For the second equation $x^3-3x+1=0$, ω denoting as before, the roots are

$$a=\omega^{\frac{1}{3}}+\omega^{\frac{8}{3}}, \text{ whence } a^2=\omega^{\frac{2}{3}}+\omega^{\frac{7}{3}}+2, =2+c,$$

$$b=\omega^{\frac{4}{3}}+\omega^{\frac{5}{3}}, \quad \text{,,} \quad b^2=\omega^{\frac{8}{3}}+\omega^{\frac{1}{3}}+2, =2+a,$$

$$c=\omega^{\frac{2}{3}}+\omega^{\frac{7}{3}}, \quad \text{,,} \quad c^2=\omega^{\frac{5}{3}}+\omega^{\frac{4}{3}}+2, =2+b.$$

The equation $x^3-5x^2+6x-1=0$, which, writing therein $x+2$ for x, gives $x^3+x^2-2x-1=0$ is considered in Hermite's *Cours d'Analyse*, Paris 1873, p. 12, and this suggested to me the foregoing investigation.

NOTE ON MR. KLEIBER'S FUNCTIONS K_i AND G_i.

By *J. W. L. Glaisher*.

§ 1. THE expansions of K and G in ascending powers of $h'-h$ given by Mr. Kleiber in § 29 of his paper, (pp. 29, 30 of this volume) do not agree with those given in Vol. XIX., pp. 146–150 (February, 1890), and it is easy to see that the former are incorrect. I proceed therefore to investigate the expansions of K_i and G_i in powers of $h'-h$.

The function K_i, §§ 2–6.

§ 2. Mr. Kleiber's identification of K and W with $P_{-\frac{1}{2}}$ and $P_{\frac{1}{2}}$ in § 11 (pp. 10, 11) is not very precise; but we may regard K_i as defined by equation (146), p. 27, viz.

$$\frac{2K_i}{\pi}=1-\frac{i^2-\frac{1}{4}}{1^2}h+\frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})}{1^2.2^2}h^2-\frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})(i^2-\frac{25}{4})}{1^2.2^2.3^2}h^3+\&c.$$

We know also that K_i satisfies the differential equation (115), viz.

$$hh'\frac{d^2u}{dh^2}+(h-h')\frac{du}{dh}+(i^2-\tfrac{1}{4})\,u=0;$$

and we have also

$$K_0=K \text{ and } K_1=2W=I+E.$$

These latter results may be easily proved by putting $i=0$ and $i=1$ in the series for K_i; we thus find

$$\frac{2K_0}{\pi}=1+\frac{1^2}{2^2}h+\frac{1^2.3^2}{2^2.4^2}h^2+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}h^3+\&c.,$$

$$\frac{2K_1}{\pi}=1-\frac{1.3}{2^2}h-\frac{1^2.3.5}{2^2.4^2}h^2-\frac{1^2.3^2.5.7}{2^2.4^2.6^2}h^3-\&c.,$$

and these series are the values of $\dfrac{2K}{\pi}$ and $\dfrac{2(I+E)}{\pi}$ respectively.

§ 3. Proceeding now to the expansion of K_i in powers of $\lambda=h'-h$ (denoted by Mr. Kleiber by x), we see that K_i satisfies the differential equation

$$(1-\lambda^2)\frac{d^2u}{d\lambda^2}-2\lambda\frac{du}{d\lambda}+(i^2-\tfrac{1}{4})\,u=0.$$

This is Legendre's equation, and we know that it is satisfied by the two series

$$A_i=1-\frac{i^2-\frac{1}{4}}{2\,!}\lambda^2+\frac{(i^2-\frac{1}{4})(i^2-\frac{25}{4})}{4\,!}\lambda^4-\frac{(i^2-\frac{1}{4})(i^2-\frac{25}{4})(i^2-\frac{81}{4})}{6\,!}\lambda^6+\&c.,$$

and

$$B_i=\lambda-\frac{i^2-\frac{9}{4}}{3\,!}\lambda^3+\frac{(i^2-\frac{9}{4})(i^2-\frac{49}{4})}{5\,!}\lambda^5-\frac{(i^2-\frac{9}{4})(i^2-\frac{49}{4})(i^2-\frac{121}{4})}{7\,!}\lambda^7+\&c.$$

These are distinct solutions of the differential equation; and we notice that neither of the series terminates where i is an integer, as is supposed to be the case.

We may put therefore

$$K_i=a_iA_i+b_iB_i,$$

where a_i and b_i are constants.

§ 4. To determine a_i we may put $h=h'$; then $\lambda=0$ and $h=\frac{1}{2}$. We thus find

$$\frac{2a_i}{\pi}=1-\frac{i^2-\frac{1}{4}}{1^2}\frac{1}{2}+\frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})}{1^2.2^2}\frac{1}{2^2}-\frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})(i^2-\frac{25}{4})}{1^2.2^2.3^2}\frac{1}{2^3}+\&c.$$

Similarly, by differentiating the relation $K_i=a_iA_i+b_iB_i$ with respect to h or λ, and putting $h=h'$, we find

$$\frac{2b_i}{\pi}=\frac{i^2-\frac{1}{4}}{1^2}\frac{1}{2}-\frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})}{1^2.2}\frac{1}{2^2}+\frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})(i^2-\frac{25}{4})}{1^2.2^2.3}\frac{1}{2^3}-\&c.$$

Thus the values of a_i and b_i are determined in the form of series.

§ 5. Putting $i=0$, we have therefore

$$K=a_0A_0+b_0B_0$$
$$=a_0\left\{1+\frac{1^2}{2.4}\lambda^2+\frac{1^2.5^2}{2.4.6.8}\lambda^4+\frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6+\&c.\right\}$$
$$+b_0\left\{\lambda+\frac{3^2}{2.4.3}\lambda^3+\frac{3^2.7^2}{2.4.6.8.5}\lambda^5+\frac{3^2.7^2.11^2}{2.4\ldots12.7}\lambda^7+\&c.\right\},$$

where $$\frac{2a_0}{\pi}=1+\frac{1^2}{2^2}\frac{1}{2}+\frac{1^2.3^2}{2^2.4^2}\frac{1}{2^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}\frac{1}{2^3}+\&c.,$$
$$\frac{2b_0}{\pi}=-\frac{1^2}{2}\frac{1}{2^2}-\frac{1^2.3^2}{2^2.4}\frac{1}{2^3}-\frac{1^2.3^2.5^2}{2^2.4^2.6}\frac{1}{2^4}-\&c.$$

Denoting by K^0 and G^0 the values of K and G when $\lambda=0$ $\left(\text{that is to say when the modulus is }\frac{1}{\sqrt{2}}\right)$, it is evident that the first series is equal to $\frac{2K^0}{\pi}$; and it is easy to prove that the second series is equal to $-\frac{2G^0}{\pi}$, for

$$\frac{2G}{\pi}=\frac{4hh'}{\pi}\frac{dK}{dh}=hh'\left\{\frac{1^2}{2}+\frac{1^2.3^2}{2^2.4}h+\frac{1^2.3^2.5^2}{2^2.4^2.6}h^2+\&c.\right\}$$

which, when $h=h'=\frac{1}{\sqrt{2}}$, becomes identical with the above series for $-\frac{2b_0}{\pi}$.

Thus $a_0=K^0$ and $b_0=-G^0$, and we may write the result in the form

$$K=K^0\left\{1+\frac{1^2}{2.4}\lambda^2+\frac{1^2.5^2}{2.4.6.8}\lambda^4+\frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6+\&c.\right\}$$
$$-2G^0\left\{\frac{1^2}{2}\lambda+\frac{3^2}{2.4.6}\lambda^3+\frac{3^2.7^2}{2.4.6.8.10}\lambda^5+\frac{3^2.7^2.11^2}{2.4.6\ldots14}\lambda^7+\&c.\right\},$$

which is the same as the expansion given on p. 147 of Vol. XIX.

§ 6 Putting $i=1$, we have

$$K_1=a_1A_1+b_1B_1$$
$$=a_1\left\{1-\frac{3}{2.4}\lambda^2-\frac{3^2.7}{2.4.6.8}\lambda^4-\frac{3^2.7^2.11}{2.4.6.8.10.12}\lambda^6-\&c.\right\}$$
$$+b_1\left\{\lambda+\frac{1^2.5}{2.4.3}\lambda^3+\frac{1^2.5^2.9}{2.4.6.8.5}\lambda^5+\frac{1^2.5^2.9^2.13}{2.4\ldots.12.7}\lambda^7+\&c.\right\},$$

where $\frac{2a_1}{\pi}$ and $\frac{2b_1}{\pi}$ are the values of the series

$$1-\frac{1.3}{2^2}h-\frac{1^2.3.5}{2^2.4^2}h^2-\frac{1^2.3^2.5.7}{2^2.4^2.6^2}h^3-\&c.,$$

and
$$\tfrac{1}{4}\left\{\frac{1.3}{2}+\frac{1^2.3.5}{2^2.4}h+\frac{1^2.3^2.5.7}{2^2.4^2.6}h^2+\&c.\right\},$$

when h is put equal to $\frac{1}{2}$.

The former of these series is equal to $\frac{2(I+E)}{\pi}$ and the latter (which is equal to the derivative of the former with respect to h multiplied by $-\frac{1}{2}$)

$$=-\tfrac{1}{2}\frac{d}{dh}\,\frac{2(I+E)}{\pi}=\frac{2}{\pi}\left\{\frac{E}{4h'}-\frac{I}{4h}\right\}.$$

Putting $h=\frac{1}{2}$, we have therefore (since $I^0=G^0-\frac{1}{2}K^0$, $E^0=G^0+\frac{1}{2}K^0$),

$$\frac{2a_1}{\pi}=\frac{4G^0}{\pi},\quad \frac{2b_1}{\pi}=\frac{K^0}{\pi},$$

giving
$$a_1=2G^0,\quad b_1=\tfrac{1}{2}K^0.$$

Thus we have finally

$$K_1=2G^0\left\{1-\frac{3}{2.4}\lambda^2-\frac{3^2.7}{2.4.6.8}\lambda^4-\frac{3^2.7^2.11}{2.4\ldots 12}\lambda^6-\&c.\right\}$$
$$+K^0\left\{\frac{1}{2}\lambda+\frac{1^2.5}{2.4.6}\lambda^3+\frac{1^2.5^2.9}{2.4.6.8.10}\lambda^5+\&c.\right\},$$

which, since $K_1=I+E$, is the same as the result given on p. 150 of Vol. XIX.

The function G_i, §§ 7, 8.

§ 7. Mr. Kleiber's functions K_i and G_i are connected by the relations

$$G_i=\frac{2hh'}{2i+1}\,\frac{dK_i}{dh},\qquad K_i=-\frac{2}{2i-1}\,\frac{dG_i}{dh}\,*.$$

* In equations (117) and (118) on p. 23, Mr. Kleiber has written these relations

$$\frac{dK_i}{dh}=\frac{2i+1}{hh'}G_i,\qquad \frac{dG_i}{dh}=-(2i-1)K_i,$$

i.e. a 2 is omitted in both. It was clearly his intention that they should be as in the text, since when $i=0$, K_i and G_i are to become K and G.

Thus we may define G_i by the series

$$\frac{2G_i}{\pi} = -(i-\tfrac{1}{2})\left\{h - \frac{i^2-\frac{1}{4}}{1^2.2}h^2 + \frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})}{1^2.2^2.3}h^3 - \frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})(i^2-\frac{25}{4})}{1^2.2^2.3^2.4}h^4 + \&c.\right\}.$$

The function G_i satisfies the differential equation

$$hh'\frac{d^2u}{dh^2} + (i^2-\tfrac{1}{4})\,u = 0,$$

and we have also

$$G_0 = G, \quad G_1 = \tfrac{1}{3}(h'I - hE).$$

These latter results may be proved as follows. Putting $i=0$ and $i=1$ in the series for $\frac{2G_i}{\pi}$, we find

$$\frac{2G_0}{\pi} = \tfrac{1}{2}\left\{h + \frac{1^2}{2^2.2}h^2 + \frac{1^2.3^2}{2^2.4^2.3}h^3 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2.4}h^4 + \&c.\right\}$$

and

$$\frac{2G_1}{\pi} = -\tfrac{1}{2}\left\{h - \frac{1.3}{2^2.2}h^2 - \frac{1^2.3.5}{2^2.4^2.3}h^3 - \frac{1^2.3^2.5.7}{2^2.4^2.6^2.4}h^4 - \&c.\right\}.$$

The former series is the development of $\frac{2G}{\pi}$, so that $G_0 = G$.

To determine the value of the latter series we find, by multiplying the known series for $\frac{2E}{\pi}$ and $\frac{2I}{\pi}$ by h and h' respectively,

$$\frac{2hE}{\pi} = hh'\left\{1 + \frac{3}{2^2}h + \frac{3^2.5}{2^2.4^2}h^2 + \frac{3^2.5^2.7}{2^2.4^2.6^2}h^3 + \&c.\right\}$$

$$\frac{2h'I}{\pi} = -hh'\left\{\frac{1}{2} + \frac{1^2.3}{2^2.4}h + \frac{1^2.3^2.5}{2^2.4^2.6}h^2 + \frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8}h^3 + \&c.\right\};$$

whence

$$\frac{2(hE-h'I)}{\pi} = hh'\left\{\frac{3}{2} + \frac{1^2.3.5}{2^2.4}h + \frac{1^2.3^2.5.7}{2^2.4^2.6}h^2 + \frac{1^2.3^2.5^2.7.9}{2^2.4^2.6^2.8}h^3 + \&c.\right\},$$

$$= 3h\left\{\frac{1}{2} - \frac{1.3}{2^2.4}h - \frac{1^2.3.5}{2^2.4^2.6}h^2 - \frac{1^2.3^2.5.7}{2^2.4^2.6^2.8}h^3 - \&c.\right\},$$

on multiplying the series by $h' = 1 - h$.

Thus

$$\frac{2G_1}{\pi} = -\tfrac{1}{3}\frac{2(hE-h'I)}{\pi},$$

giving

$$G_1 = \tfrac{1}{3}(h'I - hE).$$

§ 8. The values of G_0 and G_1 may of course be readily deduced from the relation

$$G_i = \frac{2hh'}{2i+1}\frac{dK_i}{dh};$$

for, using this formula, we have

$$G_0 = 2hh'\frac{dK}{dh} = G,$$

and

$$G_1 = \frac{2hh'}{3}\frac{dK_1}{dh} = \frac{2hh'}{3}\frac{d(I+E)}{dh}$$

$$= \frac{2hh'}{3}\left\{\frac{I}{2h} - \frac{E}{2h'}\right\} = \tfrac{1}{3}(h'I - hE),$$

as before.

The quantity $hE - h'I$ was denoted by Mr. Kleiber by H, so that $G_1 = -\tfrac{1}{3}H$. Mr. Kleiber has pointed out that the functions $W = \tfrac{1}{2}(I+E)$ and H are connected by reciprocal relations, which may be written

$$H = -4hh'\frac{dW}{dh}, \quad W = \tfrac{1}{3}\frac{dH}{dh}.$$

Expansion of G_i in powers of λ, §§ 9—12.

§ 9. The differential equation with respect to λ, which is satisfied by G_i, is

$$(1-\lambda^2)\frac{d^2u}{d\lambda^2} + (i^2 - \tfrac{1}{4})u = 0,$$

and we have

$$G_i = \alpha_i X_i + \beta_i Y_i,$$

where

$$X_i = 1 - \frac{i^2-\frac{1}{4}}{2!}\lambda^2 + \frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})}{4!}\lambda^4 - \frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})(i^2-\frac{49}{4})}{6!}\lambda^6 + \&c.^*$$

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* The series for R_n in equation (159), p. 29, should be

$$R_n = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-1)(n+1)(n+2)}{4!}x^4 - \frac{n(n-1)(n-3)(n+1)(n+2)(n+4)}{6!}x^6 + \&c.$$

$$Y_i=\lambda-\frac{i^2-\frac{1}{4}}{3!}\lambda^3+\frac{(i^2-\frac{1}{4})(i^2-\frac{25}{4})}{5!}\lambda^5-\frac{(i^2-\frac{1}{4})(i^2-\frac{25}{4})(i^2-\frac{81}{4})}{7!}\lambda^7+\&c.$$

and α_i and β_i are constants, whose values are given by the formulæ

$$\frac{2\alpha_i}{\pi}=-(i-\tfrac{1}{2})\left\{h-\frac{i^2-\frac{1}{4}}{1^2.2}h^2+\frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})}{1^2.2^2.3}h^3-\&c.\right\},$$

$$\frac{2\beta_i}{\pi}=\tfrac{1}{2}(i-\tfrac{1}{2})\left\{1-\frac{i^2-\frac{1}{4}}{1^2}h+\frac{(i^2-\frac{1}{4})(i^2-\frac{9}{4})}{1^2.2^2}h^2-\&c.\right\},$$

when h is put equal to $\frac{1}{2}$.

§ 10. For $i=0$, we have

$$\frac{2\alpha_0}{\pi}=\ \tfrac{1}{2}\left\{h+\frac{1}{2^2.2}h^2+\frac{1^2.3^2}{2^2.4^2.3}h^3+\&c.\right\},$$

$$\frac{2\beta_0}{\pi}=-\tfrac{1}{4}\left\{1+\frac{1^2}{2^2}h+\frac{1^2.3^2}{2^2.4^2}h^2+\&c.\right\},$$

when h is put equal to $\frac{1}{2}$.

The former series is the development of $\dfrac{2G}{\pi}$ and the latter is equal to $-\frac{1}{4}\dfrac{2K}{\pi}$; so that, putting $h=\frac{1}{2}$, we have

$$\alpha_0=G^0,\ \beta_0=-\tfrac{1}{4}K^0,$$

giving

$$G=G^0\left\{1+\frac{1}{2.4}\lambda^2+\frac{3^2}{2.4.6.8}\lambda^4+\frac{3^2.7^2}{2.4\ldots12}\lambda^6+\&c.\right\}$$
$$-\tfrac{1}{4}K^0\left\{\lambda+\frac{1^2}{2.4.3}\lambda^3+\frac{1^2.5^2}{2.4.6.8.5}\lambda^5+\frac{1^2.5^2.9^2}{2.4.6\ldots12.7}\lambda^7+\&c.\right\},$$

that is

$$G=G^0\left\{1+\frac{1}{2.4}\lambda^2+\frac{3^2}{2.4.6.8}\lambda^4+\frac{3^2.7^2}{2.4\ldots12}\lambda^6+\&c.\right\}$$
$$-\tfrac{1}{2}K^0\left\{\frac{1}{2}\lambda+\frac{1^2}{2.4.6}\lambda^3+\frac{1^2.5^2}{2.4.6.8.10}\lambda^5+\frac{1^2.5^2.9^2}{2.4.6\ldots14}\lambda^7+\&c.\right\},$$

which is the formula given on p. 148 of Vol. XIX.

§ 11. Similarly putting $i=1$, we have

$$G_1=\alpha_1X_1+\beta_1Y_1,$$

where

$$X_1 = 1 - \frac{3}{2.4}\lambda^2 - \frac{3.1.5}{2.4.6.8}\lambda^4 - \frac{3.1.5^2.9}{2.4.6.8.10.12}\lambda^6 - \&c.,$$

$$Y_1 = \lambda - \frac{3}{2.4.3}\lambda^3 - \frac{3^2.7}{2.4.6.8.5}\lambda^5 - \frac{3^2.7^2.11}{2.4\ldots12.7}\lambda^7 - \&c.,$$

and α_1 and β_1 are given by the equations

$$\frac{2\alpha_1}{\pi} = -\tfrac{1}{2}\left\{h - \frac{1.3}{2^2.2}h^2 - \frac{1^2.3.5}{2^2.4^2.3}h^3 - \&c.\right\},$$

$$\frac{2\beta_1}{\pi} = \tfrac{1}{4}\left\{1 - \frac{1.3}{2^2}h - \frac{1^2.3.5}{2^2.4^2}h^2 - \&c.\right\},$$

when h is put equal to $\frac{1}{2}$.

The former series $= -\frac{1}{3}\dfrac{2(hE - h'I)}{\pi}$ by § 7, and the latter, which is the derivative of the former with respect to h multiplied by $-\frac{1}{2}$, is equal to $\frac{1}{4}\dfrac{2(I+E)}{\pi}$,

Thus, putting $h = \frac{1}{2}$,

$$\frac{2\alpha_1}{\pi} = -\tfrac{1}{3}\frac{K^0}{\pi}, \quad \frac{2\beta_1}{\pi} = \tfrac{1}{4}\frac{4G^0}{\pi},$$

giving $\qquad \alpha_1 = -\frac{1}{6}K^0, \quad \beta_1 = \frac{1}{2}G^0,$

whence we find

$$G_1 = -\tfrac{1}{6}K^0\left\{1 - \frac{3}{2.4}\lambda^2 - \frac{3.1.5}{2.4.6.8}\lambda^4 - \frac{3.1^2.5^2.9}{2.4\ldots12}\lambda^6 - \&c.\right\}$$
$$+ \tfrac{1}{2}G^0\left\{\lambda - \frac{3}{2.4.3}\lambda^3 - \frac{3^2.7}{2.4.6.8.5}\lambda^5 - \frac{3^2.7^2.11}{2.4\ldots12.7}\lambda^7 - \&c.\right\},$$

that is

$$G_1 = -\tfrac{1}{2}K^0\left\{\frac{1}{3} - \frac{1}{2.4}\lambda^2 - \frac{1^2.5}{2.4.6.8}\lambda^4 - \frac{1^2.5^2.9}{2.4\ldots12}\lambda^6 - \frac{1^2.5^2.9^2.13}{2.4\ldots16}\lambda^8 - \&c.\right\}$$
$$+ G^0\left\{\frac{1}{2}\lambda - \frac{3}{2.4.6}\lambda^3 - \frac{3^2.7}{2.4.6.8.10}\lambda^5 - \frac{3^2.7^2.11}{2.4\ldots14}\lambda^7 - \&c.\right\}.$$

It will be remembered (§ 8) that $G_1 = \frac{1}{3}(h'I - hE) = -\frac{1}{3}H.$

§ 12. The above expansion of G_1 may also be readily obtained by deducing it from the expansion of K_1 in § 6 by means of the formula

$$G_1 = -\frac{1-\lambda^2}{3}\frac{dK_1}{d\lambda}.$$

Similarly we may deduce the expansion of K_1 from that of G_1 by the formula

$$K_1 = 4\frac{dG_1}{d\lambda}.$$

General values of the constants in the expansions of K_i and G_i, §§ 13, 14.

§ 13. Denoting the values of K_i and G_i when h is put equal to $\frac{1}{2}$ by K_i^0 and G_i^0, we can show that, in general,

$$K_i = K_i^0 A_i - (2i+1)\, G_i^0 B_i,$$

and

$$G_i = G_i^0 X_i + \tfrac{1}{4}(2i-1)\, K_i^0 Y_i.$$

To prove the first of these results, we see that, by putting $\lambda = 0$ in the equation

$$K_i = a_i A_i + b_i B_i,$$

we have at once $a_i = K_i^0$; and by differentiating with respect to λ and putting $\lambda = 0$, we find

$$b_1 = \left(\frac{dK_i}{d\lambda}\right)_{\lambda=0}.$$

Now

$$G_i = -\frac{1-\lambda^2}{2i+1}\frac{dK_i}{d\lambda},$$

whence, putting $\lambda = 0$,

$$G_i^0 = -\frac{1}{2i+1}\left(\frac{dK_i}{d\lambda}\right)_{\lambda=0},$$

and therefore

$$b_i = -(2i+1)\, G_i^0.$$

Treating in a similar manner the equation

$$G_i = \alpha_i X_i + \beta_i Y_i,$$

we have $\alpha_i = G_i^0$, and

$$\beta_1 = \left(\frac{dG_1}{d\lambda}\right)_{\lambda=0} = \tfrac{1}{4}(2i-1)\, K_i^0,$$

since

$$K_i = \frac{4}{2i-1}\frac{dG_i}{d\lambda}.$$

§ 14. Applying the general formulæ to the cases $i=0$ and $i=1$, we find

$$K = K^0 A_0 - G^0 B_0,$$
$$G = G^0 X_0 - \tfrac{1}{4} K^0 Y_0,$$

agreeing with the results in §§ 5 and 10,

and
$$K_1 = K_1^0 A_1 - 3 G_1^0 B_1,$$
$$G_1 = G_1^0 X_1 + \tfrac{1}{4} K_1^0 Y_1,$$

agreeing with the results in §§ 6 and 11, since

$$K_1^0 = (I+E)^0 = 2G^0,$$

and
$$G_1^0 = \tfrac{1}{3}(h'I - hE)^0 = -\tfrac{1}{6}K^0.$$

Values of the single series, §§ 15–17.

§ 15. It will be seen that in § 29 of this paper (pp. 28–30), Mr. Kleiber has only taken account of one of the two series which are involved in the expansions of K_i and G_i in powers of λ.

From p. 148 of Vol. XIX., we see that

$$1 + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9^2}{2.4\ldots12}\lambda^6 + \&c. = \frac{K'+K}{2K^0},$$

and
$$\frac{1}{2}\lambda + \frac{1^2}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8.10}\lambda^5 + \&c. = \frac{G'-G}{K^0},$$

so that the values of the series in equations (164) and (167) of Mr. Kleiber's paper should be $\dfrac{K+K'}{2K^0}$ and $\dfrac{G-G'}{K^0}$ instead of $\dfrac{2K}{\pi}$ and $\dfrac{2G}{\pi}$.

§ 16. Putting $x = \frac{1}{2}\lambda$ so that $h = \frac{1}{2} - x$ and $h' = \frac{1}{2} + x$, it is evident that we obtain as the value of the series

$$1 + \frac{1^2}{2!}x^2 + \frac{1^2.5^2}{4!}x^4 + \frac{1^2.5^2.9^2}{6!}x^6 + \&c.,$$

and
$$x + \frac{1^2}{3!}x^3 + \frac{1^2.5^2}{5!}x^5 + \frac{1^2.5^2.9^2}{7!}x^7 + \&c.,$$

the respective quantities

$$\frac{K\{\sqrt{(\frac{1}{2}+x)}\} + K\{\sqrt{(\frac{1}{2}-x)}\}}{2K\left(\frac{1}{\sqrt{2}}\right)},$$

and
$$\frac{G\{\sqrt{(\frac{1}{2}+x)}\}-G\{\sqrt{(\frac{1}{2}-x)}\}}{K\left(\frac{1}{\sqrt{2}}\right)},$$
where $K(p)$ is used to denote the value of K when the modulus is p.

§ 17. In the paper in Vol. XI. the series for K and G were derived from the expansion of a definite integral. No doubt the simplest way of obtaining these results would be to derive the series for G from the differential equation

$$(1-\lambda^2)\frac{d^2u}{d\lambda^2}-\tfrac{1}{4}u=0,$$

which is satisfied by it, and to deduce the series for K from that for G by means of the relation

$$K=-4\frac{dG}{d\lambda}.$$

Series for W and H, § 18.

§ 18. The expansions of W and H in powers of λ are respectively

$$W=G^0\left\{1-\frac{3}{2.4}\lambda^2-\frac{3^2.7}{2.4.6.8}\lambda^4-\frac{3^2.7^2.11}{2.4\ldots12}\lambda^6-\&c.\right\}$$
$$+\tfrac{1}{2}K^0\left\{\frac{1}{2}\lambda+\frac{1^2.5}{2.4.6}\lambda^3+\frac{1^2.5^2.9}{2.4.6.8.10}\lambda^5+\&c.\right\},$$

and

$$H=\tfrac{3}{2}K^0\left\{\frac{1}{3}-\frac{1}{2.4}\lambda^2-\frac{1^2.5}{2.4.6.8}\lambda^4-\frac{1^2.5^2.9}{2.4\ldots12}\lambda^6-\&c.\right\}$$
$$-3G^0\left\{\frac{1}{2}\lambda-\frac{3}{2.4.6}\lambda^3-\frac{3^2.7}{2.4.6.8.10}\lambda^5-\frac{3^2.7^2.11}{2.4\ldots14}\lambda^7-\&c.\right\},$$

whence it follows that

$$1-\frac{3}{2.4}\lambda^2-\frac{3^2.7}{2.4.6.8}\lambda^4-\frac{3^2.7^2.11}{2.4\ldots12}\lambda^6-\&c.=\frac{W-W'}{2G^0},$$
$$\frac{1}{2}\lambda+\frac{1^2.5}{2.4.6}\lambda^3+\frac{1^2.5^2.9}{2.4.6.8.10}\lambda^5+\&c.=\frac{W-W'}{K^0},$$
$$\frac{1}{3}-\frac{1}{2.4}\lambda^2-\frac{1^2.5}{2.4.6.8}\lambda^4-\frac{1^2.5^2.9}{2.4\ldots12}\lambda^6-\&c.=\frac{H'+H}{3K^0},$$
$$\frac{1}{2}\lambda-\frac{3}{2.4.6}\lambda^3-\frac{3^2.7}{2.4.6.8.10}\lambda^5-\frac{3^2.7^2.11}{2.4\ldots14}\lambda^7-\&c.=\frac{H'-H}{6G^0}.$$

The above series for W and $W\pm W'$ were given on p. 150 of Vol. XIX.

XXII

The functions E_i and I_i, § 19.

§ 19. I have not worked out the corresponding formulæ for E_i and I_i, which are considered by Mr. Kleiber in his paper. The series for $\frac{2E}{\pi}$ and $\frac{2I}{\pi}$ in equations (170) and (171) are obviously incorrect, and it would seem that they do not, as in the case of equations (164) and (167), form part of the required expansions.* The formulæ (168) and (169) are therefore also inaccurate. It may be added that the relations (123) and (124) need some modification, as when $i = 0$, they are intended to reduce to

$$\frac{dE}{dh} = \frac{I}{2h}, \quad \frac{dI}{dh} = -\frac{E}{2h'}.$$

It should be stated that Mr. Kleiber's paper was not put into type until after his death, so that he did not see any part of it in print. The paper has been printed from the manuscript without alteration, except that slips of the pen, when noticed, were corrected.

NOTE ON A PROBLEM IN THE THEORY OF NUMBERS.

By *W. W. Rouse Ball.*

THE elegant theorem on the resolution of numbers of a certain form into factors, which was given by Mr. Birch in the number of the *Messenger* for August (pp. 52–55), may be applied to determine the factors of the number 100895598169.

The partition of this number was proposed to Fermat by Mersenne; and, in a letter dated April 7, 1643, Fermat wrote to Mersenne, "Vous me demandez si le nombre 100895598169 est premier ou non, et une méthode pour découvrir dans l'espace d'un jour s'il est premier ou composé. A cette question, je réponds que ce nombre est composé et se fait du produit de ces deux: 898423 et 112303, qui sont premiers." The discovery of the method by which Fermat arrived at this result has been one of the puzzles of higher arithmetic.

Mr. Birch's theorem on the factors of a number N depends on the proposition that, if numbers x and y (such that $N>x>y$) can be found to satisfy the equation

$$x^2 = Ny + 1,$$

* The series for E and I were given on p. 149 of Vol. XXI.