THE SUM OF THE CUBES OF THE COEFFICIENTS IN $(1-x)^{2n}$.

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In Vol. xx. of the *Messenger*, p. 79, Mr. A. C. Dixon shows that the sum in question is equal to $(-1)^n 3n!/(n!)^3$, a result which, he states, had been previously noticed by Mr. Morley. I propose the following algebraical method of proof.

Lemma. The absolute term is the expansion of

$$(x-x^{-1})^{2q}(x+x^{-1})^{2q} \text{ is } (-1)^p \frac{2p!\,2q!}{p!\,q!\,(p+q)!}.$$

If we denote the absolute term by $(-1)^p A(p, q)$, then since

$$(x-x^{-1})^{2p}(x+x^{-1})^{2(q+1)} \equiv (x-x^{-1})^{2(p+1)}(x-x^{-1})^{2q} + 4(x-x^{-1})^{2p}(x+x^{-1})^{2q}$$

we have $\quad A(p, q+1) = 4A(p, q) - A(p+1, q),$

whence the lemma may be established for successive values of q by induction.

Let $\qquad (1-x)^{2n} = 1 - a_1x + a_2x^2 - a_3x^3 + \ldots.$

Then $1 + a_1^2x^2 + a_2^2x^4 + a_3^2x^6 + \ldots$ is the sum of the terms which do not contain y in the expansion of

$$(1-xy)^{2n}(1-xy^{-1})^{2n} \text{ or of } \{(1+x^2) - xy - xy^{-1}\}^{2n}.$$

That is, by the Multinomial Theorem,

$$1 + a_1^2x^2 + a_2^2x^4 + \ldots = \Sigma_{k=0}^{k=n} \frac{2n!}{(2n-2k)!\,k!\,k!}(1+x^2)^{2n-2k}x^{2k}$$

$$= \Sigma_{k=0}^{k=n} \frac{2n!}{(2n-2k)!\,k!\,k!}(x+x^{-1})^{2n-2k}x^{2n}.$$

and $1 - a_1x^{-2} + a_2x^{-4} - \ldots = (1-x^{-2})^{2n} = (x-x^{-1})^{2n}x^{-2n}$.

Hence, by multiplication, $1 - a_1^3 + a_2^3 - a_3^3 + \ldots$ is the absolute term in the expansion of

$$\Sigma_{k=0}^{k=n} \frac{2n!}{(2n-2k)!\,k!\,k!}(x+x^{-1})^{2n-2k}(x-x^{-1})^{2n}$$

$$= (-1)^n \Sigma_{k=0}^{k=n} \frac{2n!}{(2n-2k)!\,k!\,k!} \times \frac{2n!\,(2n-2k)!}{n!\,(n-k)!\,(2n-k)!}$$

$$= (-1)^n \frac{2n!}{n!\,n!} \Sigma_{k=0}^{k=n} \frac{2n!}{(2n-k)!\,k!} \times \frac{n!}{(n-k)!\,k!}.$$

But $$(1+t)^{2n} = 1 + 2nt + \ldots + \frac{2n!}{(2n-k)!\,k!}t^k + \ldots\ldots,$$

$$(1+t^{-1})^n = 1 + nt^{-1} + \ldots + \frac{n!}{(n-k)!\,k!}t^{-k} + \ldots\ldots.$$

Multiplying these two series together, we see that

$$\Sigma_{k=0}^{k=n} \frac{2n!}{(2n-k)!\,k!} \times \frac{n!}{(n-k)!\,k!} = \text{absolute term in } \frac{(1+t)^{3n}}{t^n}$$

$$= \frac{3n!}{2n!\,n!}.$$

Therefore

$$1 - a_1^3 + a_2^3 - \ldots = (-1)^n \frac{2n}{n!\,n!} \times \frac{3n!}{2n!\,n!}$$

$$= (-1)^n \frac{3n!}{(n!)^3}.$$

NOTE ON THE SIMULTANEOUS TRANSFORMATION OF TWO QUADRATIC FUNCTIONS.

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If two quadratics in n variables $x_1, x_2, x_3, \ldots x_n$,

$$u \equiv a_{11}x_1^2 + a_{22}x_2^2 + \ldots + 2a_{12}x_1x_2 + \ldots,$$

and $$v \equiv b_{11}x_1^2 + b_{22}x_2^2 + \ldots + 2b_{12}x_1x_2 + \ldots,$$

be transformed by the linear substitution,

$$x_1 \equiv l_1X_1 + l_2X_2 + \ldots,$$
$$x_2 \equiv m_1X_1 + m_2X_2 + \ldots,$$
$$\&c., \qquad \&c.,$$

they take the forms

$$U \equiv A_{11}X_1^2 + A_{22}X_2^2 + \ldots + 2A_{12}X_1X_2 + \ldots,$$
$$V \equiv B_{11}X_1^2 + B_{22}X_2^2 + \ldots + 2B_{12}X_1X_2 + \ldots.$$

It is well-known that the necessary conditions that $A_{12}, A_{13}, \ldots, B_{12}, B_{13}, \ldots$, should all vanish is that $l_1 : m_1 : n_1 : \ldots$ should be proportional to the first minors of

$$\begin{vmatrix} a_{11}+\lambda_1 b_{11}, & a_{12}+\lambda_1 b_{12}, & a_{13}+\lambda b_{13}, & \ldots, \\ a_{12}+\lambda_1 b_{12}, & a_{22}+\lambda_1 b_{22}, & \ldots & ,\ldots, \\ a_{13}+\lambda_1 b_{13}, & \ldots, & \ldots & ,\ldots, \\ \ldots & \ldots, & \ldots & ,\ldots, \end{vmatrix} = 0,$$