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Simple Proofs of Second Order Necessary Conditions for Broken Extremals

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Abstract

We consider optimal control problems with initial-final state equality and inequality constraints and mixed state-control equality constraints given by smooth functions. The mixed constraints satisfy the regularity assumption of linear independence of gradients with respect to the control. We present simple proofs of second-order necessary conditions of Pontryagin minimum for broken extremals in these problems.

1 Introduction

In this paper we study a relationship between necessary second order conditions for a week local minimum in an optimal control problem on a fixed time interval and necessary second order conditions for a Θ-week local minimum in an optimal control problem on a variable time interval. The latter type of the minimum is connected with small variations of jump points of optimal control, and the corresponding necessary conditions take these variations into account. The relationship between two types of optimality conditions is based on a simple change of time variable. As a consequence we obtain a relatively simple proof of necessary second order conditions for a Θ-week local minimum.

Let us recall conditions for weak and Θ-week minimum in a simple case. Consider the simplest problem of the calculus of variations:

$$J(x) = \int_0^1 F(t, x(t), \dot{x}(t))\, dt \to \min, \quad x(0) = a,\ x(1) = b,$$

where $x(t)$ is Lipschitz continuous, i.e., $x(\cdot) \in W^{1,\infty}$. A local minimum in the space $W^{1,\infty}$ is a *weak minimum*.

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Let $x^0(t)$ be an extremal, i.e., it satisfies the Euler equation

$$\frac{d}{dt}F_{\dot{x}}(t,x^0(t),\dot{x}^0(t)) = F_x(t,x^0(t),\dot{x}^0(t)).$$

Set $\dot{x} = u$, $w = (x,u)$. We call u the *control.* Set $u^0(t) := \dot{x}^0(t)$, $w^0(t) = (x^0(t),u^0(t))$. Let

$$w(\cdot) = (x(\cdot),u(\cdot)) \in \mathcal{W}_2 := W^{1,2} \times L^2.$$

Define a quadratic form in the space $\mathcal{W}_2$:

$$\begin{aligned}\Omega(w) &= \int_0^1 \langle F_{ww}(t,w^0(t))w(t),w(t)\rangle\,dt \\ &= \int_0^1 \left(\langle F_{xx}x(t),x(t)\rangle + 2\langle F_{xu}u(t),x(t)\rangle + \langle F_{uu}u(t),u(t)\rangle\right)dt.\end{aligned}$$

Set

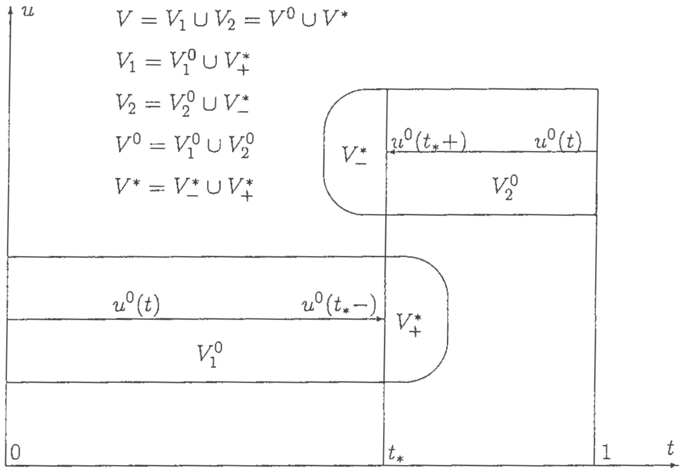
$$\mathcal{K} := \{w \in \mathcal{W}_2 : \ \dot{x} = u, \quad x(0) = x(1) = 0\}.$$

The following theorem is well-known.

Theorem 1.1. *(a) If x^0 is a weak minimum, then $\Omega(w) \geq 0$ on $\mathcal{K}$. (b) If $\Omega(w)$ is positive definite on $\mathcal{K}$, then x^0 is a (strict) weak minimum.*

Now, assume that the control $u^0(t)$ is *piecewise continuous* with one discontinuity point $t_* \in (0,1)$. Moreover, assume that $u^0(t)$ is Lipschitz continuous on each of the two intervals $(0,t_*)$ and $(t_*,1)$. Hence $x^0(t)$ is a *broken extremal* with a break at t_*. Which quadratic form corresponds to a broken extremal?

Let us change the definition of a weak local minimum. Set $\Theta := \{t_*\}$ and define a notion of a Θ-weak minimum as follows. Assuming that the control $u^0(t)$ is left-continuous, denote by $\overline{u^0(\cdot)}$ the closure of the graph of $u^0(t)$. Denote by V a neighborhood of the compact set $\overline{u^0(\cdot)}$.



Definition 1.2. x^0 is a point of a Θ-*weak minimum* if for any $\varepsilon > 0$ there exits a neighborhood V of the compact set $\overline{u^0(\cdot)}$ such that $J(x) \geq J(x^0)$ for all $x \in W^{1,1}$ such that $u(t) \in V$ a.e., where $u = \dot{x}$.

Recall the *Weierstrass-Erdmann necessary conditions for broken extremal:*

(i) $\psi(t) := -F_u(t, w^0(t))$ is continuous at t_*, i.e., $[\psi] = 0$, where $[\psi] = \psi(t_*+) - \psi(t_*-) = \psi^+ - \psi^-$ denotes the *jump* of ψ at t_*;

(ii) $H(t) := \psi(t)u^0(t) + F(t, w^0(t))$ is continuous at t_*, i.e., $[H] = 0$.

We add one more necessary condition [5]:

(iii) $D(H) \geq 0$,

where $D(H)$ is equal to minus derivative of the function

$$\Delta H(t) := (\psi(t)[u^0] + F(t, x^0(t), u^0(t_*+)) - F(t, x^0(t), u^0(t_*-)))$$

at t_* (the existence of this derivative is proved). One can show that [5]

$$D(H) = \dot{\psi}^+ \dot{x}^- - \dot{\psi}^- \dot{x}^+ + [\dot{\psi}_0],$$

where $\psi_0(t) := -H(t)$.

Denote by $P_\Theta W^{1,2}$ the Hilbert space of piecewise continuous functions $x(t)$, absolutely continuous on each of the two intervals $[0, t_*)$ and $(t_*, 1]$, and such that their first derivative is square integrable. Any $x \in P_\Theta W^{1,2}$ can have a nonzero jump

$$[x] = x(t_* + 0) - x(t_* - 0)$$

at the point t_*. Let ξ be a numerical parameter. Denote by $Z_2(\Theta)$ the space of triples $z = (\xi, x, u)$ such that

$$\xi \in \mathbb{R}, \quad x(\cdot) \in P_\Theta W^{1,2}, \quad u(\cdot) \in L^2.$$

Thus,

$$Z_2(\Theta) = \mathbb{R} \times P_\Theta W^{1,2} \times L^2.$$

In the space $Z_2(\Theta)$, define a quadratic form

$$\Omega_\Theta(z) = D(H)\xi^2 + 2[F_x]x_{\text{av}}\xi + \int_0^1 \langle F_{ww}(t, w^0(t))w(t), w(t)\rangle\, dt,$$

where $[F_x]$ is the jump of the function $F_x(t, w^0(t))$ at the point t_*, and

$$x_{\text{av}} = \frac{1}{2}\Big(x(t_*-) + x(t_*+)\Big).$$

Set

$$\mathcal{K}_\Theta = \{z \in Z_2(\Theta) : \; \dot{x} = u, \quad [x] = [u^0]\xi, \quad x(0) = x(1) = 0\}.$$

Theorem 1.3. *(a) If x^0 is a Θ-weak minimum, then $\Omega_\Theta(z) \geq 0$ on $\mathcal{K}_\Theta$. (b) If $\Omega_\Theta(z)$ is positive definite on $\mathcal{K}_\Theta$, then x^0 is a (strict) Θ-weak minimum.*

A detailed proof of this theorem (based on the so-called "method of deciphering") is given in [5]. It is rather long and technical. But it turned out that there was a relatively simple way to prove the necessary condition a) of this theorem and thus to come to the quadratic form which corresponds to a broken extremal. This way will be shown in the present paper for the general problem of the calculus of variations.

The paper is organized as follows. In section 2 we formulate the general problem of the calculus of variations on a fixed time interval and derive the second order necessary conditions for a weak local minimum in this problem, using Dubovitskii–Milyutin method of critical variations [2]. In section 3 we formulate the general problem of the calculus of variations on a variable time interval and derive the second order necessary conditions for a Θ-weak local minimum in this problem, using simple change of time variable and necessary conditions of a weak minimum obtained in section 2.

2 Necessary Second Order Condition for a Weak Local Minimum in the General Problem of the Calculus of Variations on a Fixed Time Interval

A weak local minimum in problem A Consider the following optimal control problem of Bolza type on a fixed interval of time $[t_0, t_f]$. It is required to find a pair of functions $w(t) = (x(t), u(t))$, $t \in [t_0, t_f]$, minimizing the endpoint functional

$$\mathcal{J}(w) := J(x(t_0), x(t_f)) \to \min \tag{1}$$

subject to the constraints

$$F(x(t_0), x(t_f)) \leq 0, \quad K(x(t_0), x(t_f)) = 0, \tag{2}$$

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{3}$$

$$g(t, x(t), u(t)) = 0, \tag{4}$$

$$(x(t_0), x(t_f))) \in \mathcal{P}, \quad (t, x(t), u(t)) \in \mathcal{Q}, \tag{5}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are open sets, x, u, F, K, f, and g are vector-functions. We call (1)-(5) the Problem A.

We assume that the functions J, F, and K are defined and twice continuously differentiable on $\mathcal{P}$, and the functions f and g are defined and twice continuously differentiable on $\mathcal{Q}$. It is also assumed that the gradients with respect to the control $g_{iu}(t, x, u)$, $i = 1, \ldots, d(g)$ are linearly independent at each point $(t, x, u) \in \mathcal{Q}$ such that $g(t, x, u) = 0$ (the *regularity assumption* for the equality constraint $g(t,x,u)=0$). Here g_i are the components of the vector function g and $d(g)$ is the dimension of this function.

The Problem A is considered in the space of pairs of functions $w=(x,u)$ such that the *state variable* $x(t)$ is an absolutely continuous $d(x)$-dimensional function and the *control* $u(t)$ is a bounded measurable $d(u)$-dimensional function on the interval $[t_0,t_f]$. Hence the problem is considered in the space

$$\mathcal{W}:=W^{1,1}([t_0,t_f],\mathcal{R}^{d(x)})\times L^{\infty}([t_0,t_f],\mathcal{R}^{d(u)}).$$

Define a norm in this space as a sum of the norms:

$$\|w\|:=\|x\|_{1,1}+\|u\|_{\infty}=|x(t_0)|+\int_{t_0}^{t_f}|\dot{x}(t)|\,dt+\text{ess sup}_{[t_0,t_f]}|u(t)|.$$

We say that $w\in\mathcal{W}$ is an *admissible pair* if it satisfies all constraints of the problem. Let $w^0=(x^0,u^0)\in\mathcal{W}$ be a fixed admissible pair. We say that w^0 is a *weak local minimum* if it is a local minimum in the space $\mathcal{W}$, i.e., there exists $\varepsilon>0$ such that $\mathcal{J}(w)\geq\mathcal{J}(w^0)$ for all admissible pairs $w\in\mathcal{W}$ satisfying the condition $\|w-w^0\|\leq\varepsilon$.

Necessary condition for a week local minimum We introduce the *Pontryagin function*

$$H(t,x,u,\psi)=\psi f(t,x,u) \tag{6}$$

and the *augmented Pontryagin function*

$$\bar{H}(t,x,u,\psi,\nu)=H(t,x,u,\psi)+\nu g(t,x,u), \tag{7}$$

where ψ and ν are row-vectors of the dimensions $d(x)$ and $d(g)$, respectively. For brevity we set

$$x_0=x(t_0),\quad x_f=x(t_f),\quad p=(x_0,x_f).$$

Denote by $(\mathcal{R}^{d(x)})^*$ the space of $d(x)$-dimensional row vectors. Define the *endpoint Lagrange function*

$$l(p,\alpha_0,\alpha,\beta)=\alpha_0 J(p)+\alpha F(p)+\beta K(p), \tag{8}$$

where

$$\alpha_0\in\mathcal{R},\quad \alpha\in(\mathcal{R}^{d(F)})^*,\quad \beta\in(\mathcal{R}^{d(K)})^*.$$

Introduce a tuple of Lagrange multipliers

$$\lambda=(\alpha_0,\alpha,\beta,\psi(\cdot),\nu(\cdot)) \tag{9}$$

such that $\psi(\cdot):[t_0,t_f]\rightarrow(\mathcal{R}^{d(x)})^*$ is an absolutely continuous and $\nu(\cdot):[t_0,t_f]\rightarrow(\mathcal{R}^{d(g)})^*$ is a measurable bounded function.

Denote by Λ_0 the set of all tuples λ satisfying the following conditions at the point w^0:

$$\begin{array}{l} \alpha_0 \geq 0, \ \alpha \geq 0, \ \alpha F(p^0) = 0, \ \alpha_0 + \sum_{i=1}^{d(F)} \alpha_i + \sum_{j=1}^{d(G)} |\beta_j| = 1, \\ \dot{\psi} = -\bar{H}_x, \quad \psi(t_0) = -l_{x_0}, \quad \psi(t_f) = l_{x_f}, \quad \bar{H}_u = 0, \end{array} \tag{10}$$

where $p^0 = (x^0(t_0), x^0(t_f))$, the derivatives l_{x_0} and l_{x_f} are at $(p^0, \alpha_0, \alpha, \beta)$ and the derivatives $\bar{H}_x$, $\bar{H}_u$ are at $(t, x^0(t), u^0(t), \psi(t), \nu(t))$, $t \in [t_0, t_f]$. By α_i and β_j we denote the components of row vectors α and β, respectively.

The following well-known first order necessary condition holds: if w^0 is a weak local minimum, then the set Λ_0 is nonempty. This condition is called the *local minimum principle* (or the *Euler-Lagrange equation*). From the regularity assumption for the constraint $g = 0$ and definition (10) it easily follows that Λ_0 is a finite dimensional compact set, and the projector $(\alpha_0, \alpha, \beta, \psi(\cdot), \nu(\cdot)) \to (\alpha_0, \alpha, \beta)$ is injective on Λ_0.

Now let us formulate the second order necessary condition at the point w^0. Set

$$\mathcal{W}_2 := W^{1,2}([t_0, t_f], \mathcal{R}^{d(x)}) \times L^2([t_0, t_f], \mathcal{R}^{d(u)}),$$

where $L^2([t_0, t_f], \mathcal{R}^{d(u)})$ is the space of square integrable functions, and $W^{1,2}([t_0, t_f], \mathcal{R}^{d(x)})$ is the space of absolutely continuous functions such that their first derivative is square integrable. Hence $\mathcal{W}_2$ is a Hilbert space with a scalar product

$$(w, \tilde{w}) := \langle x(t_0), \tilde{x}(t_0) \rangle + \int_{t_0}^{t_f} \langle \dot{x}, \dot{\tilde{x}} \rangle \, dt + \int_{t_0}^{t_f} \langle u, \tilde{u} \rangle \, dt.$$

Let $\mathcal{K}$ be the set of all $\bar{w} = (\bar{x}, \bar{u}) \in \mathcal{W}_2$ satisfying the following conditions:

$$\begin{array}{l} J'(p^0)\bar{p} \leq 0, \quad F_i'(p^0)\bar{p} \leq 0 \; \forall \, i \in I_F(p^0), \quad K'(p^0)\bar{p} = 0, \\ \dot{\bar{x}}(t) = f_w(t, w^0(t))\bar{w}(t), \text{ for a.a. } t \in [t_0, t_f], \\ g_w(t, w^0(t))\bar{w}(t) = 0, \text{ for a.a. } t \in [t_0, t_f], \end{array} \tag{11}$$

where $I_F(p^0) := \{i : F_i(p^0) = 0\}$ is the set of active indices. It is obvious that $\mathcal{K}$ is a convex cone in the Hilbert space $\mathcal{W}_2$. We call it the *critical cone.*

Let us introduce a quadratic form on $\mathcal{W}_2$. For $\lambda \in \Lambda_0$ and $\bar{w} = (\bar{x}, \bar{u}) \in \mathcal{W}_2$, we set

$$\Omega(\lambda, \bar{w}) = \langle l_{pp}\bar{p}, \bar{p} \rangle + \int_{t_0}^{t_f} \langle \bar{H}_{ww}\bar{w}(t), \bar{w}(t) \rangle \, dt, \tag{12}$$

where $l_{pp} = l_{pp}(p^0, \alpha_0, \alpha, \beta)$, $\bar{H}_{ww} = \bar{H}_{ww}(t, x^0(t), u^0(t), \psi(t), \nu(t))$, $\bar{p} = (\bar{x}(t_0), \bar{x}(t_f))$.

Theorem 2.1. *If the trajectory $\mathcal{T}$ yields a weak minimum, then the following Condition $\mathcal{A}$ holds: the set Λ_0 is nonempty and*

$$\max_{\lambda \in \Lambda_0} \Omega(\lambda, \bar{w}) \geq 0 \textit{ for all } \bar{w} \in \mathcal{K}.$$

Proof of the necessary condition for a week local minimum We present a short proof of this theorem omitting details. In this proof, we will use the Dubovitskii–Milyutin method of critical variations, cf. [2]. Let w^0 be a weak local minimum. Without loss of generality we assume that $J(p^0) = 0$, and $F_i(p^0) = 0$ for all $i = 1, \ldots, d(F)$. Denote by $L^1([t_0, t_f], \mathcal{R}^{d(x)})$ the space of integrable functions. Consider the operator

$$G : w = (x, u) \in \mathcal{W} \to \Big(f(w) - \dot{x},\ g(t, w),\ K(x(t_0), x(t_f)) \Big) \in \mathcal{Y}, \qquad (13)$$

where

$$\mathcal{Y} = L^1([t_0, t_f], \mathcal{R}^{d(x)}) \times L^\infty([t_0, t_f], \mathcal{R}^{d(g)}) \times \mathcal{R}^{d(K)}.$$

This operator is Frechêt continuously differentiable in a neighborhood of the point w^0, and its derivative at w^0 is a linear operator

$$G'(w^0) : w = (x, u) \in \mathcal{W} \to (f_w(t, w^0)w - \dot{x},\ g_w(t, w^0)w,\ K'(p^0)p) \in \mathcal{Y}. \qquad (14)$$

The derivative $G'(w^0)$ has a closed image (see, e.g., [4]), since the operator

$$w \in \mathcal{W} \to \big(f_w(t, w^0)w - \dot{x},\ g_w(t, w^0)w\big) \in L^1([t_0, t_f], \mathcal{R}^{d(x)}) \times L^\infty([t_0, t_f], \mathcal{R}^{d(g)})$$

is surjective (it easily follows from the regularity assumption for the constraint $g(t, w) = 0$), and the operator $w \in \mathcal{W} \to K'(p^0)p \in \mathcal{R}^{d(K)}$ is finite dimensional. Consider two possible cases: $G'(w^0)\mathcal{W} \neq \mathcal{Y}$ and $G'(w^0)\mathcal{W} = \mathcal{Y}$.

a) In the first case, the image $G'(w^0)\mathcal{W}$ is a closed subspace of $\mathcal{Y}$, not equal to $\mathcal{Y}$. Therefore, there exists a nonzero linear functional $l(w)$ vanishing on this image. The latter means that there exists a nonzero triple

$$\psi \in L^\infty([t_0, t_f], \mathcal{R}^{d(x)}), \quad \nu \in (L^\infty([t_0, t_f], \mathcal{R}^{d(g)}))^*, \quad \beta \in \mathcal{R}^{d(K)}$$

such that

$$\int_{t_0}^{t_f} \psi(f_w(t, w^0)w - \dot{x})\, dt + \langle \nu, g_w(t, w^0)w \rangle + \beta K'(p^0)p = 0 \quad \forall \bar{w} \in \mathcal{W}. \qquad (15)$$

On the subspace of $w \in \mathcal{W}$ such that $x = 0$ this condition takes the form:

$$\int_{t_0}^{t_f} \psi f_u(t, w^0)u\, dt + \langle \nu, g_u(t, w^0)u \rangle = 0 \quad \forall u \in L^\infty.$$

From this relation and the regularity assumption for g we easily obtain that the functional ν is absolutely continuous. Hence it is defined by an integrable function, which will be also denoted by ν. Then $\psi f_u(t, w^0) + \nu g_u(t, w^0) = 0$, i.e., $\bar{H}_u = 0$, where $\bar{H} = \psi f + \nu g$ (cf. (7)). It follows that ν is an essentially bounded function.

Now, setting $u = 0$ in (15) and taking into account that $\nu \in L^\infty$, we obtain

$$\begin{aligned} &\textstyle\int_{t_0}^{t_f} \big(\psi(f_x(t, w^0)x - \dot{x}) + \nu g_x(t, w^0)x\big)\ dt \\ &+\beta(K_{x_0}(p^0)x(t_0) + K_{x_f}(p^0)x(t_f)) = 0 \quad \forall x \in W^{1,1}. \end{aligned} \qquad (16)$$

It easily follows from (16) that the function ψ is absolutely continuous, and moreover it satisfies the adjoint equation $-\dot{\psi} = \bar{H}_x$ and the transversality conditions $-\psi(t_0) = l_{x_0}$ and $\psi(t_f) = l_{x_f}$ with $l = \beta K$. If $\beta = 0$, then the conditions $\psi(t_f) = l_{x_f} = 0$ and $-\dot{\psi} = \bar{H}_x$ imply that $\psi = 0$, and then the relation $\psi f_u + \nu g_u = 0$ and the full rank condition for g_u imply that $\nu = 0$. Hence $\beta \neq 0$ and we can take a triple (β, ψ, ν) with $|\beta| = 1$. Set $\alpha_0 = 0$, $\alpha = 0$, and $\hat{\lambda} = (0, 0, \beta, \psi, \nu)$. We see that thus obtained tuple $\hat{\lambda}$ belongs to the set Λ_0, and moreover, $-\hat{\lambda} \in \Lambda_0$. Then, for any element $\bar{w} \in \mathcal{W}_2$ we have: $\Omega(\lambda, \bar{w}) \geq 0$ or $\Omega(-\lambda, \bar{w}) \geq 0$. Thus, in the considered case, the set Λ_0 is nonempty and $\max_{\Lambda_0} \Omega(\lambda, \bar{w}) \geq 0$ on the whole space $\mathcal{W}_2$. Hence condition $\mathcal{A}$ trivially holds, although it is not informative in this case.

b) Now, consider the main case: $G'(w^0)\mathcal{W} = \mathcal{Y}$. The following lemma holds in this case.

Lemma 2.2. *For any $\bar{w} \in \mathcal{K} \cap \mathcal{W}$ the following system of linear equalities and inequalities is inconsistent with respect to $\tilde{w} \in \mathcal{W}$:*

$$J'(p^0)\tilde{p} + \frac{1}{2}\langle J''(p^0)\bar{p}, \bar{p}\rangle < 0, \tag{17}$$

$$F'(p^0)\tilde{p} + \frac{1}{2}\langle F''(p^0)\bar{p}, \bar{p}\rangle < 0, \tag{18}$$

$$K'(p^0)\tilde{p} + \frac{1}{2}\langle K''(p^0)\bar{p}, \bar{p}\rangle = 0, \tag{19}$$

$$f_w(t, w^0)\tilde{w} - \dot{\tilde{x}} + \frac{1}{2}\langle f_{ww}(t, w^0)\bar{w}, \bar{w}\rangle = 0, \tag{20}$$

$$g_w(t, w^0)\tilde{w} + \frac{1}{2}\langle g_{ww}(t, w^0)\bar{w}, \bar{w}\rangle = 0, \tag{21}$$

where $\tilde{p} = (\tilde{x}(t_0), \tilde{x}(t_f))$.

Proof. Assume the contrary: let there exist $\bar{w} \in \mathcal{K} \cap \mathcal{W}$ and $\tilde{w} \in \mathcal{W}$ satisfying (17)–(21). Consider the curve $w^\varepsilon = w^0 + \varepsilon\bar{w} + \varepsilon^2\tilde{w}$ parameterized by $\varepsilon > 0$. From conditions (19)–(21) it easily follows that $\|G(w^\varepsilon)\| = o(\varepsilon)$. Then, by generalized Lyusternik's theorem [1], there exists a curve $\hat{w}^\varepsilon \in \mathcal{W}$ $(\varepsilon > 0)$ such that $G(w^\varepsilon + \hat{w}^\varepsilon) = 0$ and $\|\hat{w}^\varepsilon\| = o(\varepsilon)$. Conditions (17)–(18) together with condition $\bar{w} \in \mathcal{K} \cap \mathcal{W}$ imply that $J(p^\varepsilon + \hat{p}^\varepsilon) < 0$ and $F(p^\varepsilon + \hat{p}^\varepsilon)) < 0$. Since $\|w^\varepsilon + \hat{w}^\varepsilon - w^0\| \to 0$ $(\varepsilon \to 0)$, the latter means that w^0 is not a local minimum in the problem. □

In order to analyze inconsistency of system (17)–(21), we will need the following well-known assertion (see, e.g., [4]).

Lemma 2.3. *Let X, Y be Banach spaces, $l_i : X \to \mathcal{R}$ linear functionals, a_i real numbers, $i = 1, \dots, k$, $A : X \to Y$ a liner surjective operator, $b \in Y$ a given element. The linear system*

$$l_i(x) + a_i < 0, \quad i = 1, \dots, k, \quad Ax + b = 0$$

is inconsistent (in $x \in X$) if and only if there exist numbers $\alpha_i \geq 0$ and a functional $y^* \in Y^*$ such that

$$\sum_{i=1}^{k} \alpha_i l_i + y^* A = 0, \quad \sum_{i=1}^{k} \alpha_i > 0, \quad \sum_{i=1}^{k} \alpha_i a_i + y^* b \geq 0.$$

Applying this lemma to system (17)–(21), we obtain the following result: there exist $\alpha_i \geq 0$, $i = 0, 1, \ldots, d(F)$, $\beta \in \mathcal{R}^{d(K)}$, $\psi \in L^\infty$, $\nu \in (L^\infty)^*$ such that $\sum_{i=0}^{d(F)} \alpha_i > 0$ and

$$l_p(p^0, \alpha_0, \alpha, \beta)p + \int_{t_0}^{t_f} (f_w(t, w^0)w - \dot{x})\, dt + \langle \nu, g_w(t, w^0)w \rangle = 0 \quad \forall w \in \mathcal{W}, \tag{22}$$

$$\langle l_{pp}(p^0, \alpha_0, \alpha, \beta)\bar{p}, \bar{p} \rangle + \int_{t_0}^{t_f} \langle f_{ww}(t, w^0)\bar{w}, \bar{w} \rangle\, dt + \langle \nu, (g_{ww}(t, w^0)\bar{w}, \bar{w}) \rangle \geq 0, \tag{23}$$

where l is as in (8). Without loss of generality we assume that $\sum_{i=0}^{d(F)} \alpha_i + \sum_{j=1}^{d(G)} |\beta_j| = 1$. The analysis of equation (22) is similar to that of (15). As result we prove that ν is absolutely continuous functional given by a bounded measurable function (which we also denote by ν), the function ψ is absolutely continuous, and the tuple $\lambda = (\alpha_0, \alpha, \beta, \psi, \nu)$ satisfies all conditions in the definition of Λ_0. Clearly, condition (23) means that $\Omega(\lambda, \bar{w}) \geq 0$. Thus we have proved that for any $\bar{w} \in \mathcal{K} \cap \mathcal{W}$ there exists $\lambda \in \Lambda_0$ such that $\Omega(\lambda, \bar{w}) \geq 0$ and hence $\max_{\Lambda_0} \Omega(\lambda, \cdot)$ is nonnegative on $\mathcal{K} \cap \mathcal{W}$. To get the same assertion on $\mathcal{K}$, it suffices to prove that the closure of the set $\mathcal{K} \cap \mathcal{W}$ in $\mathcal{W}_2$ is equal to $\mathcal{K}$. The latter easily can be proved using Hoffman's lemma [3]. We omit this simple proof.

3 Necessary Second Order Condition for a Θ-Weak Local Minimum in the General Problem of the Calculus of Variations on a Variable Time Interval

3.1 Problem B and a Θ-weak local minimum. Main results

Now, we consider a more general optimal control problem. Let $\mathcal{T}$ denote a *process* $(x(t), u(t) \mid t \in [t_0, t_f])$, where the state variable $x(\cdot)$ is a Lipschitz continuous function, and the control variable $u(\cdot)$ is a bounded measurable function on a time interval $\Delta = [t_0, t_f]$. The interval Δ is not fixed. For each process $\mathcal{T}$ we denote by

$$p = (t_0, x(t_0), t_f, x(t_f))$$

the vector of the endpoints of time-state variable (t, x). It is required to find $\mathcal{T}$ minimizing the functional

$$\mathcal{J}(\mathcal{T}) := J(p) \to \min \qquad (24)$$

subject to the constraints

$$F(p) \leq 0, \quad K(p) = 0, \qquad (25)$$
$$\dot{x}(t) = f(t, x(t), u(t)), \qquad (26)$$
$$g(t, x(t), u(t)) = 0, \qquad (27)$$
$$p \in \mathcal{P}, \quad (t, x(t), u(t)) \in \mathcal{Q}, \qquad (28)$$

where $\mathcal{P}$ and $\mathcal{Q}$ are open sets, x, u, F, K, f, and g are vector-functions.

We assume that the functions J, F, and K are defined and twice continuously differentiable on $\mathcal{P}$, and the functions f and g are defined and twice continuously differentiable on $\mathcal{Q}$. It is also assumed that the gradients with respect to the control $g_{iu}(t, x, u)$, $i = 1, \ldots, d(g)$ are linearly independent at each point $(t, x, u) \in \mathcal{Q}$ such that $g(t, x, u) = 0$ (the *regularity assumption* for g).

We say that a process $\mathcal{T}$ is *admissible* if it satisfies all constraints of the problem. Let $\mathcal{T} = (x(t), u(t) \mid t \in [t_0, t_f])$ be a fixed admissible process. We say that $\mathcal{T}$ is a *weak local minimum* if there exists $\varepsilon > 0$ such that $\mathcal{J}(\tilde{\mathcal{T}}) \geq \mathcal{J}(\mathcal{T})$ for each admissible process $\tilde{\mathcal{T}} = (\tilde{x}(t), \tilde{u}(t) \mid t \in [\tilde{t}_0, \tilde{t}_f])$ satisfying the conditions

$$|\tilde{t}_0 - t_0| < \varepsilon, \quad |\tilde{t}_f - t_f| < \varepsilon, \quad \max_{t \in \Delta \cap \tilde{\Delta}} |\tilde{x}(t) - x(t)| < \varepsilon,$$
$$\text{ess sup}_{t \in \Delta \cap \tilde{\Delta}} |\tilde{u}(t) - u(t)| < \varepsilon,$$

where $\tilde{\Delta} = [\tilde{t}_0, \tilde{t}_1]$.

In the sequel, we consider an admissible process $\mathcal{T} = (x(t), u(t) \mid t \in [t_0, t_f])$ such that the control $u(\cdot)$ is a piecewise continuous function on the interval Δ with the set of discontinuity points $\Theta = \{t_1, \ldots, t_s\}$, $t_0 < t_1 < \ldots < t_s < t_f$. Moreover, we assume that the control $u(\cdot)$ is Lipschitz-continuous on each interval (t_{k-1}, t_k), $k = 1, \ldots, s+1$, where $t_{s+1} := t_f$ (in this case, we say that the function $u(\cdot)$ is *piecewise Lipschitz-continuous* on Δ). Let us formulate the first-order necessary condition for optimality of the trajectory $\mathcal{T}$. Again, we introduce the Pontryagin function $H(t, x, u, \psi)$, the augmented Pontryagin function $\bar{H}(t, x, u, \psi, \nu)$, and the endpoint Lagrange function $l(p, \alpha_0, \alpha, \beta)$ defined as in (6), (7), and (8), respectively, but now $p = (t_0, x(t_0), t_f, x(t_f))$. Introduce a tuple of Lagrange multipliers

$$\lambda = (\alpha_0, \alpha, \beta, \psi(\cdot), \psi_0(\cdot), \nu(\cdot)) \qquad (29)$$

such that $\psi(\cdot) : \Delta \to (\mathcal{R}^{d(x)})^*$ and $\psi_0(\cdot) : \Delta \to \mathcal{R}^1$ are piecewise smooth functions, continuously differentiable on each interval of the set $\Delta \setminus \Theta$, and

$\nu(\cdot) : \Delta \to (\mathcal{R}^{d(g)})^*$ is a piecewise continuous function, Lipschitz continuous on each interval of the set $\Delta \setminus \Theta$.

Denote by Λ_0^Θ the set of all tuples λ satisfying the conditions:

$$\begin{aligned}
&\alpha_0 \geq 0,\ \alpha \geq 0,\ \alpha F(p) = 0,\ \alpha_0 + \sum \alpha_i + \sum |\beta_j| = 1, \\
&\dot{\psi} = -\bar{H}_x,\ \dot{\psi}_0 = -\bar{H}_t,\ \bar{H}_u = 0,\ t \in \Delta \setminus \Theta, \\
&\psi(t_0) = -l_{x_0},\ \psi(t_f) = l_{x_f},\ \psi_0(t_0) = -l_{t_0},\ \psi_0(t_f) = l_{t_f}, \\
&H(t, x(t), u(t), \psi(t)) + \psi_0(t) = 0,\ t \in \Delta \setminus \Theta.
\end{aligned} \tag{30}$$

The derivatives l_{x_0} and l_{x_f} are at $(p, \alpha_0, \alpha, \beta)$, where $p = (t_0, x(t_0), t_f, x(t_f))$, and the derivatives $\bar{H}_x$, $\bar{H}_u$, and $\bar{H}_t$ are at $(t, x(t), u(t), \psi(t), \nu(t))$, where $t \in \Delta \setminus \Theta$.

Let us give the definition of Θ-weak minimum in problem (24)-(28) on a variable interval $[t_0, t_f]$. For convenience, we assume that $u(\cdot)$ is left continuous at each point of discontinuity $t_k \in \Theta$. Denote by $\overline{u}$ the closure of the graph of $u(t)$.

Definition 3.1. The trajectory $\mathcal{T}$ affords a Θ-*weak minimum* if there exist $\varepsilon > 0$ and a neighborhood $V \subset \mathcal{R}^{d(u)+1}$ of the compact set $\overline{u}$ such that $\mathcal{J}(\tilde{\mathcal{T}}) \geq \mathcal{J}(\mathcal{T})$ for all admissible trajectories $\tilde{\mathcal{T}} = (\tilde{x}(t), \tilde{u}(t) \mid t \in [\tilde{t}_0, \tilde{t}_f])$ satisfying the conditions

(a) $|\tilde{t}_0 - t_0| < \varepsilon, \quad |\tilde{t}_f - t_f| < \varepsilon,$

(b) $\max_{\tilde{\Delta} \cap \Delta} |\tilde{x}(t) - x(t)| < \varepsilon, \quad$ where $\tilde{\Delta} = [\tilde{t}_0, \tilde{t}_f],$

(c) $(t, \tilde{u}(t)) \in V \quad$ a.e. on $[\tilde{t}_0, \tilde{t}_f]$.

The condition $\Lambda_0^\Theta \neq \emptyset$ is equivalent to the local minimum principle. It is a first-order necessary condition of Θ-weak minimum for the trajectory $\mathcal{T}$. Assume that Λ_0^Θ is nonempty. Using the definition of the set Λ_0^Θ and the full rank condition of the matrix g_u on the surface $g = 0$ one can easily prove that Λ_0^Θ is a finite-dimensional compact set, and the mapping $\lambda \mapsto (\alpha_0, \alpha, \beta)$ is injective on Λ_0^Θ.

Let us formulate a *quadratic necessary condition* of a Θ-weak minimum for the trajectory $\mathcal{T}$. First, for this trajectory, we introduce a Hilbert space $\mathcal{Z}_2(\Theta)$ and a "critical cone" $\mathcal{K} \subset \mathcal{Z}_2(\Theta)$. We denote by $P_\Theta W^{1,2}(\Delta, \mathcal{R}^{d(x)})$ the Hilbert space of piecewise continuous functions $\bar{x}(\cdot) : \Delta \to \mathcal{R}^{d(x)}$, absolutely continuous on each interval of the set $\Delta \setminus \Theta$ and such that their first derivative is square integrable. For each $\bar{x} \in P_\Theta W^{1,2}(\Delta, \mathcal{R}^{d(x)})$, $t_k \in \Theta$ we set

$$\bar{x}^{k-} = \bar{x}(t_k-), \quad \bar{x}^{k+} = \bar{x}(t_k+), \quad [\bar{x}]^k = \bar{x}^{k+} - \bar{x}^{k-}.$$

Thus $[\bar{x}]^k$ is the jump of the function $\bar{x}(t)$ at the point $t_k \in \Theta$. Similar notation we will use to denote jumps of other functions at a point $t_k \in \Theta$.

Set

$$\bar{z} = (\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_s, \bar{t}_f, \bar{x}, \bar{u}),$$

where

$$\bar{t}_k \in \mathcal{R}^1, \quad k = 0, 1, \ldots, s, \quad \bar{t}_f \in \mathcal{R}^1,$$
$$\bar{x} \in P_\Theta W^{1,2}(\Delta, \mathcal{R}^{d(x)}), \quad \bar{u} \in L^2(\Delta, \mathcal{R}^{d(u)}).$$

Thus,

$$\bar{z} \in \mathcal{Z}_2(\Theta) := \mathcal{R}^{s+2} \times P_\Theta W^{1,2}(\Delta, \mathcal{R}^{d(x)}) \times L^2(\Delta, \mathcal{R}^{d(u)}).$$

Moreover, for given $\bar{z}$ we set

$$\bar{w} = (\bar{x}, \bar{u}), \tag{31}$$
$$\bar{p} = (\bar{t}_0, \bar{x}(t_0) + \bar{t}_0 \dot{x}(t_0), \bar{t}_f, \bar{x}(t_f) + \bar{t}_f \dot{x}(t_f)). \tag{32}$$

By $I_F(p) = \{i \in \{1, \ldots, d(F)\} \mid F_i(p) = 0\}$ we denote the set of active indices of the constraints $F_i(p) \leq 0$. Set $[\dot{x}]^k = \dot{x}(t_k+) - \dot{x}(t_k-)$.

Let $\mathcal{K}$ be the set of all $\bar{z} \in \mathcal{Z}_2(\Theta)$ satisfying the following conditions:

$$\begin{array}{l} J'(p)\bar{p} \leq 0, \quad F_i'(p)\bar{p} \leq 0 \; \forall\, i \in I_F(p), \quad K'(p)\bar{p} = 0, \\ \dot{\bar{x}}(t) = f_w(t, w(t))\bar{w}(t), \text{ for a.a. } t \in [t_0, t_f], \\ [\bar{x}]^k + [\dot{x}]^k \bar{t}_k = 0, \quad k = 1, \ldots, s \\ g_w(t, w(t))\bar{w}(t) = 0, \text{ for a.a. } t \in [t_0, t_f], \end{array} \tag{33}$$

where $p = (t_0, x(t_0), t_f, x(t_f))$, $w = (x, u)$. It is obvious that $\mathcal{K}$ is a convex cone in the Hilbert space $Z_2(\Theta)$. We call it the *critical cone.*

Let us introduce a quadratic form on $\mathcal{Z}_2(\Theta)$. For $\lambda \in \Lambda_0^\Theta$ and $\bar{z} \in \mathcal{K}$, we set

$$\begin{array}{rcl} \Omega(\lambda, \bar{z}) & = & \langle l_{pp}\bar{p}, \bar{p}\rangle + \int_{t_0}^{t_f} \langle \bar{H}_{ww}\bar{w}(t), \bar{w}(t)\rangle \, dt \\ & & + \sum_{k=1}^{s} [\dot{\psi}_0 + \dot{\psi}\dot{x}]^k \bar{t}_k^2 + 2[\dot{\psi}\bar{x}]^k \bar{t}_k \\ & & + (\dot{\psi}_0(t_0) + \dot{\psi}(t_0)\dot{x}(t_0))\bar{t}_0^2 + 2\dot{\psi}(t_0)\bar{x}(t_0)\bar{t}_0 \\ & & - (\dot{\psi}_0(t_f) + \dot{\psi}(t_f)\dot{x}(t_f))\bar{t}_f^2 - 2\dot{\psi}(t_f)\bar{x}(t_f)\bar{t}_f, \end{array} \tag{34}$$

where $l_{pp} = l_{pp}(p, \alpha_0, \alpha, \beta)$, $p = (t_0, x(t_0), t_f, x(t_f))$.

Theorem 3.2. *If the trajectory $\mathcal{T}$ yields a Θ-weak minimum, then the following Condition $\mathcal{A}$ holds: the set Λ_0^Θ is nonempty and*

$$\max_{\lambda \in \Lambda_0^\Theta} \Omega(\lambda, \bar{z}) \geq 0 \textit{ for all } \bar{z} \in \mathcal{K}.$$

3.2 Proofs

The proofs are based on the quadratic necessary optimality conditions of a weak minimum, obtained for the problem on a fixed interval of time. We will give the proofs omitting some details. In order to extend the proofs to the case of a variable interval $[t_0, t_f]$ we use a simple change of the time variable. Namely, with the fixed admissible trajectory

$$\mathcal{T} = (x(t), u(t) \mid t \in [t_0, t_f])$$

in problem (24)-(28) on a variable time interval we associate a trajectory

$$\mathcal{T}^\tau = (t(\tau), x(\tau), u(\tau), v(\tau) \mid \tau \in [\tau_0, \tau_f]),$$

considered on a fixed interval $[\tau_0, \tau_f]$, where $\tau_0 = t_0$, $\tau_f = t_f$, $t(\tau) \equiv \tau$, $v(\tau) \equiv 1$. This is an admissible trajectory in the following problem on a fixed interval $[\tau_0, \tau_f]$: to minimize the cost function

$$\mathcal{J}(\mathcal{T}^\tau) := J(t(\tau_0), x(\tau_0), t(\tau_f), x(\tau_f)) \to \min \tag{35}$$

subject to the constraints

$$F(t(\tau_0), x(\tau_0), t(\tau_f), x(\tau_f)) \leq 0, \; K(t(\tau_0), x(\tau_0), t(\tau_f), x(\tau_f)) = 0, \tag{36}$$

$$\frac{dx(\tau)}{d\tau} = v(\tau) f(t(\tau), x(\tau), u(\tau)), \quad \frac{dt(\tau)}{d\tau} = v(\tau), \tag{37}$$

$$g(t(\tau), x(\tau), u(\tau)) = 0, \tag{38}$$

$$(t(\tau_0), x(\tau_0), t(\tau_f), x(\tau_f)) \in \mathcal{P}, \quad (t(\tau), x(\tau), u(\tau)) \in \mathcal{Q}. \tag{39}$$

In this problem, $t(\tau)$ and $x(\tau)$ are state variables, and $u(\tau)$ and $v(\tau)$ are control variables. For brevity, we will refer to problem (24)-(28) as problem P (on a variable interval $\Delta = [t_0, t_f]$), and to problem (35)-(39) as problem P^τ (on a fixed interval $[\tau_0, \tau_f]$). We denote by $\mathcal{A}^\tau$ the necessary quadratic condition $\mathcal{A}$ for problem P^τ on a fixed interval $[\tau_0, \tau_f]$.

Recall that the control $u(\cdot)$ is a piecewise Lipschitz-continuous function on the interval $\Delta = [t_0, t_f]$ with the set of discontinuity points $\Theta = \{t_1, \ldots, t_s\}$, where $t_0 < t_1 < \ldots < t_s < t_f$. Hence, for each $\lambda \in \Lambda_0^\Theta$, the function $\nu(t)$ is also piecewise Lipschitz-continuous on the interval Δ, and, moreover, all discontinuity points of ν belong to Θ. This easily follows from the equation $\bar{H}_u = 0$ and the full rank condition for matrix g_u. Consequently, $\dot{u}$ and $\dot{\nu}$ are bounded measurable functions on Δ. The proof of Theorem 3.2 is composed of the following chain of implications:

(i) A Θ-weak minimum is attained on the trajectory $\mathcal{T}$ in problem $P \Longrightarrow$

(ii) A weak minimum is attained on the trajectory $\mathcal{T}^\tau$ in problem $P^\tau \Longrightarrow$

(iii) Condition $\mathcal{A}^\tau$ for the trajectory $\mathcal{T}^\tau$ in problem $P^\tau \Longrightarrow$

(iv) Condition $\mathcal{A}$ for the trajectory $\mathcal{T}$ in problem P.

The first implication is readily verified, the second follows from Theorem 2.1. The verification of the third implication *(iii)* $\Rightarrow$ *(iv)* is not short and rather technical: we have to compare the sets of Lagrange multipliers, the critical cones and the quadratic forms in the both problems. This will be done below.

Comparison of the sets of Lagrange multiplies. Let us formulate the local minimum principle in problem P^τ for the trajectory $\mathcal{T}^\tau$. The endpoint Lagrange function l, the Pontryagin function H and the augmented

Pontryagin function $\bar{H}$ (all of them are equipped with the superscript τ) have the form:

$$
\begin{aligned}
&l^\tau = \alpha_0 J + \alpha F + \beta K = l, \\
&H^\tau = \psi f v + \psi_0 v = v(\psi f + \psi_0), \quad \bar{H}^\tau = H^\tau + \nu g.
\end{aligned} \tag{40}
$$

The set Λ_0^τ in problem P^τ for the trajectory $\mathcal{T}^\tau$ consists of all tuples of Lagrange multipliers $\lambda^\tau = (\alpha_0, \alpha, \beta, \psi, \psi_0, \nu)$ such that the following conditions holds:

$$
\begin{aligned}
&\alpha_0 + |\alpha| + |\beta| = 1, \\
&-\tfrac{d\psi}{d\tau} = v\psi f_x + \nu g_x, \quad -\tfrac{d\psi_0}{d\tau} = v\psi f_t + \nu g_t, \\
&\psi(\tau_0) = -l_{x_0}, \quad \psi(\tau_f) = l_{x_f}, \quad \psi_0(\tau_0) = -l_{t_0}, \quad \psi_0(\tau_f) = l_{t_f}, \\
&\bar{H}_u^\tau = v\psi f_u + \nu g_u = 0, \quad \bar{H}_v^\tau = \psi f + \psi_0 = 0.
\end{aligned} \tag{41}
$$

Recall that here $v(\tau) \equiv 1$, $t(\tau) \equiv \tau$, $\tau_0 = t_0$, $\tau_f = t_f$. In (41), the function f and the derivatives f_x, f_u, f_t, g_x g_u, g_t are taken at $(t(\tau), x(\tau), u(\tau))$, $\tau \in [\tau_0, \tau_f] \setminus \Theta$, while the derivatives l_{t_0}, l_{x_0}, l_{t_f} l_{x_f} are calculated at $(t(\tau_0), x(\tau_0), t(\tau_f), x(\tau_f)) = (t_0, x(t_0), t_f, x(t_f))$. This implies that $\Lambda_0^\tau = \Lambda_0^\Theta$.

Comparison of the critical cones. For brevity, we set $\varrho = (t, x, u, v) = (t, w, v)$. Let us define the critical cone $\mathcal{K}^\tau$ in problem P^τ for the trajectory $\mathcal{T}^\tau$. It consists of all tuples $\bar{\varrho} = (\bar{t}, \bar{x}, \bar{u}, \bar{v})$ satisfying the relations:

$$J_{t_0}\bar{t}(\tau_0) + J_{x_0}\bar{x}(\tau_0) + J_{t_f}\bar{t}(\tau_f) + J_{x_f}\bar{x}(\tau_f) \le 0, \tag{42}$$

$$F_{it_0}\bar{t}(\tau_0) + F_{ix_0}\bar{x}(\tau_0) + F_{it_f}\bar{t}(\tau_f) + F_{ix_f}\bar{x}(\tau_f) \le 0, \quad i \in I_F(p), \tag{43}$$

$$K_{t_0}\bar{t}(\tau_0) + K_{x_0}\bar{x}(\tau_0) + K_{t_f}\bar{t}(\tau_f) + K_{x_f}\bar{x}(\tau_f) = 0, \tag{44}$$

$$\frac{d\bar{x}}{d\tau} = \bar{v} f + v(f_t\bar{t} + f_x\bar{x} + f_u\bar{u}), \tag{45}$$

$$\frac{d\bar{t}}{d\tau} = \bar{v}, \tag{46}$$

$$g_t\bar{t} + g_x\bar{x} + g_u\bar{u} = 0, \tag{47}$$

where the derivatives J_{t_0}, J_{x_0}, J_{t_f} J_{x_f}, etc. are calculated at

$$(t(\tau_0), x(\tau_0), t(\tau_f), x(\tau_f)) = (t_0, x(t_0), t_f, x(t_f)),$$

while f, f_t, f_x, f_u g_t, g_x, and g_u are taken at $(t(\tau), x(\tau), u(\tau))$, $\tau \in [\tau_0, \tau_f] \setminus \Theta$. Let $\bar{\varrho} = (\bar{t}, \bar{x}, \bar{u}, \bar{v})$ be an element of the critical cone $\mathcal{K}^\tau$. We will make use of the following change of variables:

$$\tilde{x} = \bar{x} - \bar{t}\dot{x}, \quad \tilde{u} = \bar{u} - \bar{t}\dot{u}, \tag{48}$$

or briefly

$$\tilde{w} = \bar{w} - \bar{t}\dot{w}. \tag{49}$$

Since $v = 1$, $\dot{x} = f$, and $t = \tau$, equation (45) is equivalent to the equation

$$\frac{d\bar{x}}{dt} = \bar{v}\dot{x} + f_t\bar{t} + f_w\bar{w}. \tag{50}$$

Using the relation $\bar{x} = \tilde{x} + \bar{t}\dot{x}$ in this equation along with $\dot{\bar{t}} = \bar{v}$, we get

$$\dot{\tilde{x}} + \bar{t}\ddot{x} = f_t\bar{t} + f_w\bar{w}. \tag{51}$$

By differentiating the equation $\dot{x}(t) = f(t, w(t))$, we obtain

$$\ddot{x} = f_t + f_w\dot{w}. \tag{52}$$

Using this relation in (51), we get

$$\dot{\tilde{x}} = f_w\tilde{w}. \tag{53}$$

The relations

$$\bar{x} = \tilde{x} + \bar{t}\dot{x}, \quad [\bar{x}]^k = 0, \quad k = 1, \dots, s,$$

imply

$$[\tilde{x}]^k + [\dot{x}]^k\bar{t}_k = 0, \tag{54}$$

where

$$\bar{t}_k = \bar{t}(t_k), \quad k = 1, \dots, s. \tag{55}$$

Further, relation (47) can be written as $g_t\bar{t} + g_w\bar{w} = 0$. Differentiating the relation $g(t, w(t)) = 0$ we obtain

$$g_t + g_w\dot{w} = 0. \tag{56}$$

These relations along with (49) imply that

$$g_w\tilde{w} = 0. \tag{57}$$

Finally, note that since $\bar{x} = \tilde{x} + \bar{t}\dot{x}$, and $\tau_0 = t_0$, $\tau_f = t_f$, we have

$$\bar{p} = (\bar{t}_0, \bar{x}(t_0), \bar{t}_f, \bar{x}(t_f)) = (\bar{t}_0, \tilde{x}(t_0) + \bar{t}_0\dot{x}(t_0), \bar{t}_f, \tilde{x}(t_f) + \bar{t}_f\dot{x}(t_f)), \tag{58}$$

where $\bar{t}_0 = \bar{t}(t_0)$ and $\bar{t}_f = \bar{t}(t_f)$. The vector in the right hand side of the last equality has the same form as the vector $\bar{p}$ in definition (32). Consequently, all relations in definition (33) of the critical cone $\mathcal{K}$ in problem P are satisfied for the element $\tilde{z} = (\bar{t}_0, \bar{t}_1, \dots, \bar{t}_s, \bar{t}_f, \tilde{w})$. We have proved that thus obtained element $\tilde{z}$ belongs to the critical cone $\mathcal{K}$ in problem P.

Vice versa, let $(\bar{t}_0, \bar{t}_1, \dots, \bar{t}_s, \bar{t}_f, \tilde{w})$ be an element of the critical cone $\mathcal{K}$ in problem P. Let us take a Lipschitz continuous function $\bar{t}$ satisfying

$$\bar{t}(t_0) = \bar{t}_0, \quad \bar{t}(t_f) = \bar{t}_f, \quad \bar{t}(t_k) = \bar{t}_k, \ k = 1, \dots, s;$$

e.g., one can take a continuous function $\bar{t}$, affine at each interval (t_{k-1}, t_k), $k = 1, \dots, s+1$, where $t_{s+1} = t_f$. Set

$$\bar{v} = \dot{\bar{t}}, \quad \bar{w} = \tilde{w} + \bar{t}\dot{w}.$$

Then we obtain an element $(\bar{t}, \bar{w}, \bar{v})$ of the critical cone $\mathcal{K}^\tau$ (see (42)-(47)) in problem P^τ. Thus, we have proved the following lemma.

Lemma 3.3. *If $(\bar{t}, \bar{w}, \bar{v})$ is an element of the critical cone $\mathcal{K}^\tau$, as in (42)-(47), in problem P^τ for the trajectory $\mathcal{T}^\tau$ and*

$$\bar{t}_0 = \bar{t}(t_0), \quad \bar{t}_f = \bar{t}(t_f), \quad \tilde{w} = \bar{w} - \bar{t}\dot{w}, \quad \bar{t}_k = \bar{t}(t_k), \; k = 1, \dots, s, \tag{59}$$

then $(\bar{t}_0, \bar{t}_1, \dots, \bar{t}_s, \bar{t}_f, \tilde{w})$ is an element of the critical cone $\mathcal{K}$, as in (33), in problem P for the trajectory $\mathcal{T}$. Moreover, relations (59) define a surjective mapping of the critical cone $\mathcal{K}^\tau$ on the critical cone $\mathcal{K}$.

We will say that an element $(\bar{t}_0, \bar{t}_1, \dots, \bar{t}_s, \bar{t}_f, \tilde{w})$ of the critical cone $\mathcal{K}$ in problem P *corresponds* to an element $(\bar{t}, \bar{w}, \bar{v})$ of the critical cone $\mathcal{K}^\tau$ in problem P^τ if relations (59) hold.

Comparison of the quadratic forms. Let an element $(\bar{t}_0, \bar{t}_1, \dots, \bar{t}_s, \bar{t}_f, \tilde{w})$ of the critical cone $\mathcal{K}$ in problem P corresponds to an element $(\bar{t}, \bar{w}, \bar{v})$ of the critical cone $\mathcal{K}^\tau$ in problem P^τ. Assume that $\lambda \in \Lambda_0^\tau$ (recall that $\Lambda_0^\tau = \Lambda_0^\Theta$). Let us show that the quadratic form $\Omega^\tau(\lambda, \cdot)$, calculated on the element $(\bar{t}, \bar{w}, \bar{v})$ can be transformed to the quadratic form $\Omega(\lambda, \cdot)$ calculated on the corresponding element $(\bar{t}_0, \bar{t}_f, \tilde{\xi}, \tilde{w})$, i.e., the forms on these elements take equal values.

(i) Set

$$\varrho = (t, w, v), \quad \bar{\varrho} = (\bar{t}, \bar{w}, \bar{v}).$$

It follows from (40) that

$$\Omega^\tau(\lambda, \bar{\varrho}) = \langle l_{pp}^\lambda \bar{p}, \bar{p} \rangle + \int_{t_0}^{t_f} \langle \bar{H}_{\varrho\varrho}^\tau \bar{\varrho}, \bar{\varrho} \rangle \, dt, \tag{60}$$

where

$$\langle \bar{H}_{\varrho\varrho}^\tau \bar{\varrho}, \bar{\varrho} \rangle = \langle \bar{H}_{ww} \bar{w}, \bar{w} \rangle + 2\bar{H}_{tw}\bar{w}\bar{t} + \bar{H}_{tt}\bar{t}^2 + 2\bar{v}(H_w \bar{w} + H_t \bar{t}). \tag{61}$$

Since $\bar{w} = \tilde{w} + \bar{t}\dot{w}$, we have

$$\langle \bar{H}_{ww} \bar{w}, \bar{w} \rangle = \langle \bar{H}_{ww} \tilde{w}, \tilde{w} \rangle + 2\langle \bar{H}_{ww} \dot{w}, \tilde{w} \rangle \bar{t} + \langle \bar{H}_{ww} \dot{w}, \dot{w} \rangle \bar{t}^2. \tag{62}$$

$$2\bar{H}_{tw}\bar{w}\bar{t} = 2\bar{H}_{tw}\tilde{w}\bar{t} + 2\bar{H}_{tw}\dot{w}\bar{t}^2, \tag{63}$$

Moreover, using the relations

$$\begin{array}{llll} H_w = \bar{H}_w - \nu g_w, & H_t = \bar{H}_t - \nu g_t, & g_t \bar{t} + g_w \bar{w} = 0, \\ -\dot{\psi} = \bar{H}_x, & -\dot{\psi}_0 = \bar{H}_t, & \bar{H}_u = 0, & g_t + g_w \dot{w} = 0, \end{array}$$

we obtain

$$\begin{aligned} H_w \bar{w} + H_t \bar{t} &= \bar{H}_w \bar{w} + \bar{H}_t \bar{t} - \nu(g_w \bar{w} + g_t \bar{t}) \\ &= \bar{H}_w \bar{w} + \bar{H}_t \bar{t} = \bar{H}_x \bar{x} + \bar{H}_t \bar{t} \\ &= -\dot{\psi}\bar{x} - \dot{\psi}_0 \bar{t} = -\dot{\psi}(\tilde{x} + \bar{t}\dot{x}) - \dot{\psi}_0 \bar{t} \\ &= -\dot{\psi}\tilde{x} - (\dot{\psi}\dot{x} + \dot{\psi}_0)\bar{t}. \end{aligned} \tag{64}$$

Relations (61)-(64) imply

$$\begin{aligned}\langle \bar{H}^{\tau}_{\varrho\varrho}\bar{\varrho},\bar{\varrho}\rangle &= \langle \bar{H}_{ww}\tilde{w},\tilde{w}\rangle + 2\langle \bar{H}_{ww}\dot{w},\tilde{w}\rangle\bar{t} + \langle \bar{H}_{ww}\dot{w},\dot{w}\rangle\bar{t}^2 \\ &\quad +2\bar{H}_{tw}\tilde{w}\bar{t} + 2\bar{H}_{tw}\dot{w}\bar{t}^2 + \bar{H}_{tt}\bar{t}^2 \\ &\quad -2\dot{\psi}\tilde{x}\bar{v} - 2(\dot{\psi}_0 + \dot{\psi}\dot{x})\bar{t}\bar{v}.\end{aligned} \tag{65}$$

Consequently,

$$\begin{aligned}\langle \bar{H}^{\tau}_{\varrho\varrho}\bar{\varrho},\bar{\varrho}\rangle &= \langle \bar{H}_{ww}\tilde{w},\tilde{w}\rangle + 2(\langle \bar{H}_{ww}\dot{w},\tilde{w}\rangle + \bar{H}_{tw}\tilde{w})\bar{t} \\ &\quad +(\langle \bar{H}_{ww}\dot{w},\dot{w}\rangle + \bar{H}_{tw}\dot{w})\bar{t}^2 + (\bar{H}_{tw}\dot{w} + \bar{H}_{tt})\bar{t}^2 \\ &\quad -2\dot{\psi}\tilde{x}\bar{v} - 2(\dot{\psi}_0 + \dot{\psi}\dot{x})\bar{t}\bar{v}.\end{aligned} \tag{66}$$

(ii) Let us transform the terms $2(\langle \bar{H}_{ww}\dot{w},\tilde{w}\rangle + \bar{H}_{tw}\tilde{w})\bar{t}$ in (65). By differentiating the equation $-\dot{\psi} = \bar{H}_x$ with respect to t, we obtain

$$-\ddot{\psi} = \bar{H}_{tx} + (\dot{w})^*\bar{H}_{wx} + \dot{\psi}\bar{H}_{\psi x} + \dot{\nu}\bar{H}_{\nu x}.$$

Here $\bar{H}_{\psi x} = f_x$ and $\bar{H}_{\nu x} = g_x$. Therefore

$$-\ddot{\psi} = \bar{H}_{tx} + (\dot{w})^*\bar{H}_{wx} + \dot{\psi}f_x + \dot{\nu}g_x. \tag{67}$$

Similarly, by differentiating the equation $\bar{H}_u = 0$ with respect to t, we obtain

$$0 = \bar{H}_{tu} + (\dot{w})^*\bar{H}_{wu} + \dot{\psi}f_u + \dot{\nu}g_u. \tag{68}$$

Multiplying equation (67) by $\tilde{x}$ and equation (68) by $\tilde{u}$ and summing the results we get

$$-\ddot{\psi}\tilde{x} = \bar{H}_{tw}\tilde{w} + \langle \bar{H}_{ww}\dot{w},\tilde{w}\rangle + \dot{\psi}f_w\tilde{w} + \dot{\nu}g_w\tilde{w}.$$

But $f_w\tilde{w} = \dot{\tilde{x}}$ and $g_w\tilde{w} = 0$. Therefore,

$$-\ddot{\psi}\tilde{x} = \bar{H}_{tw}\tilde{w} + \langle \bar{H}_{ww}\dot{w},\tilde{w}\rangle + \dot{\psi}\dot{\tilde{x}},$$

whence

$$\bar{H}_{tw}\tilde{w} + \langle \bar{H}_{ww}\dot{w},\tilde{w}\rangle = -\frac{d}{dt}(\dot{\psi}\tilde{x}). \tag{69}$$

This implies that

$$2(\langle \bar{H}_{ww}\dot{w},\bar{w}\rangle + \bar{H}_{tw}\tilde{w})\bar{t} = -2\bar{t}\frac{d}{dt}(\dot{\psi}\tilde{x}). \tag{70}$$

(iii) Let us transform the terms $(\langle \bar{H}_{ww}\dot{w},\dot{w}\rangle + \bar{H}_{tw}\dot{w})\bar{t}^2$ in (65). Multiplying equation (67) by $\dot{x}$ and equation (68) by $\dot{u}$ and summing the results we obtain

$$-\ddot{\psi}\dot{x} = \bar{H}_{tw}\dot{w} + \langle \bar{H}_{ww}\dot{w},\dot{w}\rangle + \dot{\psi}f_w\dot{w} + \dot{\nu}g_w\dot{w}. \tag{71}$$

From (52) and (56) we get $f_w\dot{w} = \ddot{x} - f_t$, $g_w\dot{w} = -g_t$, respectively. Then (71) implies

$$\bar{H}_{tw}\dot{w} + \langle\bar{H}_{ww}\dot{w}, \dot{w}\rangle = -\frac{d}{dt}(\dot{\psi}\dot{x}) + (\dot{\psi}f_t + \dot{\nu}g_t). \tag{72}$$

Multiplying this relation by $\bar{t}^2$ we get

$$(\langle\bar{H}_{ww}\dot{w}, \dot{w}\rangle + \bar{H}_{tw}\dot{w})\bar{t}^2 = -\bar{t}^2\frac{d}{dt}(\dot{\psi}\dot{x}) + (\dot{\psi}f_t + \dot{\nu}g_t)\bar{t}^2. \tag{73}$$

(iv) Finally, let us transform the terms $\left(\bar{H}_{tw}\dot{w} + \bar{H}_{tt}\right)\bar{t}^2$ in (65). Differentiating the equation $-\dot{\psi}_0 = \bar{H}_t$ with respect to t and using the relations $\bar{H}_{\psi t} = f_t$ and $\bar{H}_{\nu t} = g_t$, we get

$$-\ddot{\psi}_0 = \bar{H}_{tt} + \bar{H}_{tw}\dot{w} + (\dot{\psi}f_t + \dot{\nu}g_t). \tag{74}$$

Consequently,

$$\left(\bar{H}_{tw}\dot{w} + \bar{H}_{tt}\right)\bar{t}^2 = -\ddot{\psi}_0\bar{t}^2 - (\dot{\psi}f_t + \dot{\nu}g_t)\bar{t}^2. \tag{75}$$

(v) Summing equations (73) and (75) we obtain

$$\langle\bar{H}_{ww}\dot{w}, \dot{w}\rangle\bar{t}^2 + \bar{H}_{tt}\bar{t}^2 + 2H_{tw}\dot{w}\bar{t}^2 = -\ddot{\psi}_0\bar{t}^2 - \bar{t}^2\frac{d}{dt}(\dot{\psi}\dot{x}). \tag{76}$$

(vi) Using relations (70) and (76) in (65) we get

$$\begin{aligned}\langle\bar{H}^{\tau}_{\varrho\varrho}\bar{\varrho}, \bar{\varrho}\rangle &= \langle\bar{H}_{ww}\tilde{w}, \tilde{w}\rangle - 2\bar{t}\frac{d}{dt}(\dot{\psi}\tilde{x}) \\ &\quad -\ddot{\psi}_0\bar{t}^2 - \bar{t}^2\frac{d}{dt}(\dot{\psi}\dot{x}) - 2\dot{\psi}\tilde{x}\bar{v} - 2(\dot{\psi}_0 + \dot{\psi}\dot{x})\bar{t}\bar{v}.\end{aligned} \tag{77}$$

But

$$\ddot{\psi}_0\bar{t}^2 + 2\bar{v}\bar{t}\dot{\psi}_0 = \frac{d}{dt}(\dot{\psi}_0\bar{t}^2), \quad \bar{t}\frac{d}{dt}(\dot{\psi}\tilde{x}) + \bar{v}(\dot{\psi}\tilde{x}) = \frac{d}{dt}\left(\bar{t}\dot{\psi}\tilde{x}\right),$$

$$2\bar{t}\bar{v}(\dot{\psi}\dot{x}) + \bar{t}^2\frac{d}{dt}(\dot{\psi}\dot{x}) = \frac{d}{dt}(\dot{\psi}\dot{x}\bar{t}^2).$$

Therefore,

$$\langle\bar{H}^{\tau}_{\varrho\varrho}\bar{\varrho}, \bar{\varrho}\rangle = \langle\bar{H}_{ww}\tilde{w}, \tilde{w}\rangle - \frac{d}{dt}\Big((\dot{\psi}\dot{x})\bar{t}^2 + \dot{\psi}_0\bar{t}^2 + 2\dot{\psi}\tilde{x}\bar{t}\Big). \tag{78}$$

We have proved the following lemma.

Lemma 3.4. *Let $\bar{\varrho} = (\bar{t}, \bar{w}, \bar{v}) \in \mathcal{K}^{\tau}$ and $(\bar{t}_0, \bar{t}_f, \tilde{\xi}, \tilde{w}) \in \mathcal{K}$ be such that the relations (59) holds, and let $\lambda \in \Lambda_0^{\Theta}$. Then formula (78) holds.*

(vi) Recall that $\tau_0 = t_0$, $\tau_f = t_f$, $t(\tau) = \tau$, $dt = d\tau$. Since the functions $\dot\psi_0$, $\dot\psi$, $\dot x$, and $\tilde x$ can have discontinuities only at the points of the set Θ, the following formula holds:

$$\begin{aligned}\int_{t_0}^{t_f} \tfrac{d}{dt}\Big((\dot\psi_0 + \dot\psi\dot x)\bar t^2 + 2\dot\psi\tilde x\bar t\Big)\, dt = \Big((\dot\psi_0 + \dot\psi\dot x)\bar t^2 + 2\dot\psi\tilde x\bar t\Big)\,\Big|_{t_0}^{t_f} \\ - \sum_{k=1}^{s} \Big([\dot\psi_0 + \dot\psi\dot x]^k\bar t(t_k)^2 + 2[\dot\psi\tilde x]^k\bar t(t_k)\Big).\end{aligned} \tag{79}$$

Formula (78) along with formula (79) gives the following transformation of quadratic form Ω^τ (60) on the element $\bar\varrho$ of the critical cone $\mathcal{K}^\tau$

$$\begin{aligned}\Omega^\tau(\lambda, \bar\varrho) \;&=\; \langle l_{pp}\bar p, \bar p\rangle + \textstyle\int_{t_0}^{t_f}\langle \bar H_{ww}\tilde w, \tilde w\rangle\, dt \\ &\quad - \Big((\dot\psi_0 + \dot\psi\dot x)\bar t^2 + 2\dot\psi\tilde x\bar t\Big)\,\Big|_{t_0}^{t_f} \\ &\quad + \sum_{k=1}^{s} \Big([\dot\psi_0 + \dot\psi\dot x]^k\bar t(t_k)^2 + 2[\dot\psi\tilde x]^k\bar t(t_k)\Big).\end{aligned} \tag{80}$$

Taking into account (59), we see that the right hand side of (80) is the quadratic form $\Omega(\lambda, \tilde z)$ (34) in problem P for the trajectory $\mathcal{T}$, where $\tilde z = (\bar t_0, \bar t_1, \ldots, \bar t_s, \bar t_f, \tilde w)$ is the corresponding element of the critical cone $\mathcal{K}$. Thus we have proved the following theorem.

Theorem 3.5. *Let $\bar\varrho = (\bar t, \tilde w, \bar v)$ be an element of the critical cone $\mathcal{K}^\tau$ in problem P^τ for the trajectory $\mathcal{T}^\tau$. Let $\tilde z = (\bar t_0, \bar t_1, \ldots, \bar t_s, \bar t_f, \tilde w)$ be the corresponding element of the critical cone $\mathcal{K}$ in problem P for the trajectory $\mathcal{T}$, i.e., relations (59) hold. Then for any $\lambda \in \Lambda_0^\tau$ the following equality holds $\Omega^\tau(\lambda, \bar\varrho) = \Omega(\lambda, \tilde z)$.*

This theorem proves the implication *(iii)* $\Rightarrow$ *(iv)* (see the beginning of this subsection). Indeed, suppose that Condition $\mathcal{A}$ holds for the trajectory $\mathcal{T}^\tau$ in problem P^τ, and let $\tilde z = (\bar t_0, \bar t_1, \ldots, \bar t_s, \bar t_f, \tilde w)$ be an arbitrary element of the critical cone $\mathcal{K}$ in problem P. Then by Lemma 3.3 there exists an element $\bar\varrho = (\bar t, \tilde w, \bar v)$ of the critical cone $\mathcal{K}^\tau$ in problem P^τ for the trajectory $\mathcal{T}^\tau$ such that relations (59) hold. Since Λ_0^Θ is a compact set and condition Condition $\mathcal{A}$ holds in problem P^τ, there exists an element $\lambda \in \Lambda_0^\Theta$ such that $\Omega^\tau(\lambda, \bar\varrho) \geq 0$. By theorem 3.5 we have $\Omega^\tau(\lambda, \bar\varrho) = \Omega(\lambda, \tilde z)$. Consequently, $\Omega(\lambda, \tilde z) \geq 0$, i.e., Condition $\mathcal{A}$ holds for the trajectory $\mathcal{T}$ in problem P. Thus we have proved the implication *(iii)* $\Rightarrow$ *(iv)*. This completes the proof of Theorem 3.2.

3.3 Equivalent formulation of main results

In [6] and [7] similar results were presented in another form. We will show that the critical cone and the quadratic form defined in the present work can be transformed to those in [6] and [7].

Let us transform the terms related to the discontinuity points t_k of the control $u(\cdot)$, $k = 1, \ldots, s$.

Lemma 3.6. *Let $\lambda \in M_0$ be an arbitrary element. Then for any $k = 1, \ldots, s$ the following formula holds*

$$[\dot\psi_0 + \dot\psi\dot x]^k \bar t(t_k)^2 + 2[\dot\psi\tilde x]^k \bar t(t_k) = D^k(\bar H)\bar\xi_k^2 - 2[\dot\psi]^k \tilde x_{\rm av}^k \bar\xi_k\,, \qquad (81)$$

where $\bar\xi_k = -\bar t(t_k)$.

Proof. Everywhere in this proof we will omit the subscript and superscript k. We will also write $\bar t$ instead of $\bar t(t_k)$. Set

$$a = D(H) = \dot\psi^+\dot x^- - \dot\psi^-\dot x^+ + [\dot\psi_0].$$

We have

$$\begin{aligned}
&[\dot\psi_0 + \dot\psi\dot x]\bar t^2 + 2[\dot\psi\tilde x]\bar t\\
&= \bar\xi^2[\dot\psi_0] + \bar\xi^2(\dot\psi^+\dot x^+ - \dot\psi^-\dot x^-) - 2\bar\xi(\dot\psi^+\tilde x^+ - \dot\psi^-\tilde x^-)\\
&= \bar\xi^2[\dot\psi_0] + \bar\xi^2(\dot\psi^+\dot x^- - \dot\psi^-\dot x^-) - 2\bar\xi\Big(\dot\psi^+(\tilde x_{av} + \tfrac12[\dot x]\bar\xi) - \dot\psi^-(\tilde x_{av} - \tfrac12[\dot x]\bar\xi\Big)\\
&= \bar\xi^2[\dot\psi_0] + \bar\xi^2(\dot\psi^+\dot x^+ - \dot\psi^-\dot x^- - \dot\psi^+[\dot x] + \psi^-[\dot x]) - 2\bar\xi[\dot\psi]\tilde x_{av}\\
&= \bar\xi^2[\dot\psi_0] + \bar\xi^2(\dot\psi^+\dot x^+ - \dot\psi^-\dot x^- - \dot\psi^+(\dot x^+ - \dot x^-) + \psi^-(\dot x^+ - \dot x^-)) - 2\bar\xi[\dot\psi]\tilde x_{av}\\
&= \bar\xi^2[\dot\psi_0] + \bar\xi^2(\dot\psi^+\dot x^- + \dot\psi^-\dot x^+) - 2\bar\xi[\dot\psi]\tilde x_{av}\\
&= D(H)\bar\xi^2 - 2[\dot\psi]\tilde x_{av}\bar\xi. \quad \square
\end{aligned} \qquad (82)$$

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