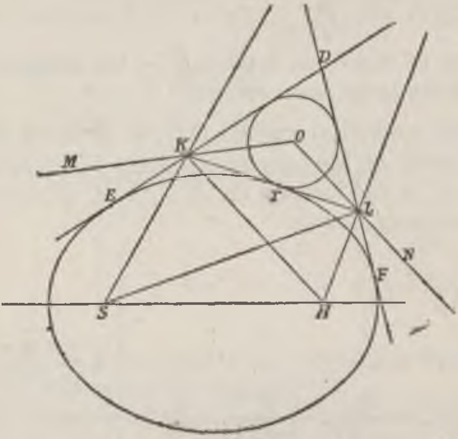
POLYGONS OF MINIMUM PERIMETER CIRCUMSCRIBED TO AN ELLIPSE.

By Prof. *G. B. Mathews.*

STEINER has stated the theorem that if a polygon of a prescribed number of sides be circumscribed to a given ellipse, its perimeter is a minimum when its vertices all lie on a confocal ellipse.* The following geometrical proof of this, if new, may be of interest.



Let S, H be the foci of an ellipse, DE, DF two fixed tangents, KTL a variable tangent, touching the ellipse in T and meeting the other tangents in K and L. Then

$$EK+KL+LF=DE+DF-(DK+DL-KL),$$

and therefore $EK+KL+LF$ will be a minimum, DE, DF being fixed, when $DK+DL-KL$ is a maximum.

Now the radius of the circle inscribed in the triangle DKL is equal to

$$\tfrac{1}{2}(DK+DL-KL)\tan\tfrac{1}{2}EDF,$$

so that this is also a maximum: and consequently the in-circle of the triangle DKL touches the ellipse at T. Let O be the centre of the circle, join SK, HK, OK, SL, HL, OL. Then by geometry

$$\angle\, OKT=OKD=EKM,$$

and $$\angle\, SKE=HKL;$$

* See Steiner's *Werke* II., pp. 618—20, or *Crelle* xlix., pp. 276—8; also *Werke* II., p. 418, or *Crelle* xxxvii., p. 189.

therefore $\angle SKM = HKO$; that is, OKM touches the confocal ellipse which passes through K. Similarly OLN touches the confocal ellipse which passes through L. These tangents intersect upon TO, which is normal to the given ellipse at T: and hence, by a well-known theorem, K, L lie on *the same* confocal.

It is now easy to infer the truth of Steiner's proposition, for if any two consecutive vertices, such as K, L, of the circumscribed polygon are not on the same confocal ellipse, we can alter the position of KTL so as to bring K, L upon the same confocal, and thus diminish the perimeter of the polygon. It is evident that at least one polygon of minimum perimeter must exist; and it follows from Poncelet's theory that there is an infinite number of such polygons: that they all have the same perimeter may be proved by considerations similar to those used in the proof of Graves's theorem.

The determination of the confocal ellipse on which the vertices lie may be effected by the aid of elliptic functions, and in the same way an expression may be found for the perimeter of any one of the series of circumscribed polygons.

Relatively to the outer ellipse, the polygon is of course an inscribed polygon of maximum perimeter. The result is rather curious from a statical point of view: namely, if an endless elastic string pass through a given number of small smooth rings free to move on a fixed smooth elliptic wire, there is an infinite number of positions of unstable equilibrium.

It may be observed that if a polygon of minimum perimeter be circumscribed to an ellipse, the points of contact of the sides will be the vertices of an inscribed polygon of maximum perimeter.

ON TWO CUBIC EQUATIONS.

By Professor *Cayley*.

Starting from the equations

$$2 + a = b^2,$$
$$2 + b = c^2,$$
$$2 + c = a^2,$$

then eliminating b, c, we find

$$(a^4 - 4a^2 + 2)^2 - (a + 2) = 0,$$

that is $$a^8 - 8a^6 + 20a^4 - 16a^2 - a + 2 = 0;$$