NOTE ON THE ORTHOTOMIC CURVE OF A SYSTEM OF LINES IN A PLANE.

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Considering *in plano* a singly infinite system of lines, then to each point of the plane there corresponds a line (not in general a unique line), and we can therefore express in terms of the coordinates (x, y) of the point the cosine-inclinations α, β of the line to the axes. The differential equation of the orthotomic curve is then $\alpha dx + \beta dy = 0$, and it is a well-known and easily demonstrable theorem that $\alpha dx + \beta dy$ is a complete differential, say it is $= dV$; the integral equation of the orthotomic curve is therefore $V = \int(\alpha dx + \beta dy)$, $=$ const., and we see further that V is a solution of the partial differential equation $\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 = 1$.

If the lines are the normals of the ellipse $\frac{X^2}{a} + \frac{Y^2}{b} = 1$, then, writing the equation of the normal at the point X, Y in the form

$$\frac{a}{X}(x - X) = \frac{b}{Y}(y - Y), = \lambda \text{ suppose},$$

we have
$$X = \frac{ax}{a + \lambda},\ Y = \frac{by}{b + \lambda};$$

and therefore
$$\frac{ax^2}{(a + \lambda)^2} + \frac{by^2}{(b + \lambda)^2} - 1 = 0,$$

which last equation determines λ as a function of x, y. We have α, β proportional to $\frac{X}{a}, \frac{Y}{b}$; or say we have $\alpha = M\frac{x}{a+\lambda}$, $\beta = M\frac{y}{b+\lambda}$, whence $\frac{1}{M^2} = \frac{x^2}{(a+\lambda)^2} + \frac{y^2}{(b+\lambda)^2}$; or, writing for convenience

$$\frac{x^2}{(a + \lambda)^2} + \frac{y^2}{(b + \lambda)^2} - \frac{k^2}{\lambda^2} = 0,$$

(viz., this equation defines k as a function of x, y and λ, that is of x and y), we have

$$\alpha = \frac{\lambda x}{k(a + \lambda)},\quad \beta = \frac{\lambda y}{k(b + \lambda)};$$

and we ought therefore to have

$$\frac{\lambda}{k}\left(\frac{xdx}{a + \lambda} + \frac{ydy}{b + \lambda}\right)$$

a complete differential, $= dV$.

This is easily verified, for from the assumed value

$$k = \lambda\left(\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} - 1\right)$$

we deduce

$$dk = 2\lambda\left(\frac{x\,dx}{a+\lambda} + \frac{y\,dy}{b+\lambda}\right) + d\lambda\left(\frac{ax^2}{(a+\lambda)^2} + \frac{by^2}{(b+\lambda)^2} - 1\right),$$

$$= 2\lambda\left(\frac{x\,dx}{a+\lambda} + \frac{y\,dy}{b+\lambda}\right);$$

and we have therefore

$$dV = \frac{\lambda}{k}\,\frac{dk}{2\lambda},\ = \tfrac{1}{2}\frac{dk}{k},$$

where k denotes a function of (x, y) defined as above; hence the equation $V=$ const. gives $k=$ const., or the equation of the orthotomic curve is given by the system of equations

$$\frac{ax^2}{(a+\lambda)^2} + \frac{by^2}{(b+\lambda)^2} - 1 = 0,$$

$$\frac{x^2}{(a+\lambda)^2} + \frac{y^2}{(b+\lambda)^2} - \frac{k^2}{\lambda^2} = 0,$$

where k is a constant; these equations (eliminating λ) give in fact the equation of the parallel curve of the ellipse, and k denotes the normal distance of a point on the curve from the ellipse. I recall that the first equation may be replaced by

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} - \frac{k}{\lambda} - 1 = 0,$$

and since the derived equation hereof in regard to λ is the second equation, we have the equation of the parallel curve in the known form

$$\text{Disct.}\ \{(\lambda+k)(\lambda+a)(\lambda+b) - (b+\lambda)x^2 - (a+\lambda)y^2\} = 0.$$

I notice further that considering k a function of x, y as above, we have

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 = \frac{1}{4k^2}\left\{\left(\frac{dk}{dx}\right)^2 + \left(\frac{dk}{dy}\right)^2\right\},$$

$$= \frac{\lambda^2}{k^2}\left\{\frac{x^2}{(a+\lambda)^2} + \frac{y^2}{(b+\lambda)^2}\right\},\ \text{that is}\ \left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 = 1,$$

as it should be.
