NOTE ON A RECURRING FORMULA FOR $\sigma(n)$.

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Value of a recurring formula, §§ 1–3.

§ 1. In Vol. xx. (1884), p. 116, of the *Quarterly Journal* it was shown that if $n \equiv 7$, mod. 8, then

$$\sigma(n) - 2\sigma(n-4) + 2\sigma(n-16) - 2\sigma(n-36) + \&c. = 0,^*$$

$\sigma(n)$ denoting the sum of the divisors of n.

As the arguments decrease more rapidly in this series than in any of the other recurring formulæ for $\sigma(n)$, it is interesting to examine the restriction with respect to the form of n.

It will be shown that the formula holds good for all uneven values of n which are not expressible as the sum of three squares; and as no number which $\equiv 7$, mod. 8, is so expressible, the theorem is necessarily true for numbers of this form.

§ 2. In the first place, suppose $n \equiv 3$, mod. 4. We find that, for such a value of n, the expression

$$\sigma(n) - 2\sigma(n-4) + 2\sigma(n-16) - 2\sigma(n-36) + \&c.$$

is equal to a quantity derived from the compositions of n as a sum of three squares in the following manner.

Consider any composition of n as the sum of three squares, all of which must be uneven since $n \equiv 3$, mod. 4. Such a composition must be of one of the three forms

$$\text{(i) } a^2 + b^2 + c^2, \quad \text{(ii) } a^2 + b^2 + b^2, \quad \text{(iii) } a^2 + a^2 + a^2,$$

a, b, c being different uneven numbers.

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* In Euler's original formula,

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \&c. = 0,$$

where $\sigma(0) = n$, the argument of the $(2r+1)^{\text{th}}$ term is less than that of the first by $\frac{1}{2}(3r^2 + r)$; in the formula,

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + \&c. = 0,$$

where $\sigma(0) = \frac{1}{3}n$, the diminution is $2r^2 + r$: while, in the formula in the text, it is $16r^2$.

For example, the eleventh term in the three series is

$$\sigma(n-40), \quad 21\sigma(n-55), \quad 2\sigma(n-400)$$

respectively.

From (i) we form the quantity

$$(-1)^{\frac{1}{2}(a-1)}8a+(-1)^{\frac{1}{2}(b-1)}8b+(-1)^{\frac{1}{2}(c-1)}8c,$$

from (ii), $$(-1)^{\frac{1}{2}(a-1)}4a+(-1)^{\frac{1}{2}(b-1)}8b,$$

and from (iii), $$(-1)^{\frac{1}{2}(a-1)}4a.*$$

We then add together the quantities so derived from all the compositions of n. This sum is equal to the above σ-expression.

As an example, let $n=27$, which is $\equiv 3$, mod. 8.

The compositions of 27 as a sum of three squares are

$$5^2+1^2+1^2, \quad 3^2+3^2+3^2.$$

The quantity formed from them is

$$20+8-12=16.$$

The theorem therefore is

$$\sigma(27)-2\sigma(23)+2\sigma(11)=16,$$

viz. $$40-48+24=16.$$

As additional examples, let $n=59$ and 75 which are both $\equiv 7$, mod. 8.

The compositions of 59 and 75 as sum of three squares are

$$7^2+3^2+1^2, \quad 5^2+5^2+3^2,$$

and $$7^2+5^2+1^2, \quad 5^2+5^2+5^2,$$

respectively.

The quantities formed from them are

$$-56-24+8+40-12=-44,$$

and $$-56+40+8+20 \quad = \quad 12$$

respectively.

The corresponding σ-theorems are therefore

$$\sigma(59)-2\sigma(55)+2\sigma(43)-2\sigma(23)=-44,$$

and $$\sigma(75)-2\sigma(71)+2\sigma(59)-2\sigma(39)+2\sigma(11)=12,$$

viz. $$60-144+88-48=-44,$$

and $$124-144+120-112+24=12.$$

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* In the formation of these numbers, we only take into account different squares occurring in the same composition, and the square root is multiplied by 8, when the other two squares in the composition are different from each other, and by 4 when they are the same.

Since a number $\equiv 3$, mod. 4 can only be the sum of three squares if it is also $\equiv 3$, mod. 8, it follows from the above theorem that if $n \equiv 7$, mod. 8, the σ-expression is necessarily equal to zero.

§ 3. If $n \equiv 1$, mod. 4, the quantity which represents the value of the σ-expression may be calculated as follows.

We write down all the compositions of n as a sum of three squares. Two of these must be even (including 0^2 as an even square) and one uneven. The different forms are

$$\text{(i)}\quad a^2+\beta^2+\gamma^2,\quad \text{(ii)}\quad a^2+\beta^2+0^2,$$

$$\text{(iii)}\quad a^2+\beta^2+\beta^2,\quad \text{(iv)}\quad a^2+0^2+0^2,$$

where a is an uneven square, and β, γ are even squares different from each other.

From (i), we form $(-1)^{\frac{1}{2}(a-1)}8a$, from (ii), $(-1)^{\frac{1}{2}(a-1)}4a$; from (iii), $(-1)^{\frac{1}{2}(a-1)}4a$; and from (iv), $(-1)^{\frac{1}{2}(a-1)}a$: and we add together the quantities so derived from all the compositions.

For example, let $n=25$. The compositions are

$$5^2+0^2+0^2,\quad 3^2+4^2+0^2,$$

from which we form

$$5-12=-7,$$

and the theorem is

$$\sigma(25)-2\sigma(21)+2\sigma(9)=-7,$$

viz.
$$31-64+26=-7.$$

As additional examples, let $n=53$ and 65. The compositions are

$$7^2+2^2+0^2,\quad 1^2+6^2+4^2,$$

and
$$7^2+4^2+0^2,\quad 5^2+6^2+2^2,\quad 1^2+8^2+0^2,$$

respectively, giving

$$-28+8=-20,$$

and
$$-28+40+4=16.$$

The corresponding σ-theorems thus are

$$\sigma(53)-2\sigma(49)+2\sigma(37)-2\sigma(17)=-20,$$

and
$$\sigma(65)-2\sigma(61)+2\sigma(49)-2\sigma(29)+2\sigma(1)=16,$$

viz.
$$54-114+76-36=-20,$$

$$84-124+114-60+2=16.$$

Expressions for the value in terms of the function E, §§ 4, 5.

§ 4. The number of the representations of a number n as the sum of two squares is equal to $4E(n)$, where $E(n)$ denotes the excess of the number of divisors of n which $\equiv 1$, mod. 4 above the number of those which $\equiv 3$, mod. 4.*

The quantity calculated in § 2 may therefore be expressed by the formula

$$4\{E(n-1) - 3E(n-9) + 5E(n-25) - \&c.\}.$$

This formula also expresses the quantity calculated in § 3 if n is not a square number; but when n is a square number r^2, we have to include the term $(-1)^{\frac{1}{2}(r-1)}$. This term may be included by allowing the formula to extend to $E(0)$, which then occurs, and putting $E(0) = \frac{1}{4}$.

Taking the examples in § 2, by putting $n = 27$, 59, 75, we have the E-expressions

$$4\{E(26) - 3E(18) + 5E(2)\},$$

$$4\{E(58) - 3E(50) + 5E(34) - 7E(10)\},$$

$$4\{E(74) - 3E(66) + 5E(50) - 7E(26)\},$$

giving

$$4(2 - 3 + 5) = 16,$$

$$4(2 - 9 + 10 - 14) = -44,$$

$$4(2 - 0 + 15 - 14) = 12,$$

respectively.

Similarly the corresponding expressions derived from the examples in § 3 are

$$4\{E(24) - 3E(16) + 5E(0)\},$$

$$4\{E(52) - 3E(44) + 5E(28) - 7E(4)\},$$

$$4\{E(64) - 3E(56) + 5E(40) - 7E(16)\},$$

giving

$$4(0 - 3 + \tfrac{5}{4}) = -7,$$

$$4(2 - 0 + 0 - 7) = -20,$$

$$4(1 - 0 + 10 - 7) = 16,$$

respectively.

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* The function $E(n)$ is considered in *Proc. Lond. Math. Soc.*, vol. xv., pp. 104–122. A table of $E(n)$ up to $n = 1000$ is given on p. 106 of that paper.

It will be noticed that when $n \equiv 1$, mod. 4, all the arguments $n-1$, $n-9$, $n-25$, ... are $\equiv 0$, mod. 4. We may therefore divide all the arguments throughout by 4, *ex. gr.*, we may replace the above expressions by

$$4\{E(6) - 3E(4) + 5E(0)\},$$

$$4\{E(13) - 3E(11) + 5E(7) - 7E(1)\},$$

$$4\{E(16) - 3E(14) + 5E(10) - 7E(4)\},$$

respectively.

§ 5. By equating the σ- and E-formulæ, we find for all uneven values of n,

$$\sigma(n) - 2\sigma(n-4) + 2\sigma(n-16) - 2\sigma(n-36) + \&c.$$
$$= 4\{E(n-1) - 3E(n-9) + 5E(n-25) - \&c.\},$$

where $E(0) = \frac{1}{4}$.

Considering separately the cases when $n \equiv 3$ and $\equiv 1$, mod. 4, we deduce the results:

(i) if $n \equiv 3$, mod. 4, then

$$\sigma(n) - 2\sigma(n-4) + 2\sigma(n-16) - 2\sigma(n-36) + \&c.$$
$$= 4\{E(n_1) - 3E(n_1-4) + 5E(n_1-12) - 7E(n_1-24) + \&c.\},$$

where $n_1 = \frac{1}{2}(n-1)$, and the numbers 4, 12, 24, ... are the quadruples of the triangular numbers. If $n \equiv 7$, mod. 8, $n_1 \equiv 3$, mod. 4, so that all the E-terms vanish, and the σ-expression is equal to zero.

(ii) if $n \equiv 1$, mod. 4,

$$\sigma(n) - 2\sigma(n-4) + 2\sigma(n-16) - 2\sigma(n-36) + \&c.$$
$$= 4\{E(n_2) - 3E(n_2-2) + 5E(n_2-6) - 7E(n_2-10) + \&c.,$$

where $n_2 = \frac{1}{4}(n-1)$ and the numbers 2, 6, 12, ... are the doubles of the triangular numbers.

The formula (i) was given in vol. XX., p. 129, of the *Quarterly Journal.*

The number of representations of a number as a sum of three squares, §§ 6–9.

§ 6. There is no simple function depending upon the divisors of n (corresponding to the function $E(n)$ in the case of two squares) by means of which we may express the number of representations of n as a sum of three squares.

It is easy to see that this number must be equal to the sum of the series

$$4\{E(n)+2E(n-1)+2E(n-4)+2E(n-9)+\&c.\},$$

for the first term is equal to the number of representations in which the square 0^2 occurs, the second is equal to the number in which 1^2 occurs, the third to the number in which 2^2 occurs, and so on.

We may, however, obtain an E-series in which the arguments descend more rapidly, when n is uneven. For if $n\equiv 3$, mod 4, the three squares in each composition must be all uneven, and the number of representations is, therefore, evidently equal to

$$8\{E(n-1)+E(n-9)+E(n-25)+\&c.\}.$$

This expression is also deducible at once from the preceding formula since in this case the alternate terms $E(n)$, $E(n-4)$, &c. vanish. If $n\equiv 7$, mod 8, all the terms vanish.

§ 7. Referring to the examples $n=27$, 59, and 75, considered in § 2, we see that the numbers of representations are $24+8=32$, $48+24=72$, and $48+8=56$ respectively, while the above formula gives in the respective cases

$$8\{E(26)+E(18)+E(2)\},$$

$$8\{E(58)+E(50)+E(34)+E(10)\},$$

and

$$8\{E(74)+E(66)+E(50)+E(26)\};$$

that is

$$8(2+1+1)=32,$$

$$8(2+3+2+2)=72,$$

and

$$8(2+0+3+2)=56.$$

§ 8. When $n\equiv 1$, mod 4, we can obtain a series in which the arguments descend much more rapidly, for in this case two squares must be even and one uneven, and it is easy to see that the number of representations must be equal to

$$6\{E(n)+2E(n-4)+2E(n-16)+\&c.\}.$$

Now it can be shown that

$$E(n)-2E(n-4)+2E(n-16)-\&c.$$

is equal to zero if n is not a square number, and is equal to $(-1)^{\frac{1}{2}(\sqrt{n}-1)}\sqrt{n}$ if n is a square number.

We may therefore express the number of representations of n by either of the formulæ

$$12\{E(n)+2E(n-16)+2E(n-64)+\&c.\}-[(-1)^{\frac{1}{2}(\sqrt{n}-1)}6\sqrt{n}],$$

or

$$24\{E(n-4)+E(n-36)+E(n-100)+\&c.\}+[(-1)^{\frac{1}{2}(\sqrt{n}-1)}6\sqrt{n}],$$

in both of which the final term, enclosed in square brackets, occurs only when n is a square number.

§ 9. The arguments in these series diminish very fast. Taking the second formula, the squares which occur in the arguments are the quadruples of the uneven squares, viz. 4, 36, 100, 196, 324, 484, 676, 900, ..., so that for example, only eight terms are required in order to express the number of representations of a number $\equiv 1$, mod. 4, which is intermediate to 900 and 1156.

Taking the examples $n = 25$, 53, and 65 considered in § 3, we see that the numbers of representations are

$$6+24=30,\quad 24+48=72,\quad 24+48+24=96,$$

and the second formula gives, for these values of n,

$$24E(21)+6\times 5 \quad =0+30=30,$$

$$24\{E(49)+E(17)\}=24(1+2)=72,$$

$$24\{E(61)+E(29)\}=24(2+2)=96.$$

The first of the two formulæ gives in these three cases

$$12\{E(25)+2E(9)\}-6\times 5=12(3+2)-30=30,$$

$$12\{E(53)+2E(37)\}=12(2+4)=72,$$

$$12\{E(65)+2E(49)+2E(1)\}=12(4+2+2)=96.$$

Taking as another example a larger square number, 121, the second formula gives as the number of representations

$$24\{E(117)+E(85)+E(21)\}-6\times 11$$

$$=24(2+4+0)-66=78.$$

The compositions of 121 are

$$11^2+0^2+0^2,\quad 9^2+6^2+4^2,\quad 7^2+6^2+6^2,$$

giving $6+48+24=78$ representations.

Relations between E-formulæ, § 10.

§ 10. By comparing the E-formula of §§ 6 and 8, we see that if $n \equiv 1$, mod. 4,

$$2\{E(n)+2E(n-1)+2E(n-4)+\&c.\}$$
$$=3\{E(n)+2E(n-4)+2E(n-16)+\&c.\}$$
$$=6\{E(n)+2E(n-16)+2E(n-64)+\&c.\}-[(-1)^{\frac{1}{2}(\sqrt{n}-1)}3\sqrt{n}]$$
$$=12\{E(n-4)+E(n-36)+E(n-100)+\&c.\}+[(-1)^{\frac{1}{2}(\sqrt{n}-1)}3\sqrt{n}],$$

the additional term, enclosed in square brackets, which occurs in the last two formulæ, being only included when n is a square number.

Formulæ connecting the functions σ and Δ', § 11.

§ 11. It can be shown that, if $\Delta'(n)$ denotes the sum of those divisors of n whose conjugates are uneven, then

$$\Delta'(n)-2\Delta'(n-1)+2\Delta'(n-4)-2\Delta'(n-9)+\&c.$$

is equal to zero or $(-1)^{n-1}n$ according as n is not, or is, a square number.

If n be uneven $\Delta'(n)=\sigma(n)$, and we may therefore write formula, when n is uneven, in the form

$$\sigma(n)+2\sigma(n-4)+2\sigma(n-16)+2\sigma(n-64)+\&c.$$
$$=2\{\Delta'(n-1)+\Delta'(n-9)+\Delta'(n-25)+\&c.\}+[(-1)^{n-1}n],$$

where the additional term, in square brackets, is only to be included when n is a square number.

Now from § 1, if $n \equiv 7$, mod. 8,

$$\sigma(n)-2\sigma(n-4)+2\sigma(n-16)-2\sigma(n-36)+\&c.=0,$$

so that, when n is of this form,

$$\Delta'(n-1)+\Delta'(n-9)+\Delta'(n-25)+\&c.$$
$$=\sigma(n)+2\sigma(n-16)+2\sigma(n-64)+\&c.$$
$$=2\sigma(n-4)+2\sigma(n-36)+2\sigma(n-100)+\&c.$$

Representations by five squares, § 12.

§ 12. It may be remarked that if $n \equiv 1$, mod. 8, the number of representations of n as a sum of five squares is equal to

$$10\{\sigma(n)+2\sigma(n-4)+2\sigma(n-16)+\&c.\},$$

and that, if $n \equiv 3$, mod. 4, the number of representations is equal to

$$20\{\sigma(n) + 2\sigma(n-4) + 2\sigma(n-16) + \&c.\}.$$

Thus, when $n \equiv 7$, mod. 8, the number of representations is equal to

$$40\{\sigma(n) + 2\sigma(n-16) + 2\sigma(n-64) + \&c.\}$$
$$= 80\{\sigma(n-4) + \sigma(n-36) + \sigma(n-100) + \&c\}.$$

Analytical Formulæ, § 13.

§ 13. The groups of analytical formulæ from which all the results contained in this note may be derived are the following.

If n denote any number and m any uneven number, then

I.

$$\{\Sigma_{-\infty}^{\infty} q^{m^2}\}\{\Sigma_{-\infty}^{\infty} q^{4n^2}\}\{\Sigma_{-\infty}^{\infty} (-1)^n q^{4n^2}\} = \Sigma_{-\infty}^{\infty} (-1)^{\frac{1}{2}(m-1)} m q^{m^2},$$

II.

$$\{\Sigma_{-\infty}^{\infty} q^{n^2}\}^2 = 1 + 4\Sigma_1^{\infty} E(n) q^n,$$
$$\{\Sigma_{-\infty}^{\infty} q^{n^2}\}\{\Sigma_{-\infty}^{\infty} q^{4n^2}\} = 2\Sigma_0^{\infty} E(4n+1) q^{4n+1},$$

III.

$$\{\Sigma_{-\infty}^{\infty} q^{m^2}\}\{\Sigma_{-\infty}^{\infty} q^{4n^2}\}^3 = 2\Sigma_0^{\infty} \sigma(4n+1) q^{4n+1},$$
$$\{\Sigma_{-\infty}^{\infty} q^{m^2}\}^3\{\Sigma_{-\infty}^{\infty} q^{4n^2}\} = 2\Sigma_0^{\infty} \sigma(4n+3) q^{4n+3},$$

IV.

$$\frac{\Sigma_{-\infty}^{\infty} (-1)^n n^2 q^{n^2}}{\Sigma_{-\infty}^{\infty} q^{n^2}} = \Sigma_1^{\infty} \Delta'(n) q^n.$$

Thus for example, the theorem in § 2 may be obtained by multiplying the second formula of III. by $\Sigma_{-\infty}^{\infty} (-1)^n q^{4n^2}$, and reducing the left-hand side by I. We thus find

$$\{\Sigma_{-\infty}^{\infty} q^{m^2}\}^2 \{\Sigma_{-\infty}^{\infty} (-1)^{\frac{1}{2}(m-1)} m q^{m^2}\}$$
$$= 2\{\Sigma_0^{\infty} \sigma(4n+3) q^{4n+3}\}\{\Sigma_{-\infty}^{\infty} (-1)^n q^{4n^2}\}.$$
