the measure $P(a)$ is described by the formulas:*

$$b(a) = b + ac + \int_{\mathbb{R}} g(x)(e^{ax} - 1)\nu(dx)$$

$$c(a) = c$$

$$\nu(a)(dx) = e^{ax}\nu(dx)$$

In the following, we will be interested in the stochastic processes of the form

$$\mathcal{S}_t = \exp(Y_t) \ , \tag{4}$$

i.e. $\mathcal{S}_t$ are the transformation of process Y_t to its exponential form.

Theorem 2. *If the constant a satisfies the following conditions*

$$b + \left(a + \frac{1}{2}\right) + \int_{\mathbb{R}} \left(e^{ax}(e^x - 1) - g(x)\right) \nu(dx) = 0 \tag{5}$$

$$\int_{\mathbb{R}} |e^{ax}(e^x - 1) - g(x)| \nu(dx) < \infty \tag{6}$$

then $(\mathcal{S}_t)_{t\in[0,T]}$ is a martingale with respect to the measure $P(a)$.

The proofs of the above theorems can be found in [6].

2.2 General method for option pricing

We use theorems and definitions from Sec. 2.1 for our main aim. We are to acquire the iterative stochastic equation (ISE) for stochastic process of movements of the underlying asset.

Let r denotes a constant risk–free interest rate and

$$\mathcal{Z}_t = e^{-rt}\mathcal{S}_t \tag{7}$$

be the discounted process of values of the underlying asset. We have to find the measure $P(a)$ equivalent to P for which $\mathcal{Z}_t$ is a martingale. To solve this problem one should use theorems 1 and 2. The next step is to find a form of the process $\mathcal{S}_t$ according to this new probabilistic measure $P(a)$.

2.2.1 Classical Black–Scholes model

Let us now illustrate this approach for the classical example of geometric Brownian motion.

We assume that the market operates in a continuous way and there are no additional transaction expenses and taxes. Let $\mathcal{W}_t$ denotes the standard Brownian motion. Let $\sigma > 0$ and $\mu > 0$ be constants which we will call the *volatility* and *drift* respectively.

For the Black–Scholes model we have $Y_t = \mu t + \sigma \mathcal{W}_t$. Then

$$\mathcal{S}_t = \mathcal{S}_0 \exp\left(\mu t + \sigma \mathcal{W}_t\right) . \tag{8}$$

In this case the form of $\mathcal{S}_t$ for the equivalent measure $P(a)$ is

$$\mathcal{S}_t = \mathcal{S}_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma \mathcal{W}_t^{P(a)}\right) , \tag{9}$$

where $\mathcal{W}_t^{P(a)}$ denotes the standard Brownian motion for the measure $P(a)$.

To use Monte Carlo methods (see Section 2.3) we should change equation (9) to the form of the iterative stochastic equation (ISE). Let $[0, T]$ denotes the life time interval for the given financial instrument. We have to discretize $[0, T]$ into the set of time moments $\mathcal{T} = \{t_0 = 0, t_1, \ldots, t_n = T\}$, where n is *number of steps*. We assume that distances between points in the set $\mathcal{T}$ are constant, i.e. $t_{i+1} - t_i = \Delta t = \text{const}$ for $i = 0, \ldots, n-1$.

From the above discretization, the equation (9) changes to the form

$$\mathcal{S}_{i+1} = \mathcal{S}_i \exp\left(\left(r - \frac{1}{2}\sigma^2\right) \Delta t + \sigma\sqrt{\Delta t}\epsilon_i\right) , \tag{10}$$

where $\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}$ are *iid* (i.e. independent, identically distributed) random variables from $N(0,1)$ distribution. This sequential form of the equations we call the iterative stochastic equations. The formula (10) is called an *Euler scheme* in the literature. It is generally known that if Δt tends to 0, the approximation given by ISE converges to the underlying originated stochastic process (see e.g. [6]).

2.2.2 Proposition of generalization of Black–Scholes model

The method presented in Sec. 2.2.1 may be used for other types of stochastic processes. Due to immense flexibility of Monte Carlo methods (see Sec. 2.3) we can use almost any kind of Levy process for modelling price movements of the underlying asset.

There are only two serious limitations. The first one is the problem of estimating additional parameters of the given process. In the case of geometrical Brownian motion (Section 2.2.1) there are only two parameters — the volatility (σ) and the risk–free rate (r). The third parameter – drift (μ) – does not appear in the appropriate equation (9). But for more complex stochastic processes there are additional necessary parameters.

The second problem arises from a necessity of solving some additional equations. These deterministic equations connect all parameters of the given process.

Let us consider next more general model of stochastic process. Assume

$$Y_t = \mu t + \sigma \mathcal{W}_t + k_1 \left(\mathcal{N}_t^{\kappa_1} - \kappa_1 t\right) + k_2 \left(\mathcal{N}_t^{\kappa_2} - \kappa_2 t\right) \ , \qquad (11)$$

where $\mu, \sigma, k_1 > 0, k_2 < 0$ are some constants, $\mathcal{W}_t$ is, as previously, Brownian motion, $\mathcal{N}_t^{\kappa_1}$ and $\mathcal{N}_t^{\kappa_2}$ are Poisson processes with intensity κ_1 and κ_2, respectively (hence $\kappa_1, \kappa_2 > 0$). All of the processes $\mathcal{W}_t, \mathcal{N}_t^{\kappa_1}, \mathcal{N}_t^{\kappa_2}$ are mutually independent.

From (5) we have

$$\mu - k_1\kappa_1 - k_2\kappa_2 - r + \frac{\sigma^2}{2} + a\sigma^2 + \kappa_1 e^{ak_1}(e^{k_1} - 1) + \kappa_2 e^{ak_2}(e^{k_2} - 1) = 0 \ . \quad (12)$$

With respect to the equivalent martingale measure $P(a)$, $\mathcal{S}_t$ is of the form

$$\mathcal{S}_t = \mathcal{S}_0 \exp\Big(\left(\mu - k_1\kappa_1 - k_2\kappa_2 + a\sigma^2\right) t + \sigma \mathcal{W}_t^{P(a)} + \\ + k_1 \mathcal{N}_t^{P(a)\ \kappa_1 \exp(ak_1)} + k_2 \mathcal{N}_t^{P(a)\ \kappa_2 \exp(ak_2)} \Big) \ , \quad (13)$$

where $\mathcal{W}_t^{P(a)}$ is the Brownian motion with respect to the new measure $P(a)$. $\mathcal{N}_t^{P(a)\ \kappa_1 \exp(ak_1)}$ and $\mathcal{N}_t^{P(a)\ \kappa_2 \exp(ak_2)}$ are the Poisson processes with respect to $P(a)$ with intensities $\kappa_1 e^{ak_1}$ and $\kappa_2 e^{ak_2}$, respectively.

From (13) it is easy to see that

$$S_t = S_u \exp\Big(\left(\mu - k_1\kappa_1 - k_2\kappa_2 + a\sigma^2\right)(t-u) + \sigma\left(\mathcal{W}_t^{P(a)} - \mathcal{W}_u^{P(a)}\right) + \\ + k_1\left(\mathcal{N}_t^{P(a)\ \kappa_1 \exp(ak_1)} - \mathcal{N}_u^{P(a)\ \kappa_1 \exp(ak_1)}\right) + \\ + k_2\left(\mathcal{N}_t^{P(a),\ \kappa_2 \exp(ak_2)} - \mathcal{N}_u^{P(a),\ \kappa_2 \exp(ak_2)}\right)\Big) \quad (14)$$

and

$$S_T = S_0 \exp\Big(\left(\mu - k_1\kappa_1 - k_2\kappa_2 + a\sigma^2\right) T + \sigma \mathcal{W}_T^{P(a)} + \\ + k_1 \mathcal{N}_T^{P(a)\ \kappa_1 \exp(ak_1)} + k_2 \mathcal{N}_T^{P(a),\ \kappa_2 \exp(ak_2)} \Big) \ . \quad (15)$$

We take advantage of these equations to argue for necessity of using Monte Carlo methods in Sec. 2.3.

A special case of model (11) is the geometrical Brownian motion with Poisson jumps. In such a model stochastic process has the form

$$Y_t = \mu t + \sigma \mathcal{W}_t + k\left(\mathcal{N}_t^{\kappa} - \kappa t\right) \ , \quad (16)$$

where μ, σ and k are some constants, $\mathcal{W}_t$ is Brownian motion and $\mathcal{N}_t^\kappa$ is a Poisson process with intensity κ (hence $\kappa > 0$). Processes $\mathcal{W}_t$ and $\mathcal{N}_t^\kappa$ are independent of each other. It is easy to see that we acquire the model described by (16) for $k_2 = 0$ in equation (11).

From (5) we have

$$\mu - k\kappa - r + \frac{\sigma^2}{2} + a\sigma^2 + \kappa e^{ak}(e^k - 1) = 0 \ . \tag{17}$$

This equation connects all parameters of the stochastic process of the form (16) and has to be solved with respect to variable a. After changing the measure according to method presented in Sec. 2.2 we have

$$\mathcal{S}_t = \mathcal{S}_0 \exp\left(\left(\mu - k\kappa + a\sigma^2\right) t + \sigma \mathcal{W}_t^{P(a)} + k\mathcal{N}_t^{P(a)\ \kappa\exp(ak)}\right) \ , \tag{18}$$

where notation has the similar meaning as in (13).

2.3 Monte Carlo methods for pricing financial instruments

We now introduce some basic concepts of Monte Carlo (abbreviated MC) methods for pricing financial instruments. They may be used for various processes of movements of the underlying asset and many types of financial instruments.

The necessity of using MC is straightforward. Assume that the process of movements of underlying asset is modelled by (11). The payment function for European–style *call* option (see e.g. [2]) is given by

$$f(\mathcal{S}_t) = (S_T - K)_+ \ , \tag{19}$$

where $(x)_+$ denotes the non–negative part of x and K is *striking price* for this option (see e.g. [4], [6]). Let C_0 denote the *price* for such an option. Mathematically, price is defined as discounted expected value of future cash flow.

Under this assumptions we calculate the price for European–style *call* option for $\sigma \neq 0$:

$$\begin{aligned}C_0 = \exp(-rT)\, E^{P(a)}((S_T - K)^+ \,|\mathcal{F}_0) = \\ = E^{P(a)}\{S_0 \exp\left(\left(\mu - k_1\kappa_1 - k_2\kappa_2 + a\sigma^2 - r\right) T + \right. \\ \left. + \sigma\mathcal{W}_T^{P(a)} + k_1\mathcal{N}_T^{P(a)\, \kappa_1 \exp(ak_1)} + k_2\mathcal{N}_T^{P(a),\, \kappa_2 \exp(ak_2)}\right) - K\}^+ = \\ = E^{P(a)}\{S_0 \exp(k_1 Z_1 + k_2 Z_2 + \sigma Z_3) - K\}^+ \,, \quad (20)\end{aligned}$$

where

$$Z_1 \sim \text{Poiss}\left(\kappa_1 e^{ak_1} T\right) \,, \quad (21)$$

$$Z_2 \sim \text{Poiss}\left(\kappa_2 e^{ak_2} T\right) \,, \quad (22)$$

$$Z_3 \sim N\left(\frac{(\mu - k_1\kappa_1 - k_2\kappa_2 + a\sigma^2 - r)\, T}{\sigma}, \sqrt{T}\right). \quad (23)$$

The cumulative distribution function of the sum $k_1 Z_1 + k_2 Z_2 + \sigma Z_3$ is given by the formula

$$\Upsilon(t) = e^{-(\kappa_1 e^{ak_1} + \kappa_2 e^{ak_2})T} \sum_{k,l=0}^{\infty} \frac{\kappa_1^k \kappa_2^l T^{k+l} e^{a(kk_1 + lk_2)}}{k!l!} \Phi_{k,l} \,, \quad (24)$$

where

$$\Phi_{k,l} = \Phi\left(\frac{t - k_1 k - k_2 l - (\mu - k_1\kappa_1 - k_2\kappa_2 + a\sigma^2 - r)\, T}{\sigma\sqrt{T}}\right). \quad (25)$$

From equations (20) – (25) the seeking price has the form

$$C_0 = \int_{\ln\left(\frac{K}{S_0}\right)}^{\infty} (S_0 \exp(x) - K)\, d\Upsilon(x)\,. \quad (26)$$

As we can see, the formula (26) can not be solved analytically. Hence, we should use simulations.

Let us assume that for a given financial instrument we know its life time interval $[0, T]$ and the initial value $\mathcal{S}_0$ of the underlying asset. This time interval is divided into n steps (see Sec. 2.2.1). The appropriate ISE formula for stochastic process of the form (11) is given by

$$\mathcal{S}_{i+1} = \mathcal{S}_i \exp\left(\left(\mu - k_1\kappa_1 - k_2\kappa_2 + a\sigma^2\right)\Delta t + \right.$$
$$\left. + \sigma\sqrt{\Delta t}\epsilon_i + k_1\rho_i^1 + k_2\rho_i^2\right) , \quad (27)$$

where $\rho_0^1, \ldots, \rho_{n-1}^1$ and $\rho_0^2, \ldots, \rho_{n-1}^2$ are *iid* random variables from Poisson distributions with intensities $\kappa_1 e^{ak_1}\Delta t$ and $\kappa_2 e^{ak_2}\Delta t$, respectively. Other notation has the same meaning as in (10).

For process (16) ISE has the form

$$\mathcal{S}_{i+1} = \mathcal{S}_i \exp\left(\left(\mu - k\kappa + a\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\epsilon_i + k\rho_i\right) , \quad (28)$$

where the notation is similar to (27).

Starting with the value $\mathcal{S}_0$, from the appropriate ISE formula (e.g. (10), (27), (28)) we obtain values $\mathcal{S}_1, \ldots, \mathcal{S}_T$. These quantities are values of the basic instrument at time moments $t_1, \ldots, t_n = T$, respectively. They form the sample path S^1 of the stochastic process modelling the underlying asset movements. In the same manner we can simulate m trajectories $S^1, S^2, \ldots S^m$ of the desired stochastic process $\mathcal{S}_t$, where m is the *number of simulations*.

The last step is the calculation of a *discounted* average of the payments sequence

$$\mathcal{C}^m = e^{-rT}\frac{1}{m}\sum_{i=1}^{m} \mathrm{FV}_T\, f(S^i) , \quad (29)$$

where r is the risk–free rate and $\mathrm{FV}_T(\mathcal{Y})$ is the *future value* of cash flow $\mathcal{Y}$ at time moment T (see e.g. [3]). For European–style *call* option (see (19)) we have more straightforward formula:

$$C^m = e^{-rT} \frac{1}{m} \sum_{i=1}^{m} f(S^i) \ . \tag{30}$$

Payments from sample paths $S^1, \ldots S^m$ form an *iid* sequence. Then, from the Strong Law of Large Numbers (in abbreviation SLLN) we have

$$C^m \xrightarrow[m \to \infty]{a.s.} C \ , \tag{31}$$

where C denotes the *present value* for the given financial instrument, defined as *discounted expected value* (see e.g. [3]).

In the following we assume that both expected value and variance of $f(\mathcal{S}_t)$ are finite, i.e. $\forall t$, $\mathbb{E} f^2(\mathcal{S}_t) < \infty$.

3 Fitting stochastic model and parameters

In this section we present a problem connected with applying the Black–Scholes model, which appears on real markets. We also present a methodology of solving this problem using Monte Carlo simulations.

3.1 Problem of *"volatility smile"*

Assuming that movements of real underlying assets follow geometrical Brownian motion (i.e. (8)), we can apply Black–Scholes formula (see e.g. [4]) to practical data from markets. Knowing r, K, S_0, T, C, where C denotes the real market price, for given European–style *call* option, we may find σ (so called *implied volatility*). To calculate this value we use inversion of Black-Scholes formula as a function of σ depending on K. As it is seen for real market data, graph of $\sigma(K)$ for fixed kind of option with various striking prices K is not the straight line, but has a "U" shape (see e.g. [1]). This observation in financial literature is called *a problem of "volatility smile"*. As it was noted, this means that real markets – i.e. real processes of movements of underlying assets – do not follow standard geometrical Brownian motion.

Therefore we should find more appropriate model of stochastic process of underlying asset for a given real market data.

3.2 Fitting process via Monte Carlo methods

As we have argued in Sec. 3.1, there is a necessity of modelling real markets via stochastic processes other than geometrical Brownian motion.

It is easy to see that stochastic process of the form (11) is a generalization of the process (16). And this last one is the generalization of the standard geometrical Brownian motion, i.e. (8). So we postulate to use the process (11) to attempt at solving the problem of "volatility smile". We present an appropriate general method of doing this via Monte Carlo methods. We also present an example of applying this method for S&P 500 option (see Sec. 3.3).

However, it is worth noted that the following method is very universal and may be used for other types of Levy processes. But we restricted our attention only to stochastic process of the form (11). The main reason of this restriction is that for a given kind of option with fixed moment T we have usually very few different striking prices K. From our point of view, we should generalize the geometrical Brownian motion as less as it is possible. Acting differently, we may have not enough information for estimation of parameters for too complicated stochastic process of underlying asset movements.

And the standard geometrical Brownian motion models behaviour of real markets pretty well.

Keeping this in mind, now we present the methodology of solving the problem of "volatility smile". Assume that for a given kind of option we have a sequence $C = (C(K_1), \ldots, C(K_p))$ of real market prices for different striking prices $K_1, \ldots, K_p$. The external market parameters r, T, S_0 are also given.

According to the method presented in Sec. 2.3 for the fixed set of parameters $(k_1, \kappa_1, k_2, \kappa_2, \sigma, \mu)$ we may simulate the prices of the given kind of option for all values $K_1, \ldots, K_p$. We acquire a sequence $C^* = (C^*(K_1), \ldots, C^*(K_p))$ of simulated prices. We may also find the l_p^1–norm between both sequences of prices C and C^*:

$$\|C - C^*\|_{l^1} = \sum_{i=1}^{p} |C(K_i) - C^*(K_i)| \ . \tag{32}$$

Applying different sets of parameters $(k_1, \kappa_1, k_2, \kappa_2, \sigma, \mu)$, we could find the appropriate sequence of simulated prices which minimizes the norm (32). We denote by $\mathcal{P}_{\min}$ this set of parameters for which the l^1 norm is minimized. Additionally, this set of parameters defines also which of models (8), (16) or (11) fits optimally to given real market data. Easily seen, this method may be used for other kinds of Levy processes. The only requirement is to use the theory presented in Sec. 2.2 and to calculate appropriate ISE formula (see Sec. 2.3).

The sequence $\mathcal{P}_{min}$ is also a solution for a problem of "volatility smile".

We may calculate the sequence of implied volatilites for a set of simulated prices given by $\mathcal{P}_{min}$.

3.3 Example of method application

In this section we present an example of applying the method developed in Sec. 3.2.

We use this general method for S&P 500 European call option. The termination date for this option was 17th of May, 2002. Data for this option is gathered in Table 1. There are values of striking prices K in the first column of this table. The real market prices and theoretical Black–Scholes prices are specified in second and third column, respectively. The risk–free rate r in this example is 0.07 and the starting value of underlying asset S_0 is 1099.1.

The value of l_p^1–norm between the theoretical prices given by Black–Scholes formula and the real market prices is

$$\|C - C_{\mathrm{BS}}\|_{l^1} = 2.084 \ . \tag{33}$$

We have found that stochastic process of the form (16) better than standard geometrical Brownian motion models the real market prices. For the set of parameters

$$\kappa = 0.25 \ , \ k = 0.1 \ , \ \sigma = 0.13 \ , \ \mu = 0.1 \ , \tag{34}$$

the norm (32) between real market data and simulated prices equals 1.74803. Comparing this value with 33 we can see that this model improved relatively fitting of the prices by 16.12 %.

The prices simulated by the process (16) with set of parameters 34 are given in the last column of the Table 1.

K	C *(real prices)*	C_{BS} *(Black–Scholes prices)*	C^* *(simulated prices)*
1050	49.2	49.1	49.4663917
1055	44.2	44.1	44.64002077
1060	39.2	39.1	39.51050215
1065	34.2	34.1	34.18952799
1070	29.3	29.11	29.41484469
1075	24.4	24.157	24.39709814
1080	19.6	19.303	19.6238879
1085	14.9	14.67	14.83853433
1090	10.7	10.452	10.6537997
1095	6.9	6.872	6.938917236
1100	4.1	4.106	4.012709425
1105	2.3	2.198	2.080002264
1110	1	1.042	0.976715763
1115	0.45	0.433	0.41267185
1120	0.25	0.157	0.216773132
1125	0.15	0.049	0.173951727
1130	0.1	0.013	0.107346476

Table 1: Real and simulated prices for S&P 500 option

4 Conclusions

The method described in our paper enables to find a stochastic process, which is fitted to a real asset better than the geometrical Brownian motion. The theoretical considerations are confirmed by the calculations for a real derivative. It is possible to find a process with the property of the "U" shape of volatility as the function of the striking price. A disadvantage is a large number of simulations required in the presented method. However it seems that the procedure is worth carrying out, if the properties of the underlying asset do not change in time. The domain of exploration of the process is not very wide and therefore the method is statistically reliable.

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