TWO NOTES ON WEIERSTRASS'S $P(u)$.

By *W. Burnside.*

1. *Forms of the addition equation.*

The addition equation for the elliptic function $P(u)$ is most simply expressed in the form

$$\begin{vmatrix} 1, & P(u)\ , & P'(u) \\ 1, & P(v)\ , & P'(v) \\ 1, & P(u+v), & P'(u+v) \end{vmatrix} = 0 \ \ldots\ldots\ldots\ldots \text{(i)}.$$

To pass from this to a form rational in $P(u)$, $P(v)$, $P(u+v)$; or in $P(u)$, $P(v)$, $P(w)$, where

$$u+v+w=0,$$

the process used by Halphen (Vol I., p. 58) may conveniently be employed.

Thus, if $x=P(u)$, $y=P(v)$, $z=P(w)$, where

$$u+v+w=0,$$

equation (i) is equivalent to stating that x, y, z are roots of

$$4X^3-g_2X-g_3-(aX+b)^2=0,$$

where a, b are arbitraries.

Hence
$$x+y+z=\tfrac{1}{4}a^2,$$
$$yz+zx+xy+\tfrac{1}{4}g_2=\tfrac{1}{2}ab,$$
$$xyz-\tfrac{1}{4}g_3=\tfrac{1}{4}b^2,$$

and therefore

$$4(x+y+z)(xyz-\tfrac{1}{4}g_3)=(yz+zx+xy+\tfrac{1}{4}g_2)^2 \ \ldots\ldots \text{(ii)},$$

which is the rational form required.

If $s_\lambda^2=(x-e_\lambda)(y-e_\lambda)(z-e_\lambda)$ $(\lambda=1, 2, 3)$, equation (ii) may be transformed by direct substitution into

$$\Sigma(e_2-e_3)^4 s_1^4-2\Sigma(e_3-e_1)^2(e_1-e_2)^2 s_2^2 s_3^2=0,$$

or taking factors

$$(e_2-e_3)s_1 \pm (e_3-e_1)s_2 \pm (e_1-e_2)s_3=0.$$

Now $$s_\lambda = \frac{\sigma_\lambda(u)\,\sigma_\lambda(v)\,\sigma_\lambda(w)}{\sigma(u)\,\sigma(v)\,\sigma(w)},$$

and by taking the particular case of u, v, w all small it may be at once verified that the signs should all be positive: hence, finally,

$$(e_2-e_3)\,\sigma_1(u)\,\sigma_1(v)\,\sigma_1(w)+(e_3-e_1)\,\sigma_2(u)\,\sigma_2(v)\,\sigma_2(w)$$
$$+(e_1-e_2)\,\sigma_3(u)\,\sigma_3(v)\,\sigma_3(w)=0 \;\ldots\ldots\; \text{(iii)}.$$

It follows from this at once that there must be three equations of the form

$$\sigma_\lambda(u)\,\sigma_\lambda(v)\,\sigma_\lambda(w)=A+Be_\lambda\,,$$

where A and B are symmetric functions of u, v, and w.

These may be obtained by using (ii) to make the right-hand side of

$$s_\lambda^2=(x-e_\lambda)(y-e_\lambda)(z-e_\lambda)$$

the square of a linear function of e_λ.

Thus, using the identity

$$e_\lambda^3=\tfrac{1}{4}g_3+\tfrac{1}{4}g_2e_\lambda,$$

and substituting from (ii) for xyz,

$$s_\lambda^2=\frac{(yz+zx+xy+\frac{1}{4}g_2)^2}{4(x+y+z)}-\tfrac{1}{4}g_2e_\lambda-(yz+zx+xy)\,e_\lambda$$
$$+(x+y+z)\,e_\lambda^2$$
$$=\frac{[yz+zx+xy+\frac{1}{4}g_2-2(x+y+z)\,e_\lambda]^2}{4(x+y+z)};$$

or

$$\frac{\sigma_\lambda(u)\,\sigma_\lambda(v)\,\sigma_\lambda(w)}{\sigma(u)\,\sigma(v)\,\sigma(w)}=\frac{yz+zx+xy+\frac{1}{4}g_2-2(x+y+z)\,e_\lambda}{2\sqrt{(x+y+z)}}\;\ldots\text{(iv)}.$$

This last equation shews incidentally that, when $u+v+w=0$, $\sqrt{\{P(u)+P(v)+P(w)\}}$ is a one-valued function.

2. *On $P(u)$ considered as a covariant of a quartic.*

In a recent memoir on Hyperelliptic functions, Prof. Klein asks and answers the following question:

If $u=\int_y^z \frac{dz}{s_z}$, where s_z^2 is any quartic function of z (so that u is a perfectly general elliptic integral of the first kind), what is $P(u)$?

He gives the result and verifies its correctness by applying the addition equation for $P(u)$.

The following purely synthetical method of answering the question is perhaps not without interest.

Let $$s^2 = a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4.$$

The general value of u at any point on the Riemann's surface defined by this equation is

$$u_0 + m\omega + m'\omega',$$

where u_0 is a particular value, ω, ω' the periods of the integral, and m, m' any integers: but

$$P(u_0 + m\omega + m'\omega') = P(u_0),$$

and hence $P(u)$ is a one-valued function on the Riemann's surface, and can therefore be expressed as a rational function of s_x, s_y, x and y. The only infinities of $P(u)$ considered as a function of u are double ones at the points $u = m\omega + m\omega'$: but these all correspond to the same point on the Riemann's surface, and hence $P(u)$ considered as a function of x must take every value twice on the surface and in particular must have a double infinity at the point corresponding to $u = 0$. Again, since $P(u)$ is an even function of u, and since interchanging x and y changes the sign of u, it must be a symmetrical function of x and y. Finally, to complete the determination it is necessary to quote the first terms in the expansion of $P(u)$, namely $P(u) = \frac{1}{u^2} +$ terms in u^2, &c.

To $u = 0$ corresponds $x = y$ and $s_x = s_y$; hence the function being symmetrical in x and y and having no infinity except a double one at this point, the most general form that can be assumed is

$$P(u=) \frac{\begin{array}{c} As_x s + (\alpha y^2 + \beta y + \gamma)\, s_x + (\alpha x^2 + \beta x + \gamma)\, s_y \\ + Bx^2y^2 + C(x^2y + xy^2) + D(x^2 + y^2) + E_1 xy + F(x+y) + G] \end{array}}{(x-y)^2}.$$

The numerator must have a double zero for $x = y$ and $s_x = -s_y$, since $P(u)$ is finite for this point. This involves

$$\alpha = \beta = \gamma = 0,$$

and $$-As_x^{\,2} + Bx^4 + 2Cx^3 + (2D + E)\,x^2 + 2Fx + G = 0,$$

for all values of x.

There results for $P(u)$ the simpler form

$$P(u) = A\,\frac{s_x s_y + f(x, y)}{(x-y)^2} + A',$$

where

$$f(x, y) = a_0 x^2 y^2 + 2a_1 (x^2 y + x y^2) + a_2 (x^2 + 4xy + y^2) + 2a_3 (x + y) + a_4.$$

Finally the first two terms in the expansion of $P(u)$ in terms of u applied to the last form give

$$A = \tfrac{1}{2}, \quad A' = 0.$$

If the terms are all made homogeneous by writing, as Prof. Klein does, x_1/x_2 and y_1/y_2 for x and y, then

$$P(u) = \frac{s_{x_1x_2}\, s_{y_1y_2} + \frac{1}{1\cdot 2}\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right)^2 s^2_{x_1x_2}}{2\,(x_1y_2 - x_2y_1)^2},$$

in which form it is obviously a covariant of the original quartic.

ON ARITHMETICAL SERIES.

(Continued from p. 19).

By Professor *Sylvester.*

Part II.*

Explicit Primes.

In this part I shall consider the asymptotic limits to the number of primes of certain *irreducible* linear forms $mz + r$ comprised between a number x and a given fractional multiple thereof kx, the method of investigation being such that the asymptotic limits determined will be unaffectd by the value of r, and will be the same for all values of m which have the same totient. The simplest case, and the foundation of all that follows, is that in which $k = 0$ and $m = 2$: this will form the subject of the ensuing chapter which may be regarded as a supplement to Tschebyscheff's celebrated memoir of 1850,† and as superseding my article thereon in Vol. IV. of the *Amer. Math. Journ.*

---

* I ought to have stated that the theorem contained in section 2 of Part I originally appeared in the form of a question (No. 10951) in the *Educational Times* for April of this year.

† Published in the St. Petersburg Transactions for 1854.