and in each of these new equations the suffixes are separated into two sets.

If one of the substitutions be applied to a single set of terms in any one of the equivalent equations already obtained, say to x_4, y_4, a new integral relation corresponding to a different value of the constant of integration will result.

Thus applying the substitution (B) to the terms x_4, y_4 in values of the equations first used, it follows that

$$\frac{y_1 + ky_2y_3y_4}{x_1 + kx_2x_3x_4} = \frac{y_4 + ky_1y_2y_3}{x_4 + kx_1x_2x_3}.$$

This equation may also be written

$$\frac{x_1y_4 - x_4y_1}{x_1y_1 - x_4y_4} + \frac{k(y_2y_3 - x_2x_3)}{1 - k^2x_2x_3y_2y_3} = 0.$$

Since $x_1 = x_2 = x_3 = 0$, $y_1 = y_2 = y_3 = 1$, $x_4 = \infty$, $y_4 = \frac{-1}{\sqrt{k}}$ are a special set of values satisfying this equation, the sum of four corresponding arguments $u_1 + u_2 + u_3 + u_4$ is $2K + iK'$.

ON THE LINEAR TRANSFORMATION OF THE ELLIPTIC DIFFERENTIAL.

By *W. Burnside*.

THE purpose of this paper is to exhibit in as simple and systematic form as possible the more important of the linear transformations of an elliptic integral of the first kind. I shall consider in order the transformation of the integral (i) into itself, (ii) into Weierstrass's normal form, (iii) into Legendre's, and (iv) into Riemann's normal form.

A single linear substitution can always be found which will transform three arbitrarily given values of a variable quantity into three other arbitrarily given values, each into each: namely, if α, β, γ, and α', β', γ' be the two sets of values, the substitution

$$\frac{x-\alpha}{x-\beta}\,\frac{\gamma'-\alpha'}{\gamma'-\beta'}=\frac{y-\alpha'}{y-\beta'}\,\frac{\gamma-\alpha}{\gamma-\beta}$$

changes α into α', β into β', and γ into γ'. Also, a single linear substitution can always be found which will interchange two pairs of values.

Thus, the equation

$$\frac{x-\alpha\,.\,x-\beta}{x-\gamma\,.\,x-\delta}=\frac{y-\alpha\,.\,y-\beta}{y-\gamma\,.\,y-\delta},$$

after casting out the factor $x-y$ is clearly linear in both x and y, while from its form it is evident that the values $\alpha, \beta, \gamma, \delta$ of x correspond to the values $\beta, \alpha, \delta, \gamma$ of y.

The elliptic integral will throughout be taken in the form

$$\frac{dx}{\sqrt{(x-\alpha\,.\,x-\beta\,.\,x-\gamma\,.\,x-\delta}}.$$

(i) To transform the integral into itself, the substitution must clearly reproduce the four values $\alpha, \beta, \gamma, \delta$ in an altered order. This may happen in three different ways. The values may be interchanged in pairs; or three may be interchanged cyclically, one remaining unaltered; or all four may be interchanged cyclically.

Four quantities may be separated into pairs in three different ways, and corresponding to these there will be the three substitutions

$$\frac{x-\alpha\,.\,x-\beta}{x-\gamma\,.\,x-\delta}=\frac{y-\alpha\,.\,y-\beta}{y-\gamma\,.\,y-\delta},$$

$$\frac{x-\alpha\,.\,x-\gamma}{x-\beta\,.\,x-\delta}=\frac{y-\alpha\,.\,y-\gamma}{y-\beta\,.\,y-\delta},$$

$$\frac{x-\varpi\,.\,x-\delta}{x-\beta\,.\,x-\gamma}=\frac{y-\alpha\,.\,y-\delta}{y-\beta\,.\,y-\gamma}.$$

On casting out the factors $x-y$, these become

$$\frac{x-p_1}{x-q_1}=-\frac{y-p_1}{y-q_1},$$

&c.,

where $p_1 = \dfrac{\alpha\beta - \gamma\delta + \sqrt{(\alpha - \gamma.\beta - \gamma.\alpha - \delta.\beta - \delta)}}{\alpha + \beta - \gamma - \delta}$,

$$q_1 = \frac{\alpha\beta - \gamma\delta - \sqrt{(\alpha - \gamma.\beta - \gamma.\alpha - \delta.\beta - \delta)}}{\alpha + \beta - \gamma - \delta},$$

and p_2, q_2, p_3, q_3 can be written down by interchanging β, γ, and δ.

These three substitutions, since each is its own reciprocal, clearly transform the differential exactly into itself, so that

$$\frac{dx}{\sqrt{(x-\alpha.x-\beta.x-\gamma.x-\delta)}} = \pm \frac{dy}{\sqrt{(y-\alpha.y-\beta.y-\gamma.y-\delta}}.$$

Considering next the second way of reproducing the four values unchanged, the substitution which changes α, β, γ into β, γ, α is

$$\frac{x-\alpha.\alpha-\beta}{x-\beta.\alpha-\gamma} = \frac{y-\beta.\gamma-\alpha}{y-\gamma.\gamma-\beta},$$

and if this leaves δ unchanged,

$$\frac{\delta-\alpha.\alpha-\beta}{\delta-\beta.\alpha-\gamma} = \frac{\delta-\beta.\gamma-\alpha}{\delta-\gamma.\gamma-\beta},$$

or $(\delta-\beta)^2(\gamma-\alpha)^2 - (\delta-\alpha)(\delta-\gamma)(\alpha-\beta)(\beta-\gamma) = 0.$

Now $(\delta-\alpha)(\beta-\gamma) + (\delta-\beta)(\gamma-\alpha) + (\delta-\gamma)(\alpha-\beta) = 0,$

so that the condition may be written

$$(\delta-\alpha)^2(\beta-\gamma)^2 + (\delta-\beta)^2(\gamma-\alpha)^2 + (\delta-\gamma)^2(\alpha-\beta)^2 = 0,$$

or $$g_2 = 0,$$

where g_2 is the quadrinvariant of the quartic

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta).$$

If then this condition, $g_2 = 0$, is satisfied there will be eight more linear substitutions which will transform the differential into itself.

[In this case, since when the substitution is repeated three times it leads to identity, the relation between the differentials will be

$$\frac{dx}{\sqrt{(x-\alpha.x-\beta.x-\gamma.x-\delta)}} = \frac{\omega\, dy}{\sqrt{(y-\alpha.y-\beta.y-\gamma.y-\delta)}},$$

where $$\omega^3 = 1].$$

Lastly, if $\alpha, \beta, \gamma, \delta$ can be interchanged cyclically, the substitution

$$\frac{x-\alpha.\delta-\beta}{x-\beta.\delta-\gamma}=\frac{y-\beta.\gamma-\alpha}{y-\gamma.\gamma-\beta}$$

must be satisfied by $x=\delta,\ y=\alpha$;

or
$$\frac{\delta-\alpha}{\delta-\gamma}=\frac{\alpha-\beta}{\beta-\gamma},$$

i.e.
$$(\delta-\alpha)(\beta-\gamma)-(\delta-\gamma)(\alpha-\beta)=0.$$

This substitution when repeated twice interchanges α and γ, and also β and δ; and when repeated three times interchanges $\alpha, \delta, \gamma, \beta$ cyclically.

Similarly the conditions that it may be possible to interchange cyclically, $\alpha, \beta, \delta, \gamma$ (or $\alpha, \gamma, \delta, \beta$), and $\alpha, \gamma, \beta, \delta$ (or $\alpha, \delta, \beta, \gamma$) will be

$$(\delta-\beta)(\gamma-\alpha)-(\delta-\gamma)(\alpha-\beta)=0,$$

and
$$(\delta-\alpha)(\beta-\gamma)-(\delta-\beta)(\gamma-\alpha)=0.$$

Hence, finally, the condition that it may be possible in one of the three independent ways to interchange the four quantities cylically is

$$[(\delta-\beta)(\gamma-\alpha)-(\delta-\gamma)(\alpha-\beta)][(\delta-\gamma)(\alpha-\beta)-(\gamma-\alpha)(\beta-\gamma)]$$
$$\times[(\delta-\alpha)(\beta-\gamma)-(\delta-\beta)(\gamma-\alpha)]=0,$$

or
$$g_3=0,$$

where g_3 is the cubinvariant of the quartic; and if this condition is satisfied, there will be four linear substitutions leading to the relation between the differentials

$$\frac{dx}{\sqrt{(x-\alpha.x-\beta.x-\gamma.x-\delta)}}=\frac{idy}{\sqrt{(y-\alpha.y-\beta.y-\gamma.y-\delta)}},$$

where
$$i^2=-1.$$

(ii) Weierstrass's normal form is

$$\frac{dy}{\sqrt{\{4(y-e_1)(y-e_2)(y-e_3)\}}}$$

with the condition $e_1+e_2+e_3=0$.

If it is possible to transform $\alpha, \beta, \gamma, \delta$ into e_1, e_2, e_3, ∞, then the substitution

$$\frac{x-\alpha.e_3-e_1}{x-\beta.e_3-e_2}=\frac{y-e_1.\gamma-\alpha}{y-e_2.\gamma-\beta}$$

must be satisfied by $x=\delta,\ y=\infty$.

Hence $(\delta-\alpha)(\beta-\gamma)(e_3-e_1)+(\delta-\beta)(\gamma-\alpha)(e_3-e_2)=0,$

or $(\delta-\alpha)(\beta-\gamma)e_1+(\delta-\beta)(\gamma-\alpha)e_2+(\delta-\gamma)(\alpha-\beta)e_3=0,$

Also $e_1 + e_2 + e_3=0;$

therefore $$\frac{e_1}{(\delta-\beta)(\gamma-\alpha)-(\delta-\gamma)(\alpha-\beta)}$$

$$=\frac{e_2}{(\delta-\gamma)(\alpha-\beta)-(\delta-\alpha)(\beta-\gamma)}=\frac{e_3}{(\delta-\alpha)(\beta-\gamma)-(\delta-\beta)(\gamma-\alpha)}.$$

The denominators here are irrational invariants of the quartic, and hence, except as regards the order, the ratics $e_1 : e_2 : e_3$ are the same for all transformations to this normal form.

If each of these fractions is equal to h, then

$$\frac{e_2e_3+e_3e_1+e_1e_2}{h^2}=-\tfrac{3}{2}\Sigma(\delta-\alpha)^2(\beta-\gamma)^2=-2^2 3^2 g_2,$$

and $$\frac{e_1e_2e_3}{h^3} = 2^4 3^3 g_3.$$

It is easy to verify that the substitution here in question may be written in the form

$$x-\delta=\frac{3h'}{y-h'\dfrac{d\log\Delta}{d\delta}},$$

where Δ is the square root of the discriminant; for this gives

$$x-\alpha=(\delta-\alpha)\frac{y-e_1}{y-h'\dfrac{d\log\Delta}{d\delta}},$$

&c.,

where $$e_1=h'\left[\frac{d\log\Delta}{d\delta}-\frac{3}{\delta-\alpha}\right]$$

$$=h'\left[\frac{1}{\delta-\beta}+\frac{1}{\delta-\gamma}-\frac{2}{\delta-\alpha}\right],$$

which is the same as the previous value, if

$$h'=h(\delta-\alpha)(\delta-\beta)(\delta-\gamma).$$

Finally, on making the substitution the relation between the differentials is

$$\frac{dx}{\sqrt{(x-\alpha . x-\beta . x-\gamma . x-\delta)}}=\sqrt{(3h)}\,\frac{dy}{\sqrt{(y-e_1 . y-e_2 . y-e_3)}}$$

$$=\frac{\sqrt{(3h)}\,dy}{\sqrt{(y^3-2^2.3^2.h^2g_2y-2^4.3^3h^3g_3)}}$$

$$=\frac{dy}{\sqrt{(4y^3-g_2y-g_3)}},$$

if $h=\frac{1}{12}$.

The correspondiug substitution for this particular value to h is

$$x-\delta=\frac{3D}{y-\frac{dD}{d\delta}},$$

where $$D\equiv\tfrac{1}{12}(\delta-\alpha)(\delta-\beta)(\delta-\gamma).$$

The three other independent transformations to this normal form may be obtained by combining the above with the three previously found substitutions which transform the differential into itself.

It is perhaps worth pointing out that this transformation is rational in the coefficients when the differential is given in the form $dx/\sqrt{\{(x-\delta)(x^3+3p_1x^2+3p_2x+p_3)\}}$, the cubic factor being supposed irreducible.

(iii) Legendre's normal form.

If $\alpha, \beta, \gamma, \delta$ transform into $1, -1, 1/k, -1/k$, then the substitution

$$\frac{x-\alpha . 1-k}{x-\beta . 1+k}=\frac{y-1 . \gamma-\alpha}{y+1 . \gamma-\beta}$$

must be satisfied by $x=\delta$, $y=-1/k$.

Hence $$\frac{\delta-\alpha . 1-k}{\delta-\beta . 1+k}=\frac{1+k . \gamma-\alpha}{1-k . \gamma-\beta},$$

or $$\frac{1-2k+k^2}{\delta-\beta . \gamma-\alpha}=-\frac{1+2k+k^2}{\delta-\alpha . \beta-\gamma}=\frac{4k}{\delta-\gamma . \alpha-\beta}$$

$$=\frac{2(1+k^2)}{(\delta-\beta)(\gamma-\alpha)-(\delta-\alpha)(\beta-\gamma)}=\frac{-(1+6k+k^2)}{(\delta-\alpha)(\beta-\gamma)-(\delta-\gamma)(\alpha-\beta)}$$

$$=\frac{-(1-6k+k^2)}{(\delta-\gamma)(\alpha-\beta)-(\delta-\beta)(\gamma-\alpha)}.$$

Combining these to form the invariants of the quartic, there results

$$\left[\frac{(1-k)^4+(1+k)^4+16k^2}{2^3.3.g_2}\right]^3 = \left[\frac{2(1+k^2)(1+6k+k^2)(1-6k+k^2)}{2^4.3^3 g_3}\right]^2,$$

or
$$\frac{(1+14k^2+k^4)^3}{(1-33k^2-33k^4-k^6)^2} = \frac{g_2^3}{3^3 g_3^2},$$

and the roots of this equation, called by Prof. F. Klein the "octohedral equation," will give the various possible values of Legendre's Modulus.

(iv) Riemann's normal form, or

$$dy/\sqrt{\{y(1-y)(1-\lambda y)}.$$

If $\alpha, \beta, \gamma, \delta$ transform into $0, 1, \infty, 1/\lambda$, then, as before,

$$\frac{x-\alpha}{x-\beta} = \frac{y}{y-1}\,\frac{\gamma-\alpha}{\gamma-\beta}$$

is satisfied by $x=\delta$ and $y=1/\lambda$.

Hence
$$\frac{\delta-\alpha.\gamma-\beta}{\delta-\beta.\gamma-\alpha} = \frac{1}{1-\lambda},$$

and
$$\frac{1}{\delta-\alpha.\beta-\gamma} = \frac{\lambda-1}{\delta-\beta.\gamma-\alpha} = \frac{-\lambda}{\delta-\gamma\ \alpha-\beta}.$$

Hence, again, as before,

$$\left[\frac{1+\lambda^2+(\lambda-1)^2}{2^3.3.g_2}\right]^3 = \left[\frac{(2\lambda-1)(\lambda^2-1)}{2^4.3^3 g_3}\right]^2,$$

or
$$\frac{4(\lambda^2-\lambda+1)^3}{(2\lambda^3-3\lambda^2-3\lambda+2)^2} = \frac{g_2^3}{3^3 g_3^2}.$$

This is a form of Prof. Klein's "dihedral equation."

The forms of the relations between the differentials in these two latter cases is

$$\frac{dx}{\sqrt{(x-\alpha.x-\beta.x-\gamma.x-\delta)}} = \frac{Mdy}{\sqrt{(1-y^2.1-k^2y^2)}},$$

and
$$\frac{dx}{\sqrt{(x-\alpha.x-\beta.x-\gamma.x-\delta)}} = \frac{Ndy}{\sqrt{(y.1-y.1-\lambda y)}}.$$

The constant factors M and N are *not* irrational invariants of the original quartic, and cannot therefore be determined by algebraical equations whose coefficients are functions of g_2 and g_3, as k and λ can.