puts $F = (1+m)E$, so that his $m = -\frac{I}{E}$, and obtains an expression for $m$ depending upon $\frac{k'}{k}$.

I may mention that Mr. F. W. Newman, in his "Elliptic Integrals" (1889), considers the ratio $\frac{E}{F}$, which he terms the 'ancilla.' He gives a formula (p. 28) which may be written

$$-\frac{I}{kK} = \tfrac{1}{2}k + \tfrac{1}{4}kk_1 + \tfrac{1}{8}kk_1k_2 + \tfrac{1}{16}kk_1k_2k_3 + \&c.,$$

where $k_1$, $k_2$, ... are successive moduli formed according to Landen's scale.

It should be added also that the differential equations satisfied by $\frac{G}{K}$, $\frac{K}{G}$, $\frac{I}{E}$, $\frac{E}{I}$ were given by Mr. Kleiber, in vol. XVIII., p. 176, of the *Messenger* (April, 1889), the modular angle $\theta$ being the independent variable.

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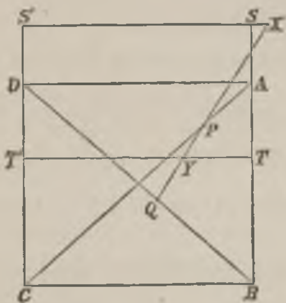
# NOTE ON RECIPROCAL LINES.

By Prof. *Cayley*.

If two lines are reciprocal in regard to a quadric surface, then any point on the one line and any point on the other line are harmonics in regard to the surface (viz. the two points and the intersections of their line of junction with the surface form a harmonic range). This is obvious, the polar plane of the first point passes through the second line, and thus the second point is a point in the polar plane of the first point, that is, the two points are harmonics.

But it is worth while to look at the theorem in a different point of view: if the first line meets the surface in the points $A$, $C$ and the second line meets the surface in the points $B$, $D$ (in order to the four points being real, the surface must of course be a skew hyperboloid), then $AB$, $BC$, $CD$, $DA$ are lines on the surface, say $AB$, $CD$ are directrices, and $BC$, $AD$ are generatrices, the two reciprocal lines being the diagonals $AC$ and $BD$ of the skew quadrilateral. Taking on $AC$ a point $P$ and on $BD$ a point $Q$, the theorem is that if $PQ$ meets the surface in the points $X$, $Y$, then the four points

$P, Q, X, Y$ are harmonics. Let the generatrix through $X$ meet $AB$, $CD$ in the points $S$, $S'$ respectively, and the




generatrix through $Y$ meet the same lines in the points $T$, $T'$ respectively. Then the four lines $CB$, $DA$, $T'T$, $S'S$, are all met by an infinite series of lines, and in particular they are met by the lines $CD$, $BA$ in the points $C$, $D$, $T'$, $S'$ and $B$, $A$, $T$, $S$ respectively. Consequently, writing $AH$ for anharmonic ratio, we have

$$AH(C, D, T', S') = AH(B, A, T, S).$$

Consider now the four lines $CA$, $DB$, $T'T$, $S'S$, these are each of them met by the lines $CD$ and $BA$, and also by the line $PQ$; therefore, by an infinity of lines (that is, they are directrices of a hyperboloid), and they are met by the last-mentioned three lines in the points $C, D, T', S'$; $A, B, T, S$; and $P, Q, Y, X$ respectively; hence

$$AH(C, D, T', S') = AH(A, B, T, S) = AH(P, Q, Y, X).$$

Comparing with the foregoing equation, it appears that

$$AH(B, A, T, S) = AH(A, B, T, S),$$

viz., the anharmonic ratio is not altered by the interchange of the two points $A, B$: this implies that the points $A, B, T, S$ are harmonics, viz. the pairs $A$, $B$ and $S$, $T$ are harmonics; and, this being so, the equation

$$AH(A, B, T, S) = AH(P, Q, Y, X)$$

shows that $P, Q, Y, X$ are harmonics, viz. that the pairs $P, Q$ and $X, Y$ are harmonics; or say $P, Q$ are harmonics in regard to the quadric surface, which is the theorem in question.