NOTE ON CLAPEYRON'S THEOREM OF THE THREE MOMENTS.

By *Karl Pearson.*

(1) CLAPEYRON in a paper published in the *Comptes rendus*, T. XLV., 1857, pp. 1076-80, has shown that the following relation holds between the moments M_3, M_4, M_5 at three successive points of support of a continuous beam, the spans between these points of support being of length l_{34} and l_{45} respectively, and there being continuous uniform loads of p_{34} and p_{45} per unit run on the two spans:

$$l_{34}M_3 + 2(l_{34} + l_{45})M_4 + l_{45}M_5 = \tfrac{1}{4}(p_{34}l_{34}^3 + p_{45}l_{45}^3) \ldots\ldots(\text{i}).$$

Heppel in a memoir published in Vol. XIX (1859-60), pp. 625-43, of the *Proceedings of the Institution of Civil Engineers*, shows that $\frac{3}{8}(W_{34}l_{34}^2 + W_{45}l_{45}^2)$ must be added to the right-hand side of (i) if there be isolated central loads W_{34} and W_{45} on the spans l_{34}, l_{45} respectively. He further shows that the reaction R_4 at the fourth point of support is given by

$$R_4 = \tfrac{1}{2}(p_{34}l_{34} + W_{34} + p_{45}l_{45} + W_{45}) + \frac{M_4 - M_3}{l_{34}} + \frac{M_4 - M_5}{l_{45}} \ldots\ldots\ldots\ldots\ldots(\text{ii}).$$

Weyrauch has generalised these results in his *Allgemeine Theorie und Berechnung der continuirlichen und einfachen Träger*, 1873, p. 8. He supposes any system of isolated loads and the points of support not necessarily on the same level. If f_4 be the depth of the fourth point of support below a given horizontal line, then the following term must be added to the left-hand side of (i), $E\omega k^2$ being the flexural rigidity of the continuous beam:

$$6E\omega k^2\left\{\frac{f_3 - f_4}{l_{34}} + \frac{f_4 - f_4}{l_{45}}\right\}.$$

For our present purposes we may accordingly write these results in the form:

$$l_{34}M_3 + 2(l_{34} + l_{45})M_4 + l_{45}M_5 + 6E\omega k^2\left\{\frac{f_3 - f_4}{l_{34}} + \frac{f_5 - f_4}{l_{45}}\right\}$$
$$= \tfrac{1}{4}(p_{34}l_{34}^3 + p_{45}l_{45}^3) + \tfrac{3}{8}(W_{34}l_{34}^2 + W_{45}l_{45}^2) \ldots\ldots\ldots(\text{iii}).$$

For R_4 we have the value given by (ii).

(2) Now the above extended form of Clapeyron's Theorem is based by its investigator on the assumption that the points of support are *rigid*. But in most practical cases these 'points of support' are metal columns as elastic as the continuous girder itself, and it seems worth while investigating whether this fact may not sensibly influence the values of the reactions as calculated from the equations (i), (ii), and (iii) above. Such is the object of the present note.

Dorna in a memoir published in *Serie* II., T. XVIII., pp. 281-8, of the *Memorie dell' Accad. delle Scienze di Torino* (1857), has gone to the opposite extreme and supposed the supported body rigid and the supports elastic. He has been followed by Ménebréa in a memoir of 1858 (*Comptes rendus*, XLVI., pp. 1056-60). Which of these two assumptions is really more approximate in practise, or is it necessary to reject both of them?

That the question is not without importance may be seen from the following case. Suppose a beam uniformly loaded with a total load W to rest on three columns of equal height and of cross-sections ω_1, ω_2, ω_1, and stretch-moduli E_1, E_2, E_1 respectively. Then Dorna's theory gives for the three reactions

$$R_1 = R_3 = \frac{E_1\omega_1}{2E_1\omega_1 + E_2\omega_2} W, \quad R_2 = \frac{E_2\omega_2}{2E_1\omega_1 + E_2\omega_2} W.$$

Supposing the supports rigid, we deduce with Clapeyron

$$R_1 = R_3 = \tfrac{3}{16} W, \quad R_2 = \tfrac{5}{8} W$$

The two theories would only agree if $\frac{E_1\omega_1}{E_2\omega_2} = \frac{3}{10}$, or since, generally, $E_1 = E_2$, if the central column had a cross-section $\frac{10}{3}$ that of either terminal column. Thus they lead as a rule to very divergent values for the reactions.

Furthermore, it is easy to show that a very slight change in the height of one of the columns does on Clapeyron's hypothesis make a great difference in the values of the reactions. For example, in the above case, let the middle column be lower than the terminal columns by $1/n$ the deflection of the beam when the middle point of support is removed, then we find (Cotterill's *Applied Mechanics* p. 331):

$$R = \tfrac{5}{8} W\left(1 - \frac{1}{n}\right).$$

Hence it would appear that a small change in the height of one of the columns would seriously influence the reactions, and such changes of height might really be produced by the columns being elastic.

(3) Let the height of the rth column be h_r, lets its stretch-modulus be E_r, and its cross-section α_r, and let us suppose the tops of all the columns at the same horizontal level before the girder is placed upon them, their differences of height depending on the sub-structure.

Then we have

$$R_r = E_r\alpha_r f_r / h_r \text{ ....(iv)}.$$

We can now easily generalise Clapeyron's theorem. Equations (iv) and (ii) give us f_r in terms of the moments, and this substituted in (iii) gives us a relation between the moments to replace Clapeyron's Theorem. It takes the following form:

$$\begin{aligned} & l_{34}M_3 + 2(l_{34}+l_{45})M_4 + l_{45}M_5 \\ & \quad + 6\left[\frac{k^2}{l_{23}l_{34}}\frac{h_3}{n_3}(M_3 - M_2) + \frac{k^2}{l_{34}^2}\left\{\frac{h_3}{n_3} + \frac{h_4}{n_4}\left(1 + \frac{l_{34}}{l_{45}}\right)\right\}(M_3 - M_4)\right. \\ & \quad \left. + \frac{k^2}{l_{45}^2}\left\{\frac{h_5}{n_5} + \frac{h_4}{n_4}\left(1 + \frac{l_{45}}{l_{34}}\right)\right\}(M_5 - M_4) + \frac{k^2}{l_{45}l_{56}}\frac{h_5}{n_5}(M_5 - M_6)\right] \\ & = \tfrac{1}{4}(p_{34}l_{34}^3 + p_{45}l_{45}^3) + \tfrac{3}{8}(W_{34}l_{34}^2 + W_{45}l_{45}^2) - \frac{3k^2}{l_{34}}\frac{h_3}{n_3}(w_{23} + w_{34}) \\ & \quad + 3\left(\frac{k^2}{l_{34}} + \frac{k^2}{l_{45}}\right)\frac{h_4}{n_4}(w_{34} + w_{45}) - \frac{3k^2}{l_{45}}\frac{h_4}{n_5}(w_{45} + w_{56}) \text{ .... (v)}, \end{aligned}$$

where $w_r =$ total load on rth span and $n_r = E_r\alpha_r/(E\omega)$.

This equation involves *five* moments, and so may be spoken of as the *theorem of the five moments*. I have written it down in the most general form that is likely to occur in any practical calculations of continuous beams; as a rule it will be simpler in form.

(4) Obviously the order of terms introduced by supposing the columns elastic is given by

$$\frac{k^2}{l_r^2}\,\frac{h_r}{l_r}\,\frac{1}{n_r}.$$

To get some idea of the value of this we note that n_r will possibly not be very different from unity, h_r/l_r will also hardly be likely to exceed this limit. Hence we may say that the terms introduced are of the order $\left(\frac{k}{l_r}\right)^2$, and in most

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practical cases this would be small. Hence to a first approximation the support moments are given by the theorem of the three moments. Let their values so calculated be given by N_r, and suppose $M_r = N_r + z_r$, z_r being small, then we have to determine the successive additions to the moments N_r, also a three moment theorem expressed by the equation:

$$l_{34}z_3 + 2(l_{34}+l_{45})z_4 + l_{45}z_3 = 6\left[\frac{k^2}{l_{23}l_{34}}\frac{h_3}{n_3}(N_2 - N_3)\right.$$
$$+\frac{k^2}{l_{34}^2}\left\{\frac{h_3}{n_3}+\frac{h_4}{n_4}\left(1+\frac{l_{34}}{l_{45}}\right)\right\}(N_4 - N_3)$$
$$+\frac{k^2}{l_{45}^2}\left\{\frac{h_5}{n_5}+\frac{h_4}{n_4}\left(1+\frac{l_{45}}{l_{34}}\right)\right\}(N_4 - N_5)$$
$$\left.+\frac{k^2}{l_{45}l_{56}}\frac{h_5}{n_5}(N_6 - N_5)\right] - \frac{3k^2}{l_{34}}\frac{h_3}{n_3}(w_{23}+w_{34})$$
$$+3\left(\frac{k^2}{l_{34}}+\frac{k^2}{l_{45}}\right)\frac{h_4}{n_4}(w_{34}+w_{45}) - \frac{3k^2}{l_{45}}\frac{h_5}{n_5}(w_{45}+w_{56}) \ldots\ldots \text{(vi)}.$$

The right-hand side of this equation consisting entirely of known quantities, we have a finite difference equation of the third order as in Clapeyron's theorem to solve.

(5) Returning to equation (v), let us ask when it will reduce to Clapeyron's three moments theorem. Obviously, if we suppose $f_1 = f_2 = f_3 = \ldots = f_r = \ldots$. That is to say, the tops of the columns in the same horizontal after strain. But by (iv) this involves

$$\frac{R_1h_1}{E_1\alpha_1} = \frac{R_2h_2}{E_2\alpha_2} = \ldots = \frac{R_rh_r}{E_r\alpha_r} = \ldots\ldots\ldots\ldots \text{(vii)}.$$

Hence: *if the reactions be calculated by the theorem of the three moments and the cross-sections of the columns be then found by* (vii), *we obtain accurate results, although the supports are supposed elastic and not rigid by this theorem.*

For the case of equal heights and equal elasticity in the columns, we have

$$\frac{R_1}{\alpha_1} = \frac{R_2}{\alpha_2} = \ldots = \frac{R_r}{\alpha_r} = \ldots\ \ldots\ldots\ldots\ldots \text{(viii)},$$

or the stress will be equal in all the columns, a distinct practical advantage.

(6) Suppose all the columns of the same form and material and all the spans equal, then (v) becomes

$$M_3 + 4M_4 + M_5 - \frac{6k^2h}{nl^3}\{4(M_3+M_5) - 6M_4 - M_2 - M_6\}$$
$$= \tfrac{1}{4}l^2(p_{34}+p_{45}) + \tfrac{3}{8}l(W_{34}+W_{45})$$
$$- \frac{3k^2h}{nl^3}(w_{23}+w_{56}-w_{34}-w_{45}) \;\ldots\ldots\; \text{(ix)}.$$

Let us apply this to the case of the uniform girder weighing W tons and resting on one central and two terminal columns of equal height, cross-section and material. We have all the M's zero except M_4 and $2lp_{34} = 2lp_{45} = W$, $w_{23} = w_{56} = 0$, and $w_{34} = w_{45} = W/2$. Hence, if $\beta = \frac{k^2h}{nl^3}$, we find

$$4M_4(1-9\beta) = \tfrac{1}{4}lW(1+12\beta),$$

or

$$M_4 = \tfrac{1}{16}lW\frac{1+12\beta}{1-9\beta}.$$

From (ii),

$$R_4 = \tfrac{1}{2}W + 2\frac{M_4}{l} = \tfrac{1}{2}W\left\{1 + \tfrac{1}{4}\frac{1+12\beta}{1-9\beta}\right\},$$

or

$$R_4 = \tfrac{5}{8}W\frac{1-4.8\beta}{1-9\beta}.$$

Similarly, we find

$$R_3 = R_5 = \tfrac{3}{16}W\frac{1-16\beta}{1-9\beta}.$$

These agree, if $\beta = 0$, with the values of the reactions given in our Art. 2 as we should expect. If $\beta = \infty$, we obtain

$$R_3 = R_4 = R_5 = W/3,$$

the values which would be given by Dorna's theory for columns of equal cross-section.

Hence Clapeyron's or Dorna's theory will be more nearly in accordance with the actual facts of the case according as β is very small or very great. For intermediate values of β the theorem of the five moments must be used. Supposing the material of the column and girder to be the same, we have

$$\beta = \frac{k^2}{l^2}\,\frac{\omega}{\alpha}\,\frac{h}{l},$$

and it is difficult to imagine a practical case in which this expression could be *very* great. For columns very high and slender as compared with the span and cross-section of the girder it might become quite sensible, however, and in this case the methods of the present note would have to be adopted; certainly not Dorna's. Thus it would appear that Clapeyron's theorem would fail to give accurate results for a massive girder or tube supported on a great number of slender columns.

For cases of this kind equation (ix) will be found easy to work with when the numerical details are given. It may be advantageously used in cases where, as in suspending bars or bracing rods, a series of slender columns either as struts or ties support a massive continuous beam, for then the three moments theorem is certainly inapplicable.

(7) As a last example, to show the relation of Clapeyron's and Dorna's hypotheses, I will investigate the case of a girder uniformly loaded resting as above on three columns of the same material as the girder, but the terminal columns having cross-sections which are n' times that of the girder, while the central column has a cross-section n times that of the girder.

Returning to equation (v), we have, if W be the weight of the girder as before and $\gamma = k^2h/l^3$,

$$4M_4 - 12\gamma\left(\frac{1}{n'} + \frac{2}{n}\right)M_4 = \tfrac{1}{4}Wl\left\{1 + 12\left(\frac{2}{n} - \frac{1}{n'}\right)\gamma\right\},$$

or

$$M_4 = \tfrac{1}{16}Wl\frac{1 + 12\left(\frac{2}{n} - \frac{1}{n'}\right)\gamma}{1 - 3\left(\frac{2}{n} + \frac{1}{n'}\right)\gamma}.$$

Hence

$$R_2 = \tfrac{1}{2}W\left\{1 + \tfrac{1}{4}\frac{1 + 12\left(\frac{2}{n} - \frac{1}{n'}\right)\gamma}{1 - 3\left(\frac{2}{n} + \frac{1}{n'}\right)\gamma}\right\}$$

$$= \tfrac{5}{8}W\frac{1 - \frac{4.8}{n'}\gamma}{1 - 3\left(\frac{2}{n} + \frac{1}{n'}\right)\gamma}.$$

Similarly

$$R_3 = R_5 = \tfrac{3}{16}W\frac{1 - \frac{16}{n}\gamma}{1 - 3\left(\frac{2}{n} + \frac{1}{n'}\right)\gamma}.$$

These results agree with those obtained in the previous article, if $n = n'$.

Now let us see with what relation between n and n' it is possible for R to equal R_4, whatever γ may be, *i.e.* even if it be small. We must have

$$10 - \frac{48}{n'}\gamma = 3 - \frac{48}{n}\gamma,$$

or

$$\frac{7}{48\gamma} = \frac{1}{n'} - \frac{1}{n}.$$

Thus n' must be less than n, or the middle column must be of greater cross-section than the terminal columns. Further, since γ is usually very small, n' must be *very much* less than n, or approximately

$$n' = \frac{48}{7}\gamma\left\{1 - \frac{48}{7n}\gamma\right\}.$$

Hence, by putting a very massive mid-column of otherwise arbitrary cross-section, and two very slender terminal columns with cross-sections about $\frac{48}{7}\gamma$ times that of the girder itself, we should have the three support-reactions nearly equal. Dorna's theory makes them equal when the three columns are of equal cross-section, which is obviously impossible for any reasonable value of γ.

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ON LATIN SQUARES.

By Prof. *Cayley*.

If in each line of a square of n^2 compartments the same n letters $a, b, c, \ldots$ are arranged so that no letter occurs twice in the same column, we have what was termed by Euler "a Latin square." Supposing that in one of the lines the letters are arranged in the natural order *abcde*..., then in the remaining lines there must be arrangements beginning with b, c, d, e, &c., respectively, and we may consider the case in which the bottom line has the arrangement *abcde*..., and in the other lines, reckoning from the bottom one in order, the arrangements begin with b, c, d, e, &c., respectively: if the number of such squares be $= N$, then, obviously, the whole