NOTE ON SERIES WHOSE COEFFICIENTS INVOLVE POWERS OF THE BERNOULLIAN NUMBERS.

By *J. W. L. Glaisher.*

§ 1. Denoting the Bernoullian numbers by $B_1, B_2, B_3, \ldots$ we know that

$$\log \frac{\sin x}{x} = -\frac{B_1}{2.2!}\,2^2x^2 - \frac{B_2}{4.4!}\,2^4x^4 - \frac{B_3}{6.6!}\,2^6x^6 - \&c.$$

By writing in this equation $\frac{1}{2}x$, $\frac{1}{3}x$, $\frac{1}{4}x$, ... for x, and adding, the right-hand member becomes

$$\begin{aligned} &- \frac{B_1}{2.2!}\,2^2x^2\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right\} \\ &- \frac{B_2}{4.4!}\,2^4x^4\left\{1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c.\right\} \\ &- \frac{B_3}{6.6!}\,2^6x^6\left\{1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \&c.\right\} \\ &\qquad \&c. \qquad\qquad\qquad \&c., \end{aligned}$$

and, since

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c. = \frac{(2\pi)^{2n}B_n}{2\,.\,(2n)!},$$

the above series

$$= -\frac{B_1^2}{2.2.(2!)^2}(4\pi x)^2 - \frac{B_2^2}{2.4.(4!)^2}(4\pi x)^4 - \frac{B_3^2}{2.6.(6!)^2}(4\pi x)^6 - \&c.$$

Now the left-hand member

$$= \log\left\{\frac{\sin x}{x}\,.\,\frac{\sin\frac{1}{2}x}{\frac{1}{2}x}\,.\,\frac{\sin\frac{1}{3}x}{\frac{1}{3}x}\ \ldots\right\},$$

and, by Euler's formula, this product

$$\begin{aligned} = &\ \cos\frac{x}{2}\cos\frac{x}{4}\ \cos\frac{x}{8}\ \cos\frac{x}{16}\ \ldots \\ &\times \cos\frac{x}{4}\cos\frac{x}{8}\ \cos\frac{x}{16}\cos\frac{x}{32}\ \ldots \\ &\times \cos\frac{x}{6}\cos\frac{x}{12}\cos\frac{x}{24}\cos\frac{x}{48}\ \ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$= \cos\frac{x}{2}\left(\cos\frac{x}{4}\right)^2\cos\frac{x}{6}\left(\cos\frac{x}{8}\right)^3\cos\frac{x}{10}\left(\cos\frac{x}{12}\right)^2\cos\frac{x}{14}\ \ldots,$$

the exponent of each cosine being equal to that of the highest power of 2 contained in the denominator of the argument.

We thus find

$$\log\left\{\cos\frac{x}{2}\left(\cos\frac{x}{4}\right)^2\cos\frac{x}{6}\left(\cos\frac{x}{8}\right)^3\dots\right\}=-\Sigma_1^\infty\frac{B_n^2}{2.2n.\{(2n)!\}^2}(4\pi x)^{2n},$$

whence, by differentiating,

$$\tfrac{1}{2}\tan\frac{x}{2}+\tfrac{2}{4}\tan\frac{x}{4}+\tfrac{1}{6}\tan\frac{x}{6}+\tfrac{3}{8}\tan\frac{x}{8}+\&c.$$

$$=2\pi\left\{\frac{B_1^2}{(2!)^2}4\pi x+\frac{B_2^2}{(4!)^2}(4\pi x)^3+\frac{B_3^2}{(6!)^2}(4\pi x)^5+\&c.\right\}.$$

If we denote by r the exponent of the highest power of 2 which is a divisor of $2n$ (*i.e.* so that $2n=2^r m$, m being an uneven number), we may write this result in the form

$$\Sigma_{n=1}^{n=\infty}\frac{r}{n}\tan\frac{x}{2n}=4\pi\Sigma_{n=1}^{n=\infty}\frac{B_n^2}{\{(2n)!\}^2}(4\pi x)^{2n-1},$$

or, multiplying by x and transposing the sides of the equation,

$$\Sigma_{n=1}^{n=\infty}\left\{\frac{B_n}{(2n)!}\right\}^2(4\pi x)^{2n}=2x\Sigma_{n=1}^{n=\infty}\frac{r}{2n}\tan\frac{x}{2n}.*$$

§2. In the preceding formula, put for x successively $\frac{1}{2}x$, $\frac{1}{3}x$, $\frac{1}{4}x$, ... The general term on the left-hand side then becomes

$$\left\{\frac{B_n}{(2n)!}\right\}^2(4\pi x)^{2n}\left\{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\&c.\right\}$$

$$=\tfrac{1}{2}\left\{\frac{B_n}{(2n)!}\right\}^3(8\pi^2 x)^{2n}.$$

To obtain the value of the right-hand member of the equation, consider the expression

$$\phi\left(\frac{x}{2}\right)+2\phi\left(\frac{x}{4}\right)+\phi\left(\frac{x}{6}\right)+3\phi\left(\frac{x}{8}\right)+\phi\left(\frac{x}{10}\right)+2\phi\left(\frac{x}{12}\right)$$

$$+\phi\left(\frac{x}{14}\right)+4\phi\left(\frac{x}{16}\right)+2\phi\left(\frac{x}{18}\right)+6\phi\left(\frac{x}{20}\right)+\&c.$$

in which the coefficient of each term is equal to the exponent of the highest power of 2 contained in the denominator.

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* I set this formula in Part II of the Mathematical Tripos, 1887 (Friday morning, June 3, Question 7).

By putting $\frac{1}{2}x$, $\frac{1}{3}x$, $\frac{1}{4}x$, ... for x, in this expression, and adding, we obtain the expression

$$\phi\left(\frac{x}{2}\right)+3\phi\left(\frac{x}{4}\right)+2\phi\left(\frac{x}{6}\right)+6\phi\left(\frac{x}{8}\right)+2\phi\left(\frac{x}{10}\right)+6\phi\left(\frac{x}{12}\right)$$
$$+2\phi\left(\frac{x}{14}\right)+10\phi\left(\frac{x}{16}\right)+3\phi\left(\frac{x}{18}\right)+6\phi\left(\frac{x}{20}\right)+\&c.,$$

The law of these coefficients, which is a rather curious one, may be stated as follows:—the coefficient of $\phi\left(\frac{x}{2n}\right)$ is equal to $\frac{1}{2}r(r+1)\,\delta_1(2n)$, where, as before, r is the exponent of the highest power of 2 contained in $2n$, and $\delta_1(2n)$ denotes the number of uneven divisors of $2n$.

If, therefore,

$$2n=2^r a^\alpha b^\beta c^\gamma\ldots,$$

where a, b, c, ... are uneven primes, then

$$\delta_1(2n)=(\alpha+1)(\beta+1)(\gamma+1)\ldots,$$

and the coefficient of $\phi\left(\frac{x}{2n}\right)$

$$=\frac{r(r+1)}{2}(\alpha+1)(\beta+1)(\gamma+1)\ldots.$$

The quantity $\frac{r(r+1)}{2}$ is the rth triangular number. Thus the coefficient of $\phi\left(\frac{x}{2n}\right)$ is equal to the product of the rth triangular number and the number of uneven divisors of $2n$.

If we denote this coefficient by $\lambda_1(2n)$, we have

$$\Sigma_{n=1}^{n=\infty}\left\{\frac{B_n}{(2n)!}\right\}^3(8\pi^2x)^{2n}=4x\Sigma_{n=1}^{n=\infty}\frac{\lambda_1(2n)}{2n}\tan\frac{x}{2n}.$$

§ 3. Proceeding as before (*i.e.* substituting $\frac{1}{2}x$, $\frac{1}{3}x$, ... for x, and adding), we find that the left-hand member of the equation becomes

$$\tfrac{1}{2}\Sigma_1^\infty\left\{\frac{B_n}{(2n)!}\right\}^4(16\pi^3x)^{2n}.$$

To obtain the value of the right-hand member, we consider the expression

$$\phi\left(\frac{x}{2}\right)+3\phi\left(\frac{x}{4}\right)+2\phi\left(\frac{x}{6}\right)+6\phi\left(\frac{x}{8}\right)+2\phi\left(\frac{x}{10}\right)+\&c.,$$

and substitute $\frac{1}{2}x$, $\frac{1}{3}x$, ... for x.

The expression obtained therefrom by addition is

$$\phi\left(\frac{x}{2}\right) + 4\phi\left(\frac{x}{4}\right) + 3\phi\left(\frac{x}{6}\right) + 10\phi\left(\frac{x}{8}\right) + 3\phi\left(\frac{x}{10}\right) + 12\phi\left(\frac{x}{12}\right)$$
$$+ 3\phi\left(\frac{x}{14}\right) + 20\phi\left(\frac{x}{16}\right) + 6\phi\left(\frac{x}{18}\right) + 12\phi\left(\frac{x}{20}\right) + \&c.$$

The coefficient of $\phi\left(\frac{x}{2n}\right)$ in this series is

$$\frac{r(r+1)(r+2)}{6}\,\delta_2(2n),$$

where r has the same meaning as before, and $\delta_2(2n)$ denote the sum of the divisors of each of the uneven divisors of $2n$.

Thus, if $1, p, q, \ldots, m$ are all the uneven divisors of $2n$, then

$$\delta_2(2n) = \delta_1(1) + \delta_1(p) + \delta_1(q) + \ldots + \delta_1(m).$$

If $$2n = 2^r a^\alpha b^\beta c^\gamma \ldots,$$

$a, b, c, \ldots$ being uneven primes, it is easy to see that

$$\delta_2(2n) = \frac{(\alpha+1)(\alpha+2)}{2} \cdot \frac{(\beta+1)(\beta+2)}{2} \cdot \frac{(\gamma+1)(\gamma+2)}{2} \ldots;$$

and therefore the coefficient of $\phi\left(\frac{x}{2n}\right)$ is

$$\frac{r(r+1)(r+2)}{6} \cdot \frac{(\alpha+1)(\alpha+2)}{2} \cdot \frac{(\beta+1)(\beta+2)}{2} \ldots;$$

Denoting this coefficient by $\lambda_2(2n)$, we have

$$\Sigma_{n=1}^{n=\infty} \left\{\frac{B_n}{(2n)!}\right\}^4 (16\pi^2 x)^{2n} = 8x\Sigma_{n=1}^{n=\infty} \frac{\lambda_2(2n)}{2n} \tan\frac{x}{2n}.$$

§ 4. It is evident that, if p and q be any two numbers which are prime to each other,

$$\delta_1(p)\,\delta_1(q) = \delta_1(pq),$$

and $$\delta_2(p)\,\delta_2(q) = \delta_2(pq).$$

These formulæ would greatly facilitate the actual calculation of the coefficients.

If n is an uneven prime number,

$$\delta_1(n) = 2, \quad \delta_2(n) = 3,$$

and $$\lambda_1(2n) = 2, \; \lambda_2(2n) = 3.$$

If $2n=2^r a$, a being an uneven prime number,

$$\lambda_1(2n)=\frac{r(r+1)}{2},\ \lambda_2(2n)=\frac{r(r+1)(r+2)}{6}.$$

§ 5. By writing $2x$ for x, and replacing $\lambda_1(2n)$ and $\lambda_2(2n)$ by other functions $\theta_1(n)$ and $\theta_2(n)$ defined below, we may write the system of formulæ in the following form:

$$\Sigma_1^\infty\left\{\frac{B_n}{(2n)!}\right\}^2(8\pi x)^{2n}\ =2x\Sigma_1^\infty\frac{\theta_0(n)}{n}\tan\frac{x}{n},$$

$$\Sigma_1^\infty\left\{\frac{B_n}{(2n)!}\right\}^3(16\pi^2 x)^{2n}=4x\Sigma_1^\infty\frac{\theta_1(n)}{n}\tan\frac{x}{n},$$

$$\Sigma_1^\infty\left\{\frac{B_n}{(2n)!}\right\}^4(32\pi^3 x)^{2n}=8x\Sigma_1^\infty\frac{\theta_2(n)}{n}\tan\frac{x}{n};$$

where, if $$n=2^s a^\alpha b^\beta c^\gamma\ldots,$$

$a, b, c, \ldots$ being uneven primes, then

$\theta_0(n)=s+1,$

$$\theta_1(n)=\frac{(s+1)(s+2)}{2}\delta_1(n)$$

$$=\frac{(s+1)(s+2)}{2}(\alpha+1)(\beta+1)(\gamma+1)\ldots,$$

$$\theta_2(n)=\frac{(s+1)(s+2)(s+3)}{6}\delta_2(n)$$

$$=\frac{(s+1)(s+2)(s+3)}{6}\cdot\frac{(\alpha+1)(\alpha+2)}{2}\cdot\frac{(\beta+1)(\beta+2)}{2}\ldots.$$

If p and q be relatively prime, we have

$$\theta_0(p)\,\theta_0(q)=\theta_0(pq),$$
$$\theta_1(p)\,\theta_1(q)=\theta_1(pq),$$
$$\theta_2(p)\,\theta_2(q)=\theta_2(pq).$$

The general law of the series is evident: the value of $\theta_r(n)$ being

$$\frac{(s+1)^{(r+1)}}{(r+1)!}\cdot\frac{(\alpha+1)^{(r)}}{r!}\cdot\frac{(\beta+1)^{(r)}}{r!}\ldots,$$

where $a^{(b)}$ denotes the factorial $a(a+1)\ldots(a+b-1)$.

§ 6. The formula, corresponding to those in the last section, in which the first powers only of the Bernoullian numbers are involved, may be written:

$$\Sigma_1^\infty \frac{B_n}{(2n)!}(2x)^{2n} = x\Sigma_1^\infty \frac{1}{2^n}\tan\frac{x}{2^n}$$
$$= 1 - x\cot x.$$

The two expressions on the right-hand side are at once seen to be equal by differentiating logarithmically Euler's formula

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\ \ldots.$$

It is perhaps worth noticing how readily the above expansion of $1 - x\cot x$ in powers of x is derivable from the expression for the Bernoullian numbers in terms of the sums of the reciprocals of the even powers of the natural numbers. For

$$1 - x\cot x = -\Sigma_1^\infty \frac{2x^2}{x^2 - n^2\pi^2}$$
$$= \Sigma_1^\infty \frac{2x^{2n}}{\pi^{2n}}\left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \ldots\right)$$
$$= \Sigma_1^\infty \frac{2x^{2n}}{\pi^{2n}} \cdot \frac{(2\pi)^{2n} B_n}{2\,(2n)!} = \Sigma_1^\infty \frac{B_n}{2n!}(2x)^{2n}.$$

§ 7. We may deduce by integration from § 5, or obtain independently as in § 1, the following formulæ:

$$\Sigma_1^\infty \frac{1}{2n}\left\{\frac{B_n}{(2n)!}\right\}^2 (8\pi x)^{2n} = -2\log \Pi_1^\infty \left(\cos\frac{x}{n}\right)^{\theta_0(n)},$$
$$\Sigma_1^\infty \frac{1}{2n}\left\{\frac{B_n}{(2n)!}\right\}^3 (16\pi^2 x)^{2n} = -4\log \Pi_1^\infty \left(\cos\frac{x}{n}\right)^{\theta_1(n)},$$
$$\Sigma_1^\infty \frac{1}{2n}\left\{\frac{B_n}{(2n)!}\right\}^4 (32\pi^3 x)^{2n} = -8\log \Pi_1^\infty \left(\cos\frac{x}{n}\right)^{\theta_2(n)},$$

&c. &c.

§ 8. By differentiating the formulæ in § 5, we find

$$\Sigma_2^\infty (2n-1)\left\{\frac{B_n}{(2n)!}\right\}^2 (8\pi x)^{2n} = 2x^2\Sigma_1^\infty \frac{\theta_0(n)}{n^2}\tan^2\frac{x}{n},$$
$$\Sigma_2^\infty (2n-1)\left\{\frac{B_n}{(2n)!}\right\}^3 (16\pi^2 x)^{2n} = 4x^2\Sigma_1^\infty \frac{\theta_1(n)}{n^2}\tan^2\frac{x}{n},$$
$$\Sigma_2^\infty (2n-1)\left\{\frac{B_n}{(2n)!}\right\}^4 (32\pi^3 x)^{2n} = 8x^2\Sigma_1^\infty \frac{\theta_2(n)}{n^2}\tan^2\frac{x}{n};$$

&c. &c.,

together with

$$\Sigma_1^\infty \frac{\theta_0(n)}{n^2} = 8B_1^2\pi^2,$$

$$\Sigma_1^\infty \frac{\theta_1(n)}{n^2} = 8B_1^3\pi^4,$$

$$\Sigma_1^\infty \frac{\theta_2(n)}{n^2} = 8B_1^4\pi^4,$$

&c. &c.

§ 9. These latter formulæ are easily verified; for $\Sigma_1^\infty \frac{\theta_0(n)}{n^2}$ may be derived from

$$1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{16^2} + \&c.,$$

by dividing it by $\frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots$, and adding; whence

$$\Sigma_1^\infty \frac{\theta_0(n)}{n^2} = \left(1 + \frac{1}{2^2} + \frac{1}{4^2} + \&c.\right)\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right)$$

$$= \left(1 - \frac{1}{2^2}\right)^{-1}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right)$$

$$= \frac{4}{3}\cdot\frac{\pi^2}{6} = \frac{2\pi^2}{9}.$$

Similarly

$$\Sigma_1^\infty \frac{\theta_1(n)}{n^4} = \left(1 + \frac{1}{2^2} + \frac{1}{4^2} + \&c.\right)\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right)^2$$

$$= \frac{2\pi^2}{9} \times \frac{\pi^2}{6} = \frac{\pi^4}{27},$$

and

$$\Sigma_1^\infty \frac{\theta_2(n)}{n^2} = \frac{\pi^4}{27} \times \frac{\pi^2}{6} = \frac{\pi^6}{162}.$$

§ 10. It may be remarked that, in general, if

$$\phi(x) = \alpha_1 x^2 + \alpha_2 x^4 + \alpha_3 x^6 + \&c.,$$

then, by substituting $\frac{1}{2}x$, $\frac{1}{3}x$, ..., for x, and adding, we find

$$\Sigma_1^\infty \alpha_n \frac{B_n}{(2n)!}(2\pi x)^{2n} = 2\Sigma_1^\infty \phi\left(\frac{x}{n}\right).$$

Similarly we find that

$$\Sigma_1^\infty \alpha_n \left\{\frac{B_n}{(2n)!}\right\}^2 (4\pi^2 x)^{2n} = 4\Sigma_1^\infty \nu_1(n)\,\phi\left(\frac{x}{n}\right),$$

where $\nu_1(n)$ denotes the number of divisors of n.

Thus, if $$n = a^\alpha b^\beta c^\gamma \ldots,$$

$a, b, c, \ldots$ being any primes, then

$$\nu_1(n) = (\alpha+1)(\beta+1)(\gamma+1)\ldots.$$

We also find that

$$\Sigma_1^\infty \alpha_n \left\{\frac{B_n}{(2n)!}\right\}^3 (8\pi^2 x)^{2n} = 8\Sigma_1^\infty \nu_2(n)\,\phi\left(\frac{x}{n}\right),$$

where $\nu_2(n)$ denotes the number of the divisors of all the divisors of n. Thus, if $1, p, q, \ldots, n$ be the divisors of n,

$$\nu_2(n) = \nu_1(1) + \nu_1(p) + \nu_1(q) + \ldots + \nu_1(n);$$

and, if $$n = a^\alpha b^\beta c^\gamma \ldots,$$

$a, b, c, \ldots$ being any primes,

$$\nu_2(n) = \frac{(\alpha+1)(\alpha+2)}{2} \cdot \frac{(\beta+1)(\beta+2)}{2} \cdot \frac{(\gamma+1)(\gamma+2)}{2} \ldots.$$

The formula involving B_n^4 is

$$\Sigma_1^\infty \alpha_n \left\{\frac{B_n}{(2n)!}\right\}^4 (16\pi^3 x)^{2n} = 16\Sigma_1^\infty \nu_3(n)\,\phi\left(\frac{x}{n}\right),$$

where, if $$n = a^\alpha b^\beta c^\gamma \ldots,$$

$$\nu_3(n) = \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{6} \cdot \frac{(\beta+1)(\beta+2)(\beta+3)}{6} \ldots;$$

and similarly we find

$$\Sigma_1^\infty \alpha_n \left\{\frac{B_n}{(2n)!}\right\}^5 (32\pi^4 x)^{2n} = 32\Sigma_1^\infty \nu_4(n)\,\phi\left(\frac{x}{n}\right),$$

where

$$\nu_4(n) = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+4)}{4!} \cdot \frac{(\beta+1)(\beta+2)\ldots(\beta+4)}{4!} \ldots,$$

the general law being evident.

Thus $\nu_1, \nu_2, \nu_3, \ldots$ all satisfy the equation

$$\nu(p)\,\nu(q) = \nu(pq),$$

p and q being prime to each other.

§ 11. In the case of the formulæ considered in §§ 1–8, we have

$$\alpha_n = \frac{B_n}{(2n)!}\,2^{2n},$$

and $$\phi(x) = x\Sigma_1^\infty \frac{1}{2^n} \tan \frac{x}{2^n}.$$

The fact that $\phi(x)$ is itself a series of terms in which the denominators are the successive powers of 2 is the cause of the distinction which occurs in the formulæ of § 5 between 2 and the other prime factors of n. In the formation of the functions θ the exponent of 2 gives rise to a factorial which is one order higher than the factorials depending upon the exponents of the other primes.

EXPANSIONS OF K, I, G, E IN POWERS OF $k'^2 - k^2$.

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THE object of this note is to give the expansion of K in ascending powers of $k'^2 - k^2$. I have also added the corresponding expansions of I, G, and E.

Expansion of K, §§ 1, 2.

§ 1. Let h and h' denote k^2 and k'^2 respectively, and let

$$\lambda = h' - h = k'^2 - k^2.$$

If therefore α be the modular angle, so that $k = \sin\alpha$, then $\lambda = \cos 2\alpha$.

We have*

$$\frac{\sqrt{\pi}}{4} K = \int_0^\infty \int_0^\infty e^{-x^4-y^4-2\lambda x^2y^2} dx\,dy,$$

whence, expanding in powers of λ,

$$\frac{\sqrt{\pi}}{4} K = \int_0^\infty \int_0^\infty e^{-x^4-y^4} \left\{1 - 2\lambda x^2y^2 + \frac{2^2\lambda^2}{2!} x^4y^4 - \frac{2^3\lambda^3}{3!} x^6y^6 + \&c.\right\}.$$

* *Proceedings of the London Mathematical Society,* vol. XIII. p. 92 (1881).