As far as I know the only remark of Fermat which bears directly on the problem of Mersenne's numbers is the one referred to above in Art. 8: namely, that the prime factors of 2^p-1 are of the form $np+1$. I think it probable that Fermat was aware of the forms of n corresponding to various numerical values of p (see Art. 9); and it is possible that these are particular or partial cases of some general theorem, from which the results enunciated by Mersenne are deducible. It is manifest, however, that Fermat's remark would be insufficient by itself to enable anyone either to answer the question asked by Mersenne or to prove the assertion given above in Art. 2; nor are such propositions on the determination of whether a given number is prime as have been enunciated subsequently sufficient for these purposes. Hence the riddle as to how Mersenne's numbers were discovered remains unsolved.

NOTE ON A FORMULA IN SPHERICAL HARMONICS.

By *R. Fujisawa.*

It is well-known that

$$\frac{1-\alpha^2}{1-2\alpha\cos\gamma+\alpha^2}=1+2\alpha\cos\gamma+2\alpha^2\cos 2\gamma+\ldots+2\alpha^n\cos n\gamma+\ldots.$$

If $(1-2\alpha\cos\gamma+\alpha^2)^{-\frac{1}{2}}$ be expanded in a series of ascending powers of α, the coefficient of α^n is denoted by $P_n(\cos\gamma)$. Thus, by definition, we have

$$\frac{1}{\sqrt{(1-2\alpha\cos\gamma+\alpha^2)}}=P_0+P_1\alpha+P_2\alpha^2+\ldots+P_n\alpha^n+\ldots.$$

Squaring,

$$\frac{1}{1-2\alpha\cos\gamma+\alpha^2}=Q+Q_1\alpha+Q_2\alpha^2+\ldots+Q_n\alpha^n+\ldots,$$

$$\text{where}\quad Q_n=\begin{array}{ll}2P_0P_n+2P_1P_{n-1}+\ldots+2P_{\frac{1}{2}(n-1)}P_{\frac{1}{2}(n+1)} & (n\text{ odd}),\\ 2P_0P_n+2P_1P_{n-1}+\ldots+2P^2_{\frac{1}{2}n} & (n\text{ even}).\end{array}$$

Multiplying by $(1-\alpha^2)$,

$$\frac{1-\alpha^2}{1-2\alpha\cos\gamma+\alpha^2}=Q_0+Q_1\alpha+(Q_2-Q_0)\alpha^2+\ldots+(Q_n-Q_{n-2})\alpha^n+\ldots.$$

Hence, we obtain

$$2\cos n\gamma=Q_n-Q_{n-2}.$$

This is the formula which appears to have hitherto escaped notice. The formula will of course still hold good after replacing $\cos\gamma$ by

$$\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi'-\phi).$$

This formula may be used with advantage on several occasions. For example,

$$2\int_0^\pi\int_0^{2\pi}\cos 2n\gamma\sin\theta\,d\theta\,d\phi=\int_0^\pi\int_0^{2\pi}(Q_{2n}-Q_{2n-2})\sin\theta\,d\theta\,d\phi$$

$$=\int_0^\pi\int_0^{2\pi}P_n^2\sin\theta\,d\theta\,d\phi-\int_0^\pi\int_0^{2\pi}P^2{}_{n-1}\sin\theta\,d\theta\,d\phi.$$

Geometrical considerations show that, on the left-hand side, we may replace γ by θ. Thus

$$4\pi\left\{\frac{1}{2n+1}-\frac{1}{2n-1}\right\}$$

$$=\int_0^\pi\int_0^{2\pi}P_n^2\sin\theta\,d\theta\,d\phi-\int_0^\pi\int_0^{2\pi}P^2{}_{n-1}\sin\theta\,d\theta\,d\phi.$$

Hence it follows that

$$\int_0^\pi\int_0^{2\pi}P_n^2\sin\theta\,d\theta\,d\phi=\frac{4\pi}{2n+1},$$

a well-known result.

Tokio, *May*, 1891.