There results for $P(u)$ the simpler form

$$P(u) = A\frac{s_x s_y + f(x, y)}{(x-y)^2} + A',$$

where

$$f(x, y) = a_0x^2y^2 + 2a_1(x^2y + xy^2) + a_2(x^2 + 4xy + y^2) + 2a_3(x+y) + a_4.$$

Finally the first two terms in the expansion of $P(u)$ in terms of u applied to the last form give

$$A = \tfrac{1}{2}, \quad A' = 0.$$

If the terms are all made homogeneous by writing, as Prof. Klein does, x_1/x_2 and y_1/y_2 for x and y, then

$$P(u) = \frac{s_{x_1x_2}\, s_{y_1y_2} + \frac{1}{1\cdot 2}\left(y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2}\right)^2 s^2_{x_1x_2}}{2\,(x_1y_2 - x_2y_1)^2},$$

in which form it is obviously a covariant of the original quartic.

ON ARITHMETICAL SERIES.

(Continued from p. 19).

By Professor *Sylvester*.

Part II.*

Explicit Primes.

In this part I shall consider the asymptotic limits to the number of primes of certain *irreducible* linear forms $mz + r$ comprised between a number x and a given fractional multiple thereof kx, the method of investigation being such that the asymptotic limits determined will be unaffectd by the value of r, and will be the same for all values of m which have the same totient. The simplest case, and the foundation of all that follows, is that in which $k = 0$ and $m = 2$: this will form the subject of the ensuing chapter which may be regarded as a supplement to Tschebyscheff's celebrated memoir of 1850,† and as superseding my article thereon in Vol. IV. of the *Amer. Math. Journ.*

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* I ought to have stated that the theorem contained in section 2 of Part I originally appeared in the form of a question (No. 10951) in the *Educational Times* for April of this year.

† Published in the St. Petersburg Transactions for 1854.

Chapter 1.

ON THE ASYMPTOTIC LIMITS TO THE NUMBER OF PRIMES INFERIOR TO A GIVEN NUMBER.

§1. *Crude determination of the asymptotic limits.*

Call the sum of the logarithms of primes not exceeding x (any real positive quantity) the prime-number-logarithmic sum, or more briefly the prime-log-sum to x, and the sum of such sums to x and all its positive integer roots the prime-log-sum-sum, which in Serret is called $\psi(x)$.

Then it follows from elementary arithmetical principles that the sum of this sum-sum to x and all its aliquot parts, *i.e.*

$$\psi(x)+\psi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{3}\right)+\dots$$

[which we may call the natural series of sum-sums and denote by $T(x)$], is identical with the logarithm of the factorial of the highest integer not exceeding x, and accordingly from Stirling's theorem may be shown to have for its asymptotic limit $x\log x - x$, the superior and inferior limits being this quantity with a residue which, as well for the one as for the other, is a known linear function of $\log x$. Serret, Vol. 2, p. 226.

If now we take two sets of positive integers,

$$p, p', p'', \dots; \ q, q', q'', \dots,$$

(together forming what may be termed a *harmonic scheme*) meaning thereby that the sum of the reciprocals of the numbers in the two sets is the same, and extend the T series over x divided by the respective numbers in each set and take the difference between the two sums thus obtained, there will result a new series of the form

$$\sum_{n=1}^{n=\infty} f(n)\,\psi\left(\frac{x}{n}\right),$$

of which the asymptotic limit will be x multiplied by

$$\Sigma\frac{\log p}{p}-\Sigma\frac{\log q}{q},$$

and the value of $f(n)$ will be

$$\Sigma\frac{n}{p:}-\Sigma\frac{n}{q:},$$

where, in general, $\frac{n}{t:}$ means 1 or 0 according as n does or does not contain t, or in other words the "denumerant" of the equation $ty = n$.

I shall call the p's and q's the *stigmata* of the scheme:

$$\Sigma \frac{\log p}{p} - \Sigma \frac{\log q}{q}$$

the stigmatic multiplier, and the new series in $\psi(x)$ a stigmatic series of sum-sums (obtained, it will be noticed, by a four-fold process of summation—viz. two infinite and two finite summations).

It is possible, in general, (as will hereafter appear) to deduce from the asymptotic value of a stigmatic series of sum-sums, superior and inferior asymptotic limits to the sum-sum itself. The *asymptotic* limits to the simple sum will then be the same as those last named [Serret, Vol. II., p. 236, formulae "(8)" and "(9)"*] and will be multiples of x: dividing these respectively by $\log x$, we obtain superior and inferior asymptotic limits to the number of primes not exceeding x (*Messenger*, May 1891, p. 9, footnote).

It is obviously simplest always to take unity as one of the stigmata; those employed by Tschebyscheff are 1, 30; 2, 3, 5; this *scheme* as I term it leads to the relation

$$\psi\left(\frac{x}{1}\right) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right)$$

$$+ \psi\left(\frac{x}{13}\right) - \psi\left(\frac{x}{15}\right) + \psi\left(\frac{x}{17}\right) - \psi\left(\frac{x}{18}\right) + \psi\left(\frac{x}{19}\right) - \psi\left(\frac{x}{20}\right)$$

$$+ \psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{24}\right) + \psi\left(\frac{x}{29}\right) - \psi\left(\frac{x}{30}\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \left(\tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3 + \tfrac{1}{5}\log 5 - \tfrac{1}{30}\log 30\right) x + \dagger,$$

the series extending to infinity but consisting of repetitions (with a difference) of the above period, obtained by adding

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* The fourth edition 1879, of Serret's, *Cours d'Algèbre Supérieure* is referred to here and throughout the paper.

† The + is used to denote that a quantity is omitted of inferior *order* of magnitude to x. The strict interpretation of the "relation" is that the sum of the stigmatic series less the stigmatic multiplier into x is intermediate to two known linear functions of $\log x$.

The stigmatic series arranged in sets in two different ways then becomes as a first arrangement

$$\psi(x) - \psi\left(\frac{x}{10}\right):$$

$$+\psi\left(\frac{x}{11}\right) + \psi\left(\frac{x}{13}\right) - \psi\left(\frac{x}{14}\right) - \psi\left(\frac{x}{15}\right);\ +\psi\left(\frac{x}{17}\right) + \psi\left(\frac{x}{19}\right)$$

$$-\psi\left(\frac{x}{20}\right) - \psi\left(\frac{x}{21}\right);\ +\psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{28}\right);\ +\psi\left(\frac{x}{29}\right) - \psi\left(\frac{x}{30}\right);$$

$$+\psi\left(\frac{x}{31}\right) - \psi\left(\frac{x}{35}\right);\ +\psi\left(\frac{x}{37}\right) - \psi\left(\frac{x}{40}\right);\ +\psi\left(\frac{x}{41}\right) - \psi\left(\frac{x}{42}\right);$$

$$+\psi\left(\frac{x}{43}\right) - \psi\left(\frac{x}{45}\right);\ +\psi\left(\frac{x}{47}\right) - \psi\left(\frac{x}{50}\right);\ +\psi\left(\frac{x}{53}\right) - \psi\left(\frac{x}{56}\right);$$

$$+\psi\left(\frac{x}{59}\right) - \psi\left(\frac{x}{60}\right);\ +\psi\left(\frac{x}{61}\right) - \psi\left(\frac{x}{63}\right);\ +\psi\left(\frac{x}{67}\right) - \psi\left(\frac{x}{70}\right);$$

$$+\psi\left(\frac{x}{71}\right) + \psi\left(\frac{x}{73}\right) - \psi\left(\frac{x}{75}\right) + \psi\left(\frac{x}{79}\right) - \psi\left(\frac{x}{80}\right) + \psi\left(\frac{x}{83}\right)$$

$$-\psi\left(\frac{x}{84}\right) + \psi\left(\frac{x}{89}\right) - \psi\left(\frac{x}{90}\right) + \psi\left(\frac{x}{97}\right) - \psi\left(\frac{x}{98}\right) - \psi\left(\frac{x}{100}\right);$$

$$+\psi\left(\frac{x}{101}\right) + \psi\left(\frac{x}{103}\right) - \psi\left(\frac{x}{105}\right) - \psi\left(\frac{x}{105}\right);\ +\psi\left(\frac{x}{107}\right)$$

$$+\psi\left(\frac{x}{109}\right) - \psi\left(\frac{x}{110}\right) - \psi\left(\frac{x}{112}\right);\ +\psi\left(\frac{x}{113}\right) - \psi\left(\frac{x}{120}\right);$$

$$+\psi\left(\frac{x}{121}\right) - \psi\left(\frac{x}{126}\right);\ +\psi\left(\frac{x}{127}\right) - \psi\left(\frac{x}{130}\right);\ +\psi\left(\frac{x}{131}\right)$$

$$-\psi\left(\frac{x}{135}\right);\ +\psi\left(\frac{x}{137}\right) + \psi\left(\frac{x}{139}\right) - \psi\left(\frac{x}{140}\right) + \psi\left(\frac{x}{143}\right)$$

$$-\psi\left(\frac{x}{147}\right) + \psi\left(\frac{x}{149}\right) - \psi\left(\frac{x}{150}\right) + \psi\left(\frac{x}{151}\right) - \psi\left(\frac{x}{154}\right)$$

$$+\psi\left(\frac{x}{157}\right) - \psi\left(\frac{x}{160}\right) + \psi\left(\frac{x}{163}\right) - \psi\left(\frac{x}{165}\right) + \psi\left(\frac{x}{167}\right)$$

$$-\psi\left(\frac{x}{168}\right)+\psi\left(\frac{x}{169}\right)-\psi\left(\frac{x}{170}\right)+\psi\left(\frac{x}{173}\right)-\psi\left(\frac{x}{175}\right)$$

$$+\psi\left(\frac{x}{179}\right)-\psi\left(\frac{x}{180}\right)+\psi\left(\frac{x}{181}\right)-\psi\left(\frac{x}{182}\right)+\psi\left(\frac{x}{187}\right)$$

$$-\psi\left(\frac{x}{189}\right)-\psi\left(\frac{x}{190}\right);\ +\psi\left(\frac{x}{191}\right)+\psi\left(\frac{x}{193}\right)-\psi\left(\frac{x}{195}\right)$$

$$-\psi\left(\frac{x}{196}\right);\ +\psi\left(\frac{x}{197}\right)+\psi\left(\frac{x}{199}\right)-\psi\left(\frac{x}{200}\right)+\psi\left(\frac{x}{209}\right)$$

$$-\psi\left(\frac{x}{210}\right)-\psi\left(\frac{x}{210}\right);\ +\psi\left(\frac{x}{211}\right)-\psi\left(\frac{x}{220}\right);$$

....

....,

the correlative arrangement being

$$\psi(x)-\psi\left(\frac{x}{10}\right)+\psi\left(\frac{x}{11}\right)+\psi\left(\frac{x}{13}\right):$$

$$-\psi\left(\frac{x}{14}\right)-\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{17}\right)+\psi\left(\frac{x}{19}\right);\ -\psi\left(\frac{x}{20}\right)-\psi\left(\frac{x}{21}\right)$$

$$+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{28}\right)+\psi\left(\frac{x}{29}\right)-\psi\left(\frac{x}{30}\right)+\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{35}\right)$$

$$+\psi\left(\frac{x}{37}\right)-\psi\left(\frac{x}{40}\right)+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)-\psi\left(\frac{x}{45}\right)$$

$$+\psi\left(\frac{x}{47}\right)-\psi\left(\frac{x}{50}\right)+\psi\left(\frac{x}{53}\right)-\psi\left(\frac{x}{56}\right)+\psi\left(\frac{x}{59}\right)-\psi\left(\frac{x}{60}\right)$$

$$+\psi\left(\frac{x}{61}\right)-\psi\left(\frac{x}{63}\right)+\psi\left(\frac{x}{67}\right)-\psi\left(\frac{x}{70}\right)+\psi\left(\frac{x}{71}\right)+\psi\left(\frac{x}{73}\right);$$

$$-\psi\left(\frac{x}{75}\right)+\psi\left(\frac{x}{79}\right);\ -\psi\left(\frac{x}{80}\right)+\psi\left(\frac{x}{83}\right);\ -\psi\left(\frac{x}{84}\right)+\psi\left(\frac{x}{89}\right);$$

$$-\psi\left(\frac{x}{90}\right)+\psi\left(\frac{x}{97}\right);\ -\psi\left(\frac{x}{98}\right)-\psi\left(\frac{x}{100}\right)+\psi\left(\frac{x}{101}\right)$$

$$+\psi\left(\frac{x}{103}\right);\ -\psi\left(\frac{x}{105}\right)-\psi\left(\frac{x}{105}\right)+\psi\left(\frac{x}{107}\right)+\psi\left(\frac{x}{109}\right);$$

$$-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right)+\psi\left(\frac{x}{113}\right)-\psi\left(\frac{x}{120}\right)+\psi\left(\frac{x}{121}\right)$$

$$-\psi\left(\frac{x}{126}\right)+\psi\left(\frac{x}{127}\right)-\psi\left(\frac{x}{130}\right)+\psi\left(\frac{x}{131}\right)-\psi\left(\frac{x}{135}\right)$$

$$+\psi\left(\frac{x}{137}\right)+\psi\left(\frac{x}{139}\right);\ -\psi\left(\frac{x}{140}\right)+\psi\left(\frac{x}{143}\right);\ -\psi\left(\frac{x}{147}\right)$$

$$+\psi\left(\frac{x}{149}\right);\ -\psi\left(\frac{x}{150}\right)+\psi\left(\frac{x}{151}\right);\ -\psi\left(\frac{x}{154}\right)+\psi\left(\frac{x}{157}\right);$$

$$-\psi\left(\frac{x}{160}\right)+\psi\left(\frac{x}{163}\right);\ -\psi\left(\frac{x}{165}\right)+\psi\left(\frac{x}{167}\right);\ -\psi\left(\frac{x}{168}\right)$$

$$+\psi\left(\frac{x}{169}\right);\ -\psi\left(\frac{x}{170}\right)+\psi\left(\frac{x}{173}\right);\ -\psi\left(\frac{x}{175}\right)+\psi\left(\frac{x}{179}\right);$$

$$-\psi\left(\frac{x}{180}\right)+\psi\left(\frac{x}{181}\right);\ -\psi\left(\frac{x}{182}\right)+\psi\left(\frac{x}{187}\right);\ -\psi\left(\frac{x}{189}\right)$$

$$-\psi\left(\frac{x}{190}\right)+\psi\left(\frac{x}{191}\right)+\psi\left(\frac{x}{193}\right);\ -\psi\left(\frac{x}{195}\right)-\psi\left(\frac{x}{196}\right)$$

$$+\psi\left(\frac{x}{197}\right)+\psi\left(\frac{x}{199}\right);\ -\psi\left(\frac{x}{200}\right)+\psi\left(\frac{x}{209}\right);\ -\psi\left(\frac{x}{210}\right)$$

$$-\psi\left(\frac{x}{210}\right)+\psi\left(\frac{x}{211}\right)-\psi\left(\frac{x}{220}\right)+\psi\left(\frac{x}{221}\right)+\psi\left(\frac{x}{223}\right):$$

....

....*

The terms in each arrangement, it will be seen, are separated by marks of punctuation into groups: omitting the first group in either of them, which may be called the outstanding group, in each of the others the sum of the coefficients is zero.

Moreover, the sum of the coefficients from the beginning of each group is always homonymous in sign, *i.e.* will be non-negative in the first and non-positive in the second arrangement: the consequence of this is that all the terms of such

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* Each of these arrangements is to be regarded as made up of the outstanding group and an infinite succession of periodic groups. In the text we have set out the outstanding group and the first period, the other periods will be formed from this one by adding to each denominator in it successive multiples of 210.

groups may be resolved into pairs, whose sum will be necessarily positive in the one and negative in the other.

Thus, *ex. gr.* in the first arrangement the last but one of the groups may be resolved into the pairs

$$\psi\left(\frac{x}{197}\right)-\psi\left(\frac{x}{200}\right);\ \psi\left(\frac{x}{199}\right)-\psi\left(\frac{x}{210}\right);\ \psi\left(\frac{x}{209}\right)-\psi\left(\frac{x}{210}\right),$$

each of which is equal to zero or a positive quantity. So the eighth group of the second arrangement is resoluble into the pairs

$$-\psi\left(\frac{x}{98}\right)+\psi\left(\frac{x}{101}\right);\ -\psi\left(\frac{x}{100}\right)+\psi\left(\frac{x}{103}\right),$$

each of which is zero or a negative quantity.

It may be as well to notice in this place that the sum of the coefficients, reckoning from the first term of the outstanding group to the term whose denominator is n, is

$$\sum_{t=0}^{t=n}\Sigma\left(\frac{t}{p:}-\frac{t}{q:}\right),$$

which by virtue of the obvious identity,

$$\sum_{t=0}^{t=n}\left(\frac{t}{i:}\right)=E\left(\frac{n}{i}\right),$$

is equal to

$$\Sigma\left\{E\left(\frac{n}{p}\right)-E\left(\frac{n}{q}\right)\right\}.$$

This formula supplies an easy and valuable test for ascertaining the correctness of the determination of the coefficients up to any given term in the series.

These observations may be extended to any harmonic scheme whatever: for it will be observed that

$$\Sigma\left\{E\left(\frac{n}{p}\right)-E\left(\frac{n}{q}\right)\right\}$$

is a periodic quantity, and therefore possesses both a maximum and a minimum; whence it is easy to see that, by taking the outstanding group of terms sufficiently extensive, all the remaining terms in either kind of arrangement may be separated into groups similar to those above set out; viz., such that the *complete* sum of the coefficients in each group from its first to its end term is zero and up to any intermediate term is *homonymous*, *i.e.* always positive in one and always negative in the other arrangement.*

The consequence of this is that the outstanding group in the first arrangement will always be less, and in the second arrangement always greater, than a function of which the principal, or, as we may call it, the asymptotic term, is the product of x by the stigmatic multiplier, say (St), (the complete function being, in each case of the form $(St)\,x$ associated with a known linear function of $\log x$. (Compare Serret, Vol. II, p. 232).

The importance of this observation will become apparent in a subsequent section.

In the case before us (*i.e.* for the scheme in the Key of 7) confining our attention to the principal term of either limit, the first arrangement leads immediately (Serret, p. 234) to the superior asymptotic limit $\frac{10}{9}Dx$.

As regards the inferior limit, we have

$$\psi(x) + \psi\left(\frac{x}{13}\right) > Dx,$$

$$\psi(x) > Dx - \tfrac{1}{13}\cdot\tfrac{10}{9}Dx > \tfrac{107}{117}Dx.\dagger$$

---

* *Ex. gr.* from the harmonic scheme 1, 15; 2, 3, 5, 30, we may derive a stigmatic series under the two forms of arrangement

$$\psi(x) - \psi\left(\frac{x}{6}\right) : + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right); + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right); + \psi\left(\frac{x}{13}\right) + \psi\left(\frac{x}{17}\right) - \psi\left(\frac{x}{18}\right)$$

$$+ \psi\left(\frac{x}{19}\right) - \psi\left(\frac{x}{20}\right) + \psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{24}\right) + \psi\left(\frac{x}{29}\right) - \psi\left(\frac{x}{30}\right) - \psi\left(\frac{x}{30}\right); + \psi\left(\frac{x}{31}\right) - \psi\left(\frac{x}{36}\right) : \text{\&c.},$$

$$\psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right) + \psi\left(\frac{x}{13}\right) + \psi\left(\frac{x}{17}\right) : - \psi\left(\frac{x}{18}\right)$$

$$+ \psi\left(\frac{x}{19}\right); - \psi\left(\frac{x}{20}\right) + \psi\left(\frac{x}{23}\right); - \psi\left(\frac{x}{24}\right) + \psi\left(\frac{x}{29}\right); - \psi\left(\frac{x}{30}\right) - \psi\left(\frac{x}{30}\right) + \psi\left(\frac{x}{31}\right)$$

$$- \psi\left(\frac{x}{36}\right) + \psi\left(\frac{x}{37}\right) - \psi\left(\frac{x}{40}\right) + \psi\left(\frac{x}{41}\right) - \psi\left(\frac{x}{42}\right) + \psi\left(\frac{x}{43}\right) + \psi\left(\frac{x}{47}\right) : \text{\&c.}$$

In the above arrangements the groups are separated by semicolons and the period is marked out by the colons. In this instance it will be observed that minimum and maximum values of $E(n) + E\left(\frac{n}{15}\right) - E\left(\frac{n}{2}\right) - E\left(\frac{n}{3}\right) - E\left(\frac{n}{5}\right) - E\left(\frac{n}{30}\right)$ are 0 and 2, and accordingly in the first arrangement the outstanding group has to be continued until the sum of the coefficients of the terms which it contains is 0, and in the second until such sum is 2.

Writing $\qquad Q = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{5}\log 5 + \frac{1}{30}\log 30 - \frac{1}{15}\log 15 = \cdot 96750\ldots,$

we may deduce from the above, the asymptotic coefficients $\frac{6}{5}Q$ and $Q - \frac{1}{17}\cdot\frac{6}{5}Q$; *i.e.* 1·1610... and ·8992....

† Compare the determination of the limits for the harmonic scheme 1; 2, 3, 6 (*Am. J.*, Vol. IV, pp. 243, 244).

Substituting for D its value ·9787955, we obtain the asymptotic limits 1·0873505 and ·8951370.

The corresponding values got from the Tschebyscheffian scheme (1, 30; 2, 3, 5) being 1·1055504 and ·9212920, which are the $\frac{6}{5}A$ and A of Serret.

We know *aliunde* that the true asymptotic values are each of them presumably unity. The superior value above obtained by the new scheme is thus seen to be better, and the inferior value worse than those given by Tschebyscheff's scheme. But these values correspond to what may be termed the *crude* determination of the limits which the schemes are capable of affording. The contraction of these asymptotic limits by a method of continual successive approximation will form the subject of the following section.*

§ 2. *On a method of obtaining continually contracting asymptotic limits to* $\frac{\psi(x)}{x}$.

To fix the ideas let us consider the scheme (1, 30; 2, 3, 5) which leads to the stigmatic series

$$(1)-(6)+(7)-(10)+(11)-(12)+(13)-(15)+(17)-(18)+(19)$$
$$-(20)+(23)-(24)+(29)-(30)+(31)\ldots,$$

in which for brevity (n) is used to denote $\psi\left(\frac{x}{n}\right)$.

The sum of this series is, we know, intermediate between

$$Dx+R(\log x) \text{ and } D_1x+R_1(\log x),$$

where $\quad D=·9212920\ldots,\quad D_1=1·1055504\ldots=\frac{6}{5}D,$

and R_1, R_2 signify two known quantities which for uniformity may both be regarded as quadratic functions of $\log x$ (in the first of which the coefficient of $(\log x)^2$ is zero. (Serret pp. 233, 235).

Omitting every pair of consecutive terms $-(m)+(\mu)$ in which $\frac{\mu}{m}<\frac{6}{5}$, and using $[\psi(x)]$ to signify the asymptotic value of $\psi(x)$, we find

$$[\psi(x)]>Dx+\left[\psi\left(\frac{x}{24}\right)\right]-\left[\psi\left(\frac{x}{29}\right)\right]>Dx+D\frac{x}{24}-D_1\frac{x}{29},$$

say $\qquad > D'x$

* By the method about to be explained, it should be noticed, we may not merely improve upon the results obtained by the *crude* method from certain harmonic schemes (which form a very restricted class) but may also obtain limits to $\psi(x)\div x$ from harmonic schemes which without its aid would be absolutely sterile (see p. 101).

Similarly, omitting every consecutive pair of terms $(m)-(\mu)$ in which $\frac{\mu}{m} < \frac{6}{5}$, we find

$$[\psi(x)] < Dx + D_1\frac{x}{6} - D\frac{x}{7} + D_1\frac{x}{10},$$

say $$< D'_1 x$$

If instead of $[\psi(x)]$ we had deduced limits to $\psi(x)$ in the manner indicated above, we should have found

$$\psi(x) > D'x + R'(\log x), \quad \psi(x) < D_1'x + R_1'(\log x);$$

the added terms being each of them quadratic functions of $\log x$.

Repeating this process we shall obtain

$$[\psi(x)] > D''x, \quad [\psi(x)] < D_1''x,$$

where $D'' = D + \frac{1}{24}D' - \frac{1}{29}D_1'$, $D'' = D + \frac{1}{6}D_1' - \frac{1}{7}D' + \frac{1}{10}D_1'$.

Similarly we may write

$$[\psi(x)] > D'''x, \quad [\psi(x)] < D_1'''x,$$

where

$$D''' = D + \frac{1}{24}D'' - \frac{1}{29}D_1'', \quad D_1''' = D + \frac{1}{6}D_1'' - \frac{1}{7}D'' + \frac{1}{10}D_1'',$$

and so on.

If then we write for $D, D', D'', \ldots, v_0, v_1, v_2, \ldots,$ and for $D_1, D_1', D_1'', \ldots, u_0, u_1, u_2, \ldots,$ we shall find in general

$$[\psi(x)] > v_i x, \quad [\psi(x)] < u_i x;$$

where
$$v_{i+1} = D + \frac{v_i}{24} - \frac{u_i}{29},$$

$$u_{i+1} = D + \left(\frac{1}{6} + \frac{1}{10}\right)u_i - \frac{1}{7}v_i;$$

the complete statement of the inequalities being

$$\psi(x) > v_i x + R^{(i)}(\log x), \quad \psi(x) < u_i x + R_1^{(i)}(\log x),$$

where it is to be noticed that the supplemental terms always remain *quadratic* functions of $\log x$.

[The result thus obtained differs in this particular from that stated by me in the *Amer. Math. Jour.* (Vol. IV., p. 241); the process therein employed giving as supplemental terms rational integral functions of continually rising degrees of $\log x$. I am indebted to Mr. Hammond for drawing my attention to this simple but important circumstance which had strangely escaped my attention previously]. To integrate the equations in u, v we have only to write

$$v_i = V_i + F, \quad u_i = U_i + E,$$
$$F(1 - \tfrac{1}{24}) + \tfrac{1}{29}E = D, \quad V_i = C_1\rho_1^i + C_2\rho_2^i,$$
$$\tfrac{1}{7}F + (1 - \tfrac{1}{6} - \tfrac{1}{10})E = D, \quad U_i = K_1\rho_1^i + K_2\rho_2^i;$$

and to take for ρ_1, ρ_2 the two roots of the equation

$$\begin{vmatrix} \rho - \frac{1}{24}, & \frac{1}{29} \\ \frac{1}{7} & \rho - \frac{1}{6} - \frac{1}{10} \end{vmatrix} = \rho^2 - (\tfrac{1}{6} + \tfrac{1}{10} + \tfrac{1}{24})\rho + \tfrac{1}{24}(\tfrac{1}{6} + \tfrac{1}{10}) - \tfrac{1}{203} = 0,$$

i.e.
$$\rho^2 - \tfrac{37}{120}\rho + \tfrac{113}{18270} = 0.$$

The roots of this equation being each less than 1, on making $i = \infty$ we obtain $v_\infty = F$, $u_\infty = E$, where E, F are deduced from the two algebraic equations

$$\tfrac{23}{24}F + \tfrac{1}{29}E = D,$$
$$\tfrac{1}{7}F + \tfrac{11}{15}E = D.$$

This gives

$$\frac{E}{F} = (\tfrac{23}{24} - \tfrac{1}{7}) \div (\tfrac{11}{15} - \tfrac{1}{29}) = \frac{137 \times 145}{304 \times 56} = \frac{19865}{17024} = q$$

(Compare *Amer. Math. Jour.*, Vol. IV., p. 242)

$$E = \tfrac{59595}{55099}D = 1.0765779\ldots,$$
$$F = \tfrac{51072}{55099}D = \cdot 9226107\ldots;$$

whence we may infer that $\psi(x)$ may be made intermediate between two known functions $u_i x + r(\log x)$, $v_i x + s(\log x)$, where u_i, v_i may be brought *indefinitely near* to the numbers

$$1\cdot 0765779\ldots, \quad \cdot 9226107\ldots;$$

and the supplemental terms are quadratic functions of $\log x$ depending upon the value of i that may be employed. We may, therefore (subject to an obvious interpretation), treat E and F as asymptotic limits to $\dfrac{\psi(x)}{x}$.*

* For the complete analytical determination of the limits to $\psi(x)$ see § 3 of this chapter.

By making i sufficiently great u_i, v_i may be brought indefinitely near to E, F: furthermore, when the superior and inferior limits of $\psi(x) \div x$ are expressed as functions of x and i of the form mentioned in the text, these limits may, by taking x sufficiently great, be brought indefinitely near to u_i, v_i, and therefore to E, F, which I therefore speak of throughout as asymptotic limits to $\psi(x) \div x$. But more strictly the optimistic limits actually arrived at are E' as little as we please greater than E, and F' as little as we please less than F.

If we examine the ratio of the denominators m, μ of any pair of consecutive terms throughout the entire infinite series, whether of the form $(m)-(\mu)$ or $-(m)+(\mu)$, we shall find that $\frac{\mu}{m}$ is always less than q, (viz. 1·16688...) except in the case of the pairs that have been retained in forming the equations between E and F, from which we may infer that if any of the discarded pairs had been retained we should have obtained values of E and F respectively greater and less than those above set forth.

If, on the other hand, q had turned out to be so much less than $\frac{6}{5}$ as to cause $\frac{\mu}{m}$ in any rejected pair to be greater than q, in such case in order to obtain a value of E the least, and of F the greatest, capable of being extracted from the given scheme, it would have been necessary to take account of every such pair and perform the calculations afresh, thereby obtaining a new value of q (say q') less than the former one; we should then have had to continue the process of examining the rejected pairs and reinstating those (if any) whose denominators furnished a ratio $\frac{\mu}{m}$ greater than q', thereby obtaining a still smaller value q''. Repeating these operations *toties quoties* we should at last arrive at a value of q superior to every ratio $\frac{\mu}{m}$ throughout the entire stigmatic series; the corresponding values of the asymptotic limits would then be the best capable of being deduced from the given scheme.

Per contra had we retained at the start any of the discarded pairs of terms, we should have found for q a value greater than the value of $\frac{\mu}{m}$ in some of the terms retained, which would be a sure indication that the retention of those terms had led to a greater value of q than was necessary; those pairs would then have to be omitted; the q calculated from the reformed equations would be diminished by so doing and the resulting values of E, F would be the best attainable, provided that care was taken at the outset that no rejected pair gave a larger value to $\frac{\mu}{m}$ than any pair that had been retained.

In the case we have considered initial asymptotic limits (viz. D and D_1) to $\frac{\psi(x)}{x}$ were obtained from the scheme itself, but it will not always be possible to do this when we are dealing with any harmonic scheme.

Thus *ex. gr.* from the fact that the minor arrangement of the stigmatic series corresponding to the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105] has (1) + (13) for its outstanding group (see p. 104), we may deduce that $\psi(x) + \psi\left(\frac{x}{13}\right)$ has Nx for its inferior asymptotic limit, but are unable from this arrangement to obtain an initial inferior asymptotic limit to $\psi(x)$ itself, and still less shall we be able to obtain an initial superior asymptotic limit to $\psi(x)$ from the major arrangement of the same stigmatic series. It is therefore important to notice that the final asymptotic limits arrived at by the method explained in this section, depend only on the stigmatic multiplier and the coefficients of the stigmatic series, being quite independent of the *initial* values employed, so that in the general case we may start from any given asymptotic limits to $\frac{\psi(x)}{x}$, *however obtained*, without thereby producing any effect in the final result. The limits $u_0 = 2\log 2$ and $v_0 = \log 2$ obtained from the scheme [1 ; 2, 2] will do as well as any others for our initial asymptotic limits to $\frac{\psi(x)}{x}$, and we may, by substituting these limits in the retained portion of the stigmatic series, arrive at new limits u_1, v_1 which in their turn will give rise to fresh limits u_2, v_2 and so on. We shall in this way obtain a pair of difference equations (connecting u_{i+1}, v_{i+1} with u_i, v_i) which will be of the same form as those given on p. 98, and it is to be noticed that in the solution of these equations, viz.

$$u_i = C\rho^i + C_1\rho_1^{\,i} + E,\ v_i = K\rho^i + K_1\rho_1^{\,i} + F,$$

only the values of C, C_1, K, K_1 will depend on the initial values of u, v; so that, provided the roots of the quadratic in ρ (which are always real) are each less than unity, we may, by taking i sufficiently great, make u_i and v_i approach as near as we please to E and F respectively; *i.e.* as near as we please to two quantities whose values depend solely on the stigmatic series employed.

The positive and negative divergences from unity of the E and F previously found are respectively

·0765779..., ·0773893...;

these divergences as found by Tschebyscheff being

·1055504..., ·0787080,

which is already an important gain; but by varying the scheme we shall obtain still better results.

Let us apply the method of indefinite successive approximation to the scheme in the key of 7 treated of in the preceding section, viz. [1, 6, 70; 2, 3, 5, 7, 210], for which the stigmatic multiplier (the D of p. 91), viz.

$\frac{1}{2}\log 2+\frac{1}{3}\log 3+\frac{1}{5}\log 5+\frac{1}{7}\log 7+\frac{1}{210}\log 210-\frac{1}{6}\log 6-\frac{1}{70}\log 70$ is ·9787955... .

Preliminary calculations having served to satisfy me that the asymptotic ratio $\frac{E}{F}$ (the q) for this system was not likely to differ much from 1·10, which may be called the *regulator* I form the corresponding equations for E and F by retaining only those pairs $(m)-(\mu)$ in the stigmatic series for which $\frac{\mu}{m}$ is greater than 1·10.

As previously explained no *error* can result whatever regulator we employ; the worst that can happen will be that the result will not be the best attainable from the scheme, and such imperfection can be ascertained by means of the method previously explained; the result, if the best possible, will prove itself to be so, and if not the best, will indicate whether the regulator (or criterion of retention) has been taken too small or too great.

Let us examine separately the two arrangents set out in the previous section, the first being employed to obtain by successive approximations the superior, and the second the inferior, limit.

Consider 1° the periodic part of the first arrangement: in the group $(11)+(13)-(14)-(15)$, the pair $(13)-(14)$ being rejected, $(11)-(15)$ remains. Similarly, in the following group $(19)-(20)$ being rejected, $(17)-(21)$ remains; in the third and fourth groups $(23)-(28)$ and $(31)-(35)$ are to be retained. In the following group, all the consecutive pairs from (73) to (98) both inclusive are to be rejected, leaving $(71)-(100)$ available. [The corresponding pair to this in the next period, viz. $(281)-(310)$ gives $\frac{310}{281}$ which is less than the assumed regulator.] All the groups in the first period, following $-(100)$, will have to be rejected until we come to the group beginning with (137), which leads to the available pair $(137)-(190)$: in the next period all the ratios will be too small with the exception of $(347)-(400)$ which must be retained, but the term corresponding to this in the third period, viz. $(557)-(610)$, will have to be neglected.

Hence, in approximating to the superior limit, we may write

$$u_{i+1} = M + \left(\tfrac{1}{10} + \tfrac{1}{15} + \tfrac{1}{21} + \tfrac{1}{28} + \tfrac{1}{35} + \tfrac{1}{100} + \tfrac{1}{190} + \tfrac{1}{400}\right) u_i$$
$$- \left(\tfrac{1}{11} + \tfrac{1}{17} + \tfrac{1}{23} + \tfrac{1}{31} + \tfrac{1}{71} + \tfrac{1}{137} + \tfrac{1}{347}\right) v_i.$$

2°. In the second arrangement, the first group in the periodic part being $-(14) - (15) + (17) + (19)$, and $\tfrac{17}{15}$ (and *à fortiori* $\tfrac{19}{14}$) exceeding the regulator, all these terms are to be preserved.

In addition to these, we shall find in the first period the available couples $-(20) + (73)$ and $-(110) + (139)$, and in the second period $-(230) + (283)$; no other couples will be available, and accordingly, we shall have

$$v_{i+1} = M + \left(\tfrac{1}{10} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{20} + \tfrac{1}{110} + \tfrac{1}{230}\right) v_i$$
$$- \left(\tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} + \tfrac{1}{19} + \tfrac{1}{73} + \tfrac{1}{139} + \tfrac{1}{283}\right) u_i.$$

If then we write a, b for the coefficients of u_i, $-v_i$ in the first, and c, d for the coefficients of v_i, $-u_i$ in the second of the above equations, and make $u_i = U_i + E$, $v_i = V_i + F$, we shall obtain

$$u_i = C\rho^i + C_1\rho_1^{\,i} + E,$$
$$v_i = K\rho^i + K_1\rho_1^{\,i} + F,$$

where ρ, ρ_1 are the roots of the equation

$$\begin{vmatrix} \rho - a, & b \\ d\;\;, & \rho - c \end{vmatrix} = 0,$$

i.e.
$$\rho^2 - (a + c)\rho + (ac - bd) = 0,$$

and E, F are subject to the equations

$$(1 - a)E + bF = M,$$
$$dE + (1 - c)F = M,$$

which give

$$E = \frac{1 - b - c}{(1 - a)(1 - c) - bd} M, \quad F = \frac{1 - a - d}{(1 - a)(1 - c) - bd} M.$$

On performing the calculations, we shall find

$$a = {\cdot}29633\ldots, \quad b = {\cdot}24973\ldots,$$
$$c = {\cdot}30153\ldots, \quad d = {\cdot}30371\ldots,$$
$$1 - b - c = {\cdot}44873\ldots, \quad 1 - a - d = {\cdot}39995\ldots,$$
$$ac = {\cdot}08935\ldots, \qquad bd = {\cdot}07584\ldots,$$
$$a + c = {\cdot}59786\ldots, \quad (1 - a)(1 - c) - bd = {\cdot}41563\ldots,$$

ρ, ρ_1 will therefore be the roots of

$$\rho^2 - {\cdot}59786\rho + {\cdot}01350 = 0,$$

which are each less than unity.

Also $E = 1{\cdot}0567265...$, $F = {\cdot}9418543...$,

$$q = \frac{1-b-c}{1-a-d} = 1{\cdot}12196\ldots.$$

This last number being *greater* than the assumed regulator 1·10, and *less* than any of the retained ratios "$\left[\frac{\mu}{m}\right]$," it follows that no better limits than E, F can be extracted from the scheme [1, 6, 70; 2, 3, 5, 7, 210]; or (as we may phrase it) E, F are the optimistic asymptotic limits to that scheme.

Obviously, there is no reason to suppose that these are the closest asymptotic limits that can be obtained from the infinite choice of schemes at our disposal: on the contrary, there is every reason to suppose that these limits may by schemes in higher and higher keys be brought to coincide as nearly as may be desired to each other and to unity.

We shall presently obtain by aid of a new scheme a better result than the E, F of the preceding investigation. But first it should be observed that instead of forming the difference equations in u, v from the two arrangements, say the major and minor, of one and the same stigmatic series, (the former meaning the one used to find the superior and the latter the inferior) asymptotic limit, we may take these two arrangements, if we please, from two distinct series corresponding to two different schemes.

I have had calculated, from beginning to end, the value of the coefficient of each term in the stigmatic series of sums-sums corresponding to the first natural period, containing 2310 terms of the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105], the stigmatic multiplier to which, viz.

$$\tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3 + \tfrac{1}{5}\log 5 + \tfrac{1}{7}\log 7 + \tfrac{1}{11}\log 11 + \tfrac{1}{105}\log 105$$

$$- \tfrac{1}{6}\log 6 - \tfrac{1}{10}\log 10 - \tfrac{1}{210}\log 210 - \tfrac{1}{231}\log 231 - \tfrac{1}{1155}\log 1155.$$

is ·9909532... (say N).

This stigmatic series, though too long for printing at full in the restricted space of this Journal, is given later on in a condensed tabular form (see Table A, p. 107). I will proceed to describe its essential features and the use made of it to bring the asymptotic limits closer together. The maximum and minimum sums of its coefficients are 2 and -2: the first terms being $(1)+(13)-(14)-(15)$, the maximum is first reached at the second term; so that the outstanding group in the minor arrangement will be $(1)+(13)$. But the minimum sum, -2, is not reached before the term whose argument is (616). The outstanding group in the major arrangement will therefore contain a very great number of terms, and there might be some trouble in handling the groups, so as to secure the greatest possible advantage. For this reason, I have thought it sufficient for the present to combine the major arrangement of the scheme [1, 6, 70; 2, 3, 5, 7, 210] with the minor one of the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105].

Maintaining the regulator still at the same value as before, viz. 1·10, the major arrangement will remain unaltered from what it was in the preceding case. In the minor arrangement there will be found to exist the following 17 available pairs, all of which, except the last belong to the first period (the last one belonging to the second period), viz.

$$(14)-(19),\ (15)-(17),\ (21)-(31),\ (33)-(41),\ (44)-(53),$$
$$(63)-(73),\ (84)-(97),\ (105)-(241),\ (110)-(131),$$
$$(195)-(223),\ (315)-(481),\ (525)-(703),\ (735)-(943),$$
$$(945)-(1231),\ (1484)-(1693),\ (1694)-(2323),$$
$$(4004)-(4633).$$

We accordingly may write

$$u_{i+1} = M + au_i - bv_i$$
$$v_{i+1} = N + \gamma v_i - \delta u_i$$

where

$$a = \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28} + \frac{1}{35} + \frac{1}{100} + \frac{1}{190} + \frac{1}{400},$$

$$b = \frac{1}{11} + \frac{1}{17} + \frac{1}{23} + \frac{1}{31} + \frac{1}{71} + \frac{1}{137} + \frac{1}{347},$$

$$\gamma = \frac{1}{14} + \frac{1}{15} + \frac{1}{21} + \frac{1}{33} + \frac{1}{44} + \frac{1}{63} + \frac{1}{84} + \frac{1}{105} + \frac{1}{110}$$
$$+ \frac{1}{195} + \frac{1}{315} + \frac{1}{525} + \frac{1}{735} + \frac{1}{945} + \frac{1}{1484} + \frac{1}{1694} + \frac{1}{4004},$$

$$\delta = \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{31} + \frac{1}{41} + \frac{1}{53} + \frac{1}{73} + \frac{1}{97} + \frac{1}{131} + \frac{1}{223}$$
$$+ \frac{1}{241} + \frac{1}{481} + \frac{1}{703} + \frac{1}{943} + \frac{1}{1231} + \frac{1}{1693} + \frac{1}{2323} + \frac{1}{4633},$$

from which, writing

$$(1-a)E+bF=M,$$

$$\delta E+(1-\gamma)F=N,$$

we shall find

$$u_i=C\rho^i+C_1\rho_1^i+E,$$

$$v_i=K\rho^i+K_1\rho_1^i+F,$$

where ρ, ρ_1 are the roots of

$$\begin{vmatrix} \rho-a, & b \\ \delta, & \rho-\gamma \end{vmatrix}=0,$$

i.e. $$\rho^2-(a+\gamma)\rho+a\gamma-b\delta=0.$$

The values of a, b; γ, δ are respectively

$$\cdot 2963346\ldots,\quad \cdot 2497346\ldots;\quad \cdot 2992774\ldots,\quad \cdot 3107808\ldots,$$

from which we see that ρ, ρ_1 being each less than unity the values of u_∞, v_∞ will be E, F,

where $$E=\frac{(1-\gamma)M-bN}{(1-a)(1-\gamma)-b\delta},$$

$$F=\frac{(1-a)N-\delta M}{(1-a)(1-\gamma)-b\delta},$$

and on performing the calculation it will be found that

$$E=1\cdot 0551851\ldots,\quad F=\cdot 9461974.$$

Also $$q=\frac{E}{F}=1\cdot 11518\ldots,$$

which being greater than the assumed regulator, but less than any of the retained ratios $\frac{\mu}{m}$, the results thus obtained are *optimistic*, *i.e.* no better can be found without having recourse to some other harmonic scheme.

The advance made upon the determination of the asymptotic limits beyond what was known previously is already remarkable. Tschebyscheff's asymptotic numbers stood at

$$1\cdot 1055504\ldots,$$

$$\cdot 9212920\ldots,$$

corresponding to a divergence from unity

$$\cdot 1055504\ldots \text{ in excess,}$$

and $$\cdot 0787080\ldots \text{ in defect;}$$

by the combined effect of scheme variation and successive substitution we have succeeded in reducing these divergences to

$$\cdot 0551851\ldots \text{ in excess,}$$

and $$\cdot 0538026\ldots \text{ in defect;}$$

in which it will be noticed that the divergence for the superior limit is only a little more than half the original one.

The mean of the two limits, it will be seen, is now less than

$$1\cdot 0007.$$

The annexed table, in which for brevity $\bar{c}$ is written for $-c$, gives in a condensed form the stigmatic series to the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105].

The coefficients, for all the terms $\psi\left(\frac{x}{m}\right)$ from $m=1$ to $m=1155$ (the half modulus), are written down in regular batches of 10. The coefficients for the ensuing terms up to 2309 can be got from these by the formula $c_{1155+i}=c_{1155-i}$, the term following will have the coefficient zero; the rest of the infinite series is then known from the formula $c_{i+2310i}=c_i$.

Table A.

The coefficients of the first 1155 *terms of the stigmatic series to* [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105].*

1000000000	$001\bar{1}\bar{1}01010$	$\bar{1}\bar{1}10000\bar{1}10$	$10\bar{1}0\bar{1}01000$
$\bar{1}11\bar{1}\bar{1}01000$	$0010\bar{1}\bar{1}0010$	$10\bar{1}00\bar{1}100\bar{1}$	$1010\bar{1}0\bar{1}010$
$001\bar{1}000\bar{1}10$	$0000001\bar{1}\bar{1}0$	$1010\bar{3}0101\bar{1}$	$0\bar{1}10000000$
$00000\bar{1}1000$	$1\bar{1}00\bar{1}0101\bar{1}$	$000000\bar{1}010$	$100\bar{2}001000$
$0010\bar{2}01\bar{1}10$	$0010\bar{1}\bar{1}0010$	$1\bar{1}000000\bar{1}0$	$1010\bar{1}\bar{1}1\bar{1}10$
$000000000\bar{1}$	$100000000\bar{1}$	$101\bar{1}\bar{1}01010$	$\bar{1}010000\bar{1}10$
$1\bar{1}00\bar{1}01000$	$1\bar{1}00\bar{1}01000$	$001\bar{1}0\bar{1}0010$	$10\bar{1}0\bar{1}0100\bar{1}$
$1010\bar{1}\bar{1}0010$	$001\bar{1}00\bar{1}010$	$0000001\bar{2}00$	$1010\bar{3}01000$
$0\bar{1}10000000\bar{1}$	$10000\bar{1}1000$	$0000\bar{1}0101\bar{1}$	$0\bar{1}1000\bar{1}010$
$10\bar{1}\bar{1}001000$	$001\bar{1}\bar{1}01\bar{1}10$	$0010\bar{2}00010$	$1\bar{1}000\bar{1}10\bar{1}0$

$1010\bar{1}\bar{1}0010$ $0000000\bar{1}1\bar{1}$ $10000000\bar{1}0$ $101\bar{1}\bar{1}0101\bar{1}$

$\bar{1}010000\bar{1}10$ $0000\bar{1}01000$ $1\bar{1}10\bar{1}01000$ $00000\bar{1}0010$

$10\bar{1}\bar{1}00100\bar{1}$ $1010\bar{2}00010$ $001\bar{1}0\bar{1}0010$ $0000000\bar{1}00$

$1010\bar{3}01\bar{1}10$ $0\bar{1}100000\bar{1}0$ $10000\bar{1}100\bar{1}$ $1000\bar{1}0101\bar{1}$

$\bar{1}01000\bar{1}010$ $1\bar{1}0\bar{1}001000$ $0000\bar{1}01\bar{1}10$ $001\bar{1}\bar{1}00010$

$1\bar{1}00\bar{1}010\bar{1}0$ $1010\bar{1}\bar{2}1010$ $000000\bar{1}01\bar{1}$ $1000000\bar{1}00$

$101\bar{1}\bar{1}01000$ $\bar{1}010000\bar{1}1\bar{1}$ $1000\bar{1}01000$ $0\bar{1}10\bar{1}01000$

$0\bar{1}100\bar{1}0010$ $10\bar{1}000100\bar{1}$ $101\bar{1}\bar{1}00010$ $001\bar{1}\bar{1}00010$

$00000\bar{1}1\bar{1}00$ $1010\bar{3}00010$ $0\bar{1}10000\bar{1}00$ $10000\bar{1}10\bar{1}0$

$1000\bar{1}0101\bar{2}$ $001000\bar{1}010$ $000\bar{1}001000$ $0\bar{1}10\bar{1}01\bar{1}10$

$0000\bar{1}00010$ $1\bar{1}0\bar{1}0010\bar{1}0$ $1010\bar{2}\bar{1}1010$ $00000\bar{1}001\bar{1}$

$100000\bar{1}000$ $101\bar{1}\bar{1}01\bar{1}10$ $\bar{1}010000\bar{1}00$ $1000\bar{1}0100\bar{1}$

$1\bar{1}10\bar{1}01000$ $\bar{1}0100\bar{1}0010$ $1\bar{1}\bar{1}000100\bar{1}$ $1000\bar{1}00010$

$001\bar{1}000010$ $0000\bar{1}01\bar{1}00$ $1010\bar{3}\bar{1}1010$ $0\bar{1}1000\bar{1}000$

$10000\bar{1}1\bar{1}00$ $1000\bar{1}0100\bar{1}$ $001000\bar{1}01\bar{1}$ $100\bar{1}001000$

$\bar{1}010\bar{1}01\bar{1}10$ $0\bar{1}10\bar{1}00010$ $1\bar{1}\bar{1}00010\bar{1}0$ $101\bar{1}\bar{1}\bar{1}1010$

$0000\bar{1}0001\bar{1}$ $10000\bar{1}0000$ $101\bar{1}\bar{1}00010$ $\bar{1}010000\bar{2}10$

$1000\bar{1}010\bar{1}0$ $1\bar{1}10\bar{1}0100\bar{1}$ $00100\bar{1}0010$ $00\bar{1}000100\bar{1}$

$1\bar{1}10\bar{1}00010$ $000\bar{1}000010$ $000\bar{1}001\bar{1}00$ $1010\bar{2}$†

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* This table is to be read off in lines. The first three lines set out in full (omitting the *null* terms) will mean

$$\psi(x)+\psi\left(\frac{x}{13}\right)-\psi\left(\frac{x}{14}\right)-\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{17}\right)+\psi\left(\frac{x}{19}\right)-\psi\left(\frac{x}{21}\right)-\psi\left(\frac{x}{22}\right)$$

$$+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{28}\right)+\psi\left(\frac{x}{29}\right)+\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{33}\right)-\psi\left(\frac{x}{35}\right)+\psi\left(\frac{x}{37}\right)$$

$$+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)-\psi\left(\frac{x}{44}\right)-\psi\left(\frac{x}{45}\right)+\psi\left(\frac{x}{47}\right)+\psi\left(\frac{x}{53}\right)$$

$$-\psi\left(\frac{x}{55}\right)-\psi\left(\frac{x}{56}\right)+\psi\left(\frac{x}{59}\right)+\psi\left(\frac{x}{61}\right)-\psi\left(\frac{x}{63}\right)-\psi\left(\frac{x}{66}\right)+\psi\left(\frac{x}{67}\right)$$

$$-\psi\left(\frac{x}{70}\right)+\psi\left(\frac{x}{71}\right)+\psi\left(\frac{x}{73}\right)-\psi\left(\frac{x}{75}\right)-\psi\left(\frac{x}{77}\right)+\psi\left(\frac{x}{79}\right)$$

$$+\psi\left(\frac{x}{83}\right)-\psi\left(\frac{x}{85}\right)-\psi\left(\frac{x}{88}\right)+\psi\left(\frac{x}{89}\right)+\psi\left(\frac{x}{97}\right)-\psi\left(\frac{x}{98}\right)-\psi\left(\frac{x}{99}\right)+\psi\left(\frac{x}{101}\right)$$

$$+\psi\left(\frac{x}{103}\right)-3\psi\left(\frac{x}{105}\right)+\psi\left(\frac{x}{107}\right)+\psi\left(\frac{x}{109}\right)-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right)+\psi\left(\frac{x}{113}\right)$$

† By actual summation it will be found as stated towards the end of p. 104 that the sum reckoned from the beginning of the positive and negative integers in this table always lies between 2 and -2 (both inclusive).

If we confine our attention exclusively to the outstanding group of the Major Arrangement, which extends to the 616^{th} term inclusive, *without taking advantage* of any of the other groups, we shall find, on making $E = 1{\cdot}0551851$, $F = {\cdot}9461974$, and N (the stigmatic multiplier) $= {\cdot}9909532$.

$$\frac{[\psi(x)]}{x} < N + \left(\frac{1}{15}+\frac{1}{22}+\frac{1}{28}+\frac{1}{35}+\frac{1}{45}+\frac{1}{56}+\frac{1}{66}+\frac{1}{77}+\frac{1}{88}+\frac{1}{99}\right.$$
$$\left.+\frac{1}{105}+\frac{1}{126}+\frac{1}{525}+\frac{1}{616}\right)E$$
$$-\left(\frac{1}{17}+\frac{1}{23}+\frac{1}{29}+\frac{1}{37}+\frac{1}{47}+\frac{1}{59}+\frac{1}{71}+\frac{1}{79}+\frac{1}{89}+\frac{1}{113}+\frac{1}{227}\right)F$$

$< 1{\cdot}0542390\ldots$ which is inferior in value to E.

This is enough to assure us that a better result than the one last found would be obtained by using the above scheme to furnish the major as well as the minor arrangement, instead of combining it, as we have done, with the scheme [1, 6, 70; 2, 3, 5, 7, 210].

Mr. Hammond has been good enough to work out for me in the annexed scholium the *complete* approximation to the limits to $\psi(x)$ given by the original scheme of Tschebyscheff [1, 30; 2, 3, 5]: this approximation preserves precisely the same form as that obtained by the crude method, and, although it lies a little out of the track which I had marked out for myself in this paper, will I think, besides being possibly valuable for future purposes in a more or less remote future, serve as an example to clear up any obscurity that may have pervaded the previous exposition of the purely asymptotic portion of these limits.*

§ 3. *Scholium. Containing an example of the complete* i^{th} *approximation to the limits to the prime-log-sum-sum to* x.

Using S to denote the stigmatic series

$$\psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \ldots,$$

we have the inequalities

$$\left.\begin{array}{l} S > Ax - \frac{5}{2}\log x - 1 \\ S < Ax + \frac{5}{2}\log x \end{array}\right\} \text{(Serret, p. 233)},$$

* In the paragraph commencing at the foot of p. 94 in the preceding number, a theorem (too simple to require a formal proof) is tacitly assumed which virtually amounts to saying :

If an equal number of black and white beads be strung upon a wire, in such a way that on telling them all, from left to right, more white than black ones are never told off, then the whole number of beads, as they stand, may be sorted into pairs, in each of which a black bead lies to the left of a white one.

which, as explained in the preceding section, may be replaced by

$$\psi(x) > Ax - \tfrac{5}{2}\log x - 1 + \psi\left(\frac{x}{24}\right) - \psi\left(\frac{x}{29}\right) \ldots\ldots\ldots\ldots(1),$$

$$\psi(x) < Ax + \tfrac{5}{2}\log x + \psi\left(\frac{x}{6}\right) - \psi\left(\frac{x}{7}\right) + \psi\left(\frac{x}{10}\right) \ldots\ldots\ldots(2).$$

If now we assume

$$\psi(x) > p_i Ax + q_i(\log x)^2 + r_i(\log x) + s_i \ldots\ldots(3),$$

$$\psi(x) < t_i Ax + u_i(\log x)^2 + v_i(\log x) + w_i \ldots\ldots(4),$$

we obtain, by combining these inequalities with (1),

$$\begin{aligned}\psi(x) > Ax &\qquad - \tfrac{5}{2}\log x \qquad - 1\\ &+ \tfrac{1}{24}p_i Ax + q_i(\log x - \log 24)^2 + r_i(\log x - \log 24) + s_i\\ &- \tfrac{1}{29}t_1 Ax - u_i(\log x - \log 29)^2 - v_i(\log x - \log 29) - w_i.\end{aligned}$$

Say $\psi(x) > p_{i+1}Ax + q_{i+1}(\log x)^2 + r_{i+1}(\log x) + s_{i+1}$,

where

$p_{i+1} = \tfrac{1}{24}p_i - \tfrac{1}{29}t_i + 1$,

$q_{i+1} = q_i - u_i$,

$r_{i+1} = r_i - v_i + 2u_i\log 29 - 2q_i\log 24 - \tfrac{5}{2}$,

$s_{i+1} = s_i - w_i + q_i(\log 24)^2 - u_i(\log 29)^2 - r_i\log 24 + v_i\log 29 - 1$.

Similarly, combining (3) and (4) with (2), we find

$$\begin{aligned}\psi(x) < Ax &\qquad + \tfrac{5}{2}\log x\\ &+ \tfrac{1}{6}t_i Ax + u_i(\log x - \log 6)^2 + v_i(\log x - \log 6) + w_i\\ &- \tfrac{1}{7}p_i Ax - q_i(\log x - \log 7)^2 - r_i(\log x - \log 7) - s_i\\ &+ \tfrac{1}{10}t_i Ax + u_i(\log x - \log 10)^2 + v_i(\log x - \log 10) + w_i.\end{aligned}$$

Say $\psi(x) < t_{i+1}Ax + u_{i+1}(\log x)^2 + v_{i+1}(\log x) + w_{i+1}$,

where

$t_{i+1} = \tfrac{4}{15}t_i - \tfrac{1}{7}p_i + 1$,

$u_{i+1} = 2u_i - q_i$,

$v_{i+1} = 2v_i - r_i + 2q_i\log 7 - 2u_i\log 60 + \tfrac{5}{2}$,

$w_{i+1} = 2w_i - s_i - q_i(\log 7)^2 + u_i\{(\log 6)^2 + (\log 10)^2\} + r_i\log 7 - v_i\log 60$.

These, together with the 4 given above, constitute a set of 8 difference equations for the determination of p_i, q_i, r_i, s_i, t_i, u_i, v_i, w_i. Their initial values are furnished by the inequalities

$$\left.\begin{aligned}\psi(x) &> Ax - \tfrac{5}{2}\log x - 1\\ \psi(x) &< \tfrac{6}{5}Ax + \frac{5}{4\log 6}(\log x)^2 + \tfrac{5}{4}\log x + 1\end{aligned}\right\}\text{ (Serret, p. 236),}$$

which give

$$p_0 = 1,\ q_0 = 0\qquad,\ r_0 = -\tfrac{5}{2},\ s_0 = -1,$$
$$t_0 = \tfrac{6}{5},\ u_0 = \frac{5}{4\log 6},\ v_0 = \ \tfrac{5}{4},\ w_0 = \ 1.$$

The values of p_i, t_i will be found to be

$$p_i = \frac{1}{50999}\left\{51072 - 36\tfrac{1}{2}(\rho^i + \rho_1^i) - 47\tfrac{211}{2320}\left(\frac{\rho^i - \rho_1^i}{\rho - \rho_1}\right)\right\},$$
$$t_i = \frac{1}{50999}\left\{59595 + 801\tfrac{9}{10}(\rho^i + \rho_1^i) + 190\tfrac{2397}{2800}\left(\frac{\rho^i - \rho_1^i}{\rho - \rho_1}\right)\right\},$$

where ρ, ρ_1 are the roots of the equation

$$(\rho - \tfrac{4}{15})(\rho - \tfrac{1}{24}) = \frac{1}{203},$$

and it is easy to verify that these values (which agree with the general ones, involving arbitrary constants, obtained in the preceding section) satisfy the initial conditions

$$p_0 = 1,\ p_1 = \tfrac{1}{24}p_0 - \tfrac{1}{29}t_0 + 1 = 1\tfrac{1}{3480},$$
$$t_0 = \tfrac{6}{5},\ t_1 = \tfrac{4}{15}t_0 - \tfrac{1}{7}p_0 + 1 = 1\tfrac{31}{175}.$$

The values of q_i and u_i, obtained from the equations

$$q_{i+1} = q_i - u_i,\quad u_{i+1} = 2u_i - q_i,$$

with the initial conditions $q_0 = 0$, $u_0 = \dfrac{5}{4\log 6}$, are

$$q_i = -\frac{5}{4\log 6}\left(\frac{\alpha^i - \alpha^{-i}}{\alpha - \alpha^{-1}}\right),$$
$$u_i = \frac{5}{8\log 6}\left(\alpha^i + \alpha^{-i} + \frac{\alpha^i - \alpha^{-i}}{\alpha - \alpha^{-1}}\right),$$

where α, α^{-1} are the roots of the equation

$$\alpha^2 - 3\alpha + 1 = 0.$$

The values of r_i, s_i, v_i, w_i are linear functions of q_i, u_i whose coefficients are linear functions of i in the case of r_i, v_i and quadratic functions of i in the case of s_i, w_i.

Thus we find, when the constants are properly determined,

$$r_i = -(2\log 6 + \lambda i)\,u_i + \{\kappa - \lambda - 2\log 29 + \log 6 - (\kappa+\lambda)\,i\}\,q_i,$$

$$v_i = \ \ (3\log 6 - \kappa i)\,u_i + (2\log 10 + \lambda - 2\kappa - \lambda i)\,q_i - \tfrac{5}{2},$$

where $\qquad \kappa = \tfrac{2}{5}\log\left(\dfrac{24^3.60^2}{7.29}\right),\ \lambda = \tfrac{2}{5}\log\left(\dfrac{24^4.60}{7^3.29^3}\right).$

The substitution of these values of r_i and v_i in the equations for determining s_i and w_i, will give a pair of equations of the form

$$s_{i+1} = s_i - w_i \ + (a+bi)\,q_i + (c+di)\,u_i - (1 + \tfrac{5}{2}\log 29),$$

$$w_{i+1} = 2w_i - s_i + (e+fi)\,q_i + (g+hi)\,u_i - \tfrac{5}{2}\log 60,$$

where a, b, c, d, e, f, g, h are known constants, and q_i, u_i are known linear functions of α^i, α^{-i}.

Ex. gr. the value of a is

$$(\log 24)^2 - (\kappa - \lambda - 2\log 29 + \log 6)\log 24 + (2\log 10 + \lambda - 2\kappa)\log 29.$$

From these equations we should obtain a result of the form

$$s_i = Q_1\alpha^i + R_1\alpha^{-i} + C_1,$$

$$w_i = Q_2\alpha^i + R_2\alpha^{-i} + C_2,$$

in which C_1, C_2 are constants and Q_1, Q_2, R_1, R_2 quadratic functions of i, but the complete determination of these would occupy too much space to be given here.

Sequel to Part 2, *Chapter* 1, § 2.

Since § 2 of this chapter was sent to press I have had asymptotic limits to $\psi(x) \div x$ computed by means of a scheme whose stigmata contain simply and in combination all the prime numbers up to 13 inclusive. The numerical results obtained on the one hand and on the other the process employed to determine *à priori* (so as to save the labor of working out the 30030 terms of a complete period) the minimum and maximum values (-1 and 4) of the sum of the coefficients of any number of consecutive terms (the first included) in the stigmatic series proper to the scheme, appear to me too noteworthy to be consigned to oblivion.

This calculation differs from those that precede it in the circumstance that it does not attempt to give the *optimistic* limits which the scheme will afford, notwithstanding which the limits actually obtained will be found to be each of them materially closer to unity than the optimistic limits furnished by any of the preceding schemes.

The scheme I adopt is [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001], which satisfies the necessary condition that the sums of the reciprocals of the numbers on the two sides of the semicolon are equal to one another.

The first thing to be done is to discover the maximum and minimum values of

$$S_n = E\left(\frac{n}{1}\right) + E\left(\frac{n}{6}\right) + E\left(\frac{n}{10}\right) + E\left(\frac{n}{14}\right) + E\left(\frac{n}{105}\right)$$
$$- E\left(\frac{n}{2}\right) - E\left(\frac{n}{3}\right) - E\left(\frac{n}{5}\right) - E\left(\frac{n}{7}\right) - E\left(\frac{n}{11}\right)$$
$$- E\left(\frac{n}{13}\right) - E\left(\frac{n}{385}\right) - E\left(\frac{n}{1001}\right).$$

On taking n equal to 66, it will be found that the value of S_n is -1: I shall proceed to show that this is the minimum, in other words that $-S_n$ cannot be so great as 2.

Denote the fractional part of any quantity x by $F(x)$: if $-S_n$ is not less than 2, then it may be shown that *à fortiori*

$$F\left(\frac{n}{6}\right) + F\left(\frac{n}{10}\right) + F\left(\frac{n}{14}\right)$$
$$- F\left(\frac{n}{2}\right) - F\left(\frac{n}{3}\right) - F\left(\frac{n}{5}\right) - F\left(\frac{n}{7}\right) + F\left(\frac{n}{105}\right),$$

say $Q(n) + F\left(\frac{n}{105}\right)$ must be not less than 2, and therefore $Q(n)$ must be greater than 1: now it is not difficult to show that $Q(n)$ is only greater than unity when

$$n = 106 + 210\kappa \text{ or } n = 136 + 210\kappa$$

(κ being a positive integer). But corresponding to these two values it will be found that

$$Q(106) + F\left(\frac{106}{105}\right) = \tfrac{1}{3} + \tfrac{2}{5} + \tfrac{3}{7} + \tfrac{1}{105},$$

$$Q(136) + F\left(\frac{136}{105}\right) = \tfrac{1}{3} + \tfrac{2}{5} + \tfrac{2}{7} + \tfrac{31}{105},$$

so that on either supposition $Q(n) + F\left(\frac{n}{105}\right)$ is less than 2.

Hence the minimum value of S_n is -1, and consequently, since the stigmatic excess is here $8 - 5$, the maximum value, as appears from the footnote below, will be $8 - 5 + 1$, *i.e.* 4.* [By the stigmatic excess for any scheme I mean the number of stigmata in the right-hand less the number of those in the left-hand set. This excess is obviously equal to the coefficient, with its sign changed, of $\psi\left(\frac{x}{\mu}\right)$ in the stigmatic series, where μ is any common multiple of the stigmata.]

It will be found, on summing up the numbers in Table B, that S_n first attains the value 4 when $n = 1891$, and the value -1 when $n = 66$.

For the inferior limit the outstanding group consists of all the terms up to 1891 inclusive, and for the superior limit all the terms up to 66 inclusive. But in obtaining this limit advantage has been taken of the next three groups, which end with 78, 418, and 2068 respectively. Thus the extreme limit of the following table is 2068, instead of being 30030 (*i.e.* 2.3.5.7.11.13) which is the number of terms in a complete period. It contains the coefficients of the first 2068 terms of the stigmatic series for the scheme [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001] written down in horizontal order in regular batches of ten, as was done in table A for the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105] with the unimportant difference that (for typographical convenience) negative coefficients are indicated by dots instead of by bars placed over them.

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* If we call c_n the coefficient of $\psi\left(\frac{x}{n}\right)$ and S_n the sum of such coefficients up to c_n inclusive (regarding c_0 and S_0 as zero), and take μ the least common multiple of the stigmata, we have, obviously,

$$S_\mu = 0,\ c_n = c_{\mu-n}, \text{ and } (S_n + S_{\mu-1-n}) - (S_{n-1} + S_{\mu-n}) = c_n - c_{\mu-n} = 0.$$

Consequently,

$$S_n + S_{\mu-1-n} = S_0 + S_{\mu-1} = -c_\mu = \eta \text{ (the stigmatic excess).}$$

This is a valuable formula of verification, and moreover gives a rule for finding either the maximum or minimum coefficient-sum when the other sum is given: for if S_n has the maximum value, $S_{\mu-1-n} = \eta - S_n$; if this is not the minimum let S_n' be less than $\eta - S_n$, then $S_{\mu-1-n}'$ will be greater than S_n, contrary to hypothesis. Hence the minimum value of a coefficient-sum may be found by subtracting the maximum from the stigmatic excess and *vice versâ.*

[I may perhaps be allowed to add that this theorem suggests a generalization of itself, which I think it is safe to anticipate may be formally deduced from it, viz.:

If $a_1, a_3, \ldots, a_n$; $a_1, a_2, \ldots, a_\nu$ *be any given positive quantities (integer or fractional, rational or irrational) such that* $\Sigma a = \Sigma \alpha$, *and if* $-m$, M *be the least and greatest values that* $\Sigma E(\alpha x) - \Sigma E(ax)$ *can assume when* x *is any positive quantity whatever, then* $M - m = \nu - n$].

Table B.

The coefficients of the first 2068 *terms of the stigmatic series to*
[1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001].

1000000000	0000i01010	ii100i0010	10i0i010i0
101ii01000	0i10i00010	10i0ii1000	1010i0ii10
0010000i10	i00000i0i0	101ii0101i	001000i000
000000100i	1i00i01010	00i000i010	100i0i1000
00102̇01000	0010ii0010	1i000000i0	10102̇01i10
0000000i01	100000000i	0010i01010	2̇01i000010
1i00i00000	1000i0100i	001i000010	102̇0i01000
1010i2̇0010	001000i000	0000001i00	1i10i01000
0010i0000i	1000001i00	0000i01010	ii1000i010
10ii001000	001ii00010	00103̇0001i	10000i10i0
1000i00010	00000i0i11	10000000 2̇0	1010i0101i
ii10000010	00002̇01000	1i10i01i00	0000000010
00ii001000	101i2̇00010	00100ii010	000000000i
1010i01i10	00000000i0	10000i100i	1000i01000
i01000i010	12̇00001000	00002̇01010	001ii00i10
1000i010i0	0010ii1010	000i00i011	100000ii00
1010i0100i	i01000001i	10i0i01000	0010ii1000
0i10000000	102̇0001000	1i1ii00010	00102̇00010
00000i1i00	1010i00010	i010000i00	100i0010i0
1000i00012̇	001000i01i	0000001000	0i00i01010
0000ii0010	100i00102̇0	10102̇01010	0i000i0011
1000i0i000	1010i012̇10	i010000000	0000i0100i
101ii01000	i01000i010	1ii000100i	1000i00010
000i000010	0000ii1000	1010ii1000	001000i000
1i00001i00	10002̇01000	001000ii1i	1000001000

3̇0101̇01010 01̇11̇1̇00010 101̇000001̇0 1011̇1̇01011̇
00001̇00011 101̇001̇0000 10101̇1̇0010 1̇0100001̇00
10001̇0101̇0 11̇101̇01001̇ 00101̇00010 001̇00011̇00
11̇101̇00010 1̇000000010 0002̇001000 10103̇00010
001001̇0001̇ 1000000000 101̇01̇01̇110 001001̇1̇000

100000101̇1̇ 00101̇01010 1̇1̇101̇00010 11̇001̇0101̇0
10001̇011̇10 0001̇000011 00001̇00000 1011̇1̇1̇1010
1̇010002̇010 10001̇011̇01̇ 10101̇0101̇0 0000000011̇
101̇001̇1000 00101̇00000 01̇10000010 01̇1̇0001000
1011̇2̇01010 00101̇001̇00 100001̇1000 00001̇00010
0011̇001̇1̇10 1000000001̇0 00101̇01012̇ 00101̇00010
001̇000101̇0 11̇101̇1̇1010 001̇0000001 1001̇000000
11̇102̇01010 1̇0101̇1̇0010 10001̇001̇00 10101̇011̇00
1̇010000000 101̇1̇001002̇ 10101̇01̇010 1̇010000011̇
01̇00001000 101̇01̇01010 0011̇01̇0000 10001̇0101̇0

10001̇1̇1010 01̇10002̇010 10001̇011̇00 00101̇011̇00
00101̇00011̇ 000000101̇0 0011̇1̇01010 01̇00001̇011
101̇0000001̇ 1011̇1̇01010 1̇0001̇00010 10001̇2̇1000
10101̇0001̇0 00100001̇10 11̇1̇000101̇0 10102̇00011̇
00100001̇10 1̇000001000 01̇101̇01010 0001̇000000
1001̇000000 10002̇01011̇ 001001̇1̇010 101̇0000000
00101̇1̇11̇10 00101̇0001̇0 100000101̇1̇ 11̇101̇01010
1̇0001̇00011 11̇000001̇00 10001̇01010 2̇011̇000010
1001̇3̇01000 10101̇1̇0000 0010001̇011̇ 101̇00011̇00
10001̇00000 001001̇0011̇ 000000101̇0 00101̇01010

3̇010000000 101̇01̇01000 1001̇1̇011̇10 00101̇01̇010
000001̇1000 0011̇1̇00010 00101̇01̇1̇

In Tables I and II below, in addition to pairs of numbers $-(\eta)+(\eta+\theta)$, meaning $-\psi\left(\frac{x}{\eta}\right)+\psi\left(\frac{x}{\eta+\theta}\right)$, and $+(\eta)-(\eta+\theta)$ meaning $+\psi\left(\frac{x}{\eta}\right)-\psi\left(\frac{x}{\eta+\theta}\right)$, there will be found the unpaired numbers (15) and (66) in the one and (19), (229) and (1891) in the other: to understand how these are got, it should be observed that S_n (the sum of the first n numbers in Table B) first becomes 0 when $n=15$, first becomes -1 when $n=66$ and first becomes 2, 3, 4 when $n=19, 229, 1891$ respectively.*

Table I.

				+ (15)
−	(17)	+	(22)	
−	(19)	+	(21)	
−	(23)	+	(26)	
−	(29)	+	(35)	
−	(41)	+	(45)	
−	(47)	+	(52)	
−	(59)	+	(65)	+ (66)
−	(67)	+	(78)	
−	(79)	+	(418)	
−	(107)	+	(135)	
−	(210)	+	(275)	
−	(289)	+	(385)	
−	(419)	+	(2068)	
−	(521)	+	(585)	
−	(629)	+	(795)	
−	(839)	+	(936)	
−	(1049)	+	(1144)	
−	(1717)	+	(1925)	

Table II.

+	(15)	−	(17)	− (19)
+	(21)	−	(31)	
+	(26)	−	(29)	
+	(33)	−	(43)	
+	(44)	−	(61)	
+	(63)	−	(73)	
+	(65)	−	(71)	
+	(75)	−	(103)	− (229)
+	(242)	−	(271)	
+	(285)	−	(323)	
+	(385)	−	(421)	
+	(385)	−	(439)	
+	(440)	−	(493)	
+	(494)	−	(571)	
+	(770)	−	(841)	
+	(1155)	−	(1273)	− (1891)

* Call Σ the sum of the infinite series given by Table B: it may then easily be verified that $\{\psi(x)-\Sigma\}-\left\{\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{66}\right)\right\}$ may be resolved into term-pairs of the form $-\psi\left(\frac{x}{\eta}\right)+\psi\left(\frac{x}{\eta+\theta}\right)$ that shall contain among them all those in Table I, and

$$\{\psi(x)-\Sigma\}+\left\{\psi\left(\frac{x}{19}\right)+\psi\left(\frac{x}{229}\right)+\psi\left(\frac{x}{1891}\right)\right\}$$

The reasoning employed in dealing with previous schemes serves to show that superior and inferior asymptotic limits to $\psi(x) \div x$, which we shall call E_1, F_1 in order to distinguish them from the corresponding optimistic limits (E, F), may be found from the equations

$$\left.\begin{aligned} E_1 &= M + aE_1 - bF_1 \\ F_1 &= M + cF_1 - dE_1 \end{aligned}\right\}.$$

where a is the sum of the reciprocals of the numbers occurring in Table I. with the sign +
b „ „ „ −
c „ „ Table II. „ +
d „ „ „ −

and M is the stigmatic multiplier.

viz. $$a = \frac{1}{15} + \frac{1}{21} + \frac{1}{22} + \ldots\ldots + \frac{1}{2068} = \cdot 33352\ldots,$$

$$b = \frac{1}{17} + \frac{1}{19} + \frac{1}{23} + \ldots + \frac{1}{1717} = \cdot 30580\ldots,$$

$$c = \frac{1}{15} + \frac{1}{21} + \frac{1}{26} + \ldots + \frac{1}{1155} = \cdot 26966\ldots,$$

$$d = \frac{1}{17} + \frac{1}{19} + \frac{1}{29} + \ldots\ldots + \frac{1}{1891} = \cdot 27742\ldots,\text{*}$$

and $$M = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{5}\log 5 + \frac{1}{7}\log 7$$

$$+ \frac{1}{11}\log 11 + \frac{1}{13}\log 13 + \frac{1}{385}\log 385 + \frac{1}{1001}\log 1001$$

$$- \frac{1}{6}\log 6 - \frac{1}{10}\log 10 - \frac{1}{14}\log 14 - \frac{1}{105}\log 105 = \cdot 98859\ldots.$$

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into term-pairs of the form $+\psi\left(\frac{x}{\eta}\right) - \psi\left(\frac{x}{\eta+\theta}\right)$ that shall contain among them all those in Table II above.

[The maximum value of S_n is here 4: if it had been 2, then instead of 3 unpaired positive terms appended to $\{\psi(x) - \Sigma\}$ there would have been but 1. This is what happens for the scheme [1, 15; 2, 3, 5, 30] given in the footnote on p. 96: and accordingly, we see that $\{\psi(x) - \Sigma\} + \psi\left(\frac{x}{17}\right)$, for that scheme, is resoluble into paired terms of the form $+\psi\left(\frac{x}{\eta}\right) - \psi\left(\frac{x}{\eta+\theta}\right)$. So again, the minimum being 0 (instead of -1), there will be but 1 unpaired negative term to append to $\{\psi(x) - \Sigma\}$, and accordingly, we see that $\{\psi(x) - \Sigma\} - \psi\left(\frac{x}{6}\right)$ in that scheme is resoluble into term-pairs of the form $-\psi\left(\frac{x}{\eta}\right) + \psi\left(\frac{x}{\eta+\theta}\right)$].

* The above values of a, b, c, d give $a + c = \cdot 603\ldots$ and $ac - bd = \cdot 005\ldots$, and consequently the roots of the "characteristic" equation $\rho^2 - (a+c)\rho + (ac - bd) = 0$ satisfy the necessary condition of being each less than unity in absolute value.

$$\text{Hence}\quad E_1 = \frac{(1-c-b)\,M}{(1-a)(1-c)-bd} = 1{\cdot}04423\ldots,$$

$$F_1 = \frac{(1-a-d)\,M}{(1-a)(1-c)-db} = {\cdot}95695\ldots,$$

[so that the mean of E_1 and F_1 is less than ·0006], and

$$\frac{E_1}{F_1} = 1{\cdot}09120\ldots.*$$

Thus then, (see footnote to p. 9 of the May number) by taking x sufficiently great, the number of primes not exceeding x, multiplied by $\log x$ and divided by x, may always be made to lie between the numbers

1·04423... and ·95695...,

the divergences of which from unity are

·04423... and ·04304... (as against

Tschebyscheff's ·10555... and ·07807...).

These divergences, there is little doubt, would become even more nearly equal than they are, if any one should feel inclined to undertake the very laborious task of extracting the *optimistic* values (E, F) from the scheme employed.

In order to understand this necessarily abbreviated sketch of a method more easy to think out and apply than to find language to express, I must not conceal that a careful study of the several schemes given, and of the principles embodied in the calculations relating to them, is a *sine quâ non*. It may somewhat lighten the burden thrown upon the reader, if I add a few words concerning one or two points, perhaps inadequately explained in what precedes.

Let μ be the least common multiple of the stigmata of any given harmonic scheme and S_n the sum of the coefficients of $\psi(x)$, $\psi\left(\frac{x}{2}\right)$, $\psi\left(\frac{x}{3}\right)$,......$\psi\left(\frac{x}{n}\right)$ in the corresponding stigmatic series. Then from the last formula (p. 95) combined with the equation which connects the stigmata, it follows that

$$S_\mu = 0,\ S_{n+\mu} = S_n$$

* In tables I and II above, the ratio $\frac{\eta+\theta}{\eta}$ is greater than 1·09120... for every pair of terms except $-(1049)+(1144)$ in table I. In the case of this pair, we have $\frac{1144}{1049} = 1{\cdot}0905\ldots$, which shows that the exclusion of it from that table would have led to asymptotic limits better (but very slightly so) than those arrived at in the text.

Hence an infinite number of values of n will give S_n its greatest value; the difference of these values will be of the form $k\mu - \mu'$ where μ' may, and in general will, besides zero have various other values less than μ, thus giving rise to the collections of terms called *groups* (see p. 94) of which the period of μ terms will be composed. The same will be true when we substitute the word *least* for *greatest.*

If now i be taken *any* number such that S_i has its greatest value it may be shown that the sum of all the terms in the stigmatic series subsequent to the one containing $\psi\left(\frac{x}{i}\right)$ will be *negative* or zero, and similarly when S_i has its least value such sum will be *positive* or zero;* consequently when i is properly determined we can find immediately a superior limit in the one case and an inferior limit in the other, to the sum of the first i terms of the series.

I will conclude this portion of the subject with the remark that from the values of E_1 and F_1 it is easy to infer that if μ is equal to or less than $(\cdot 95695\ldots)\, k - (1\cdot 04423\ldots)$, and x exceeds a certain ascertainable number whose value depends on k and μ, then between x and kx there will be found more than $\mu \frac{x}{\log x}$ primes.†

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* The reason of this is that the sum of all the terms beyond the ith may be separated into partial sums, each containing μ terms, which ultimately vanish. If now $\gamma_1(k\mu+i+1)+\gamma_2(k\mu+i+2)+\ldots\ldots\gamma_\mu(k\mu+i+\mu)$ be one of them, then $\gamma_1+\gamma_2+\ldots\ldots+\gamma_t$ will be zero when $t=\mu$, and will have a constant algebraical sign (or else be zero) when $t<\mu$; from which it follows (see footnote p. 109 where, be it observed, a coefficient $+\lambda$ or $-\lambda$ is supposed to be represented by a *sequence* of λ black or λ white beads) that each partial sum may be decomposed into an aggregate of quantities of the form $+(\eta)-(\eta+\theta)$ or $-(\eta)+(\eta+\theta)$ according as the first coefficient in each such sum is positive or negative, and will therefore, if not zero, have the same algebraical sign as that coefficient has, viz. $-$ or $+$ according as S_i has its greatest or least value.

† In order that μ may be positive (which ensures the existence of *some* primes between x and kx, when x exceeds a certain limit) it is only necessary to take $k>1\cdot 09120\ldots$ (which differs very little from $\frac{12}{11}$), whereas if we limited ourselves to the results of the oft-quoted memoir of 1850, we could not prove the existence of prime numbers between x and kx, for a given value of x, however great, unless k exceeds $\frac{6}{5}$.