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O. Hryniewicz

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 8373578

fax: (+48) (22) 8372772

Kierownik Pracowni zgłaszający pracę:
Prof. dr hab. inż. Olgierd Hryniewicz

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Olgierd Hryniewicz

Systems Research Institute, Warsaw, Poland,
hryniewi@ibspan.waw.pl,
WWW home page: http://ibspan.waw.pl/~hryniewi/

Abstract. In the paper we present the most important, from theoretical and practical points of view, applications of fuzzy sets in the evaluation of reliability. We discuss fuzzy probabilistic reliability models that can be used for the unified analysis of probabilistic randomness and fuzzy vagueness which are both present in reliability data. We also present an alternative possibilistic approach that is based on the theory of possibility proposed by L.A.Zadeh. Large part of the paper is devoted to the problem of the statistical analysis of imprecisely reported (fuzzy) reliability data.

1 Introduction

Theory of reliability is more than fifty years old. Its basic concepts were established in the 1950s as useful tools for the analysis of complex technical systems. The rapid development of the theory of reliability was closely related to the importance of its main field of applications - military and space. For this reason the origins of the research in the area of reliability are still not well known. Ralph A.Evans, one of the founders of the IEEE Transactions on Reliability, wrote in an Editorial in this journal that all important theoretical results published in the 1960s and 1970s had been already obtained even in the 1950s, and for many years remained classified. The authors of the most important publications on reliability from those years belonged to the group of the most important scientists working in theory of probability, mathematical statistics, electronics and computer sciences.

When we look at the theory of reliability as the application of a basic mathematical theory, we could see without any doubt that it should be reagarded as one of the most important applications of the theory of probability. All important events which are of interest for the theory and practice of reliability have undoubtedly stochastic character, and all processes that lead to failures can be described by stochastic processes. Therefore, the theory of probability has been for many years used as the only tool for description, prediction and optimization of reliability. As the consequence of applying that approach, mathematical statistics has been used for the analysis reliability data.

In its initial phase of development, statistical methods used in the area of reliability were based on a classical approach to statistics. Classical concepts of statistics, such as estimators, confidence intervals and tests of hypotheses, that have their interpretations in terms of frequencies, were widely used in the analysis of reliability data. However, together with a continuous improvement of reliability of components and systems these classical methods became not sufficient for practical applications. Therefore, new statistical methods that were based on the Bayesian paradigm found their applications both in theory and practice of reliability. It is worthy noting that in that time the Bayesian approach to statistics was heavily attacked by the majority of the statistical community. However, practical successes of this approach have resulted nowdays with common acceptance of the Bayesian methodology in the area of reliability.

During the last fifteen years we have wittnessed a similar situation in the case of the application of the theory of fuzzy sets in the area of reliability. First, in the early 1980s the quality of components used mainly in the aerospace industry became so high that the probabilities of their failures had the order of magnitude close to 10^{-7} and less. Classical statistical methods of estimation, based on the observation of a random sample, are not applicable in that case. On the other hand, the methods based on the Bayesian approach are usually too complicated to be used in practice. As the result of these difficulties researchers and practitioners working in the area of reliability were able to provide only *imprecisely* defined values of probabilities of failures. In order to describe those imprecise values of probabilities they proposed to use the theory of fuzzy sets introduced by Lotfi A. Zadeh in the 1960s. Moreover, this new methodology appeared to be very usefull in all cases where the information related to reliability were based on imprecise expert opinions, imprecisely reported reliability data, etc. Another impulse for the development of the fuzzy reliability methodology was given in the investigation of complex man–machine systems, and complex multistate systems with imprecise definitions of failures. New methods for the reliability analysis that are based on the theory of fuzzy sets (and the related theory of possibility) and its mixture with the theory of probability have been proposed during last fifteen years, and are now ready for practical applications. An excellent overview of the problems mentioned above can be found in the paper by Cai [5].

The number of papers devoted to the applications of fuzzy sets in the analysis of reliability has become quite large, and it is rather impossible to present a comprehensive review of all of them in one paper. The readers who are interested in a broad introduction to the problem are encouraged to read collections of papers on that topic edited by Onisawa and Kacprzyk [43] and Misra [33]. Therefore, we have decided to give a rather general overview of the main results in this area. In the second section of the paper we consider problems related to the reliability analysis of systems with the usage of imprecise probabilities. In the third section of the paper we present the most important applications of the theory of possibility in the area of reliability. The fourth section is devoted to another very important from a practical point of view problem: statistical analysis of imprecise reliability data in both classical and Bayesian frameworks. Throughout the paper we present only main ideas and results that have been published in a few selected papers. The reader is encouraged, however, to find other related results that have been already published in the papers referenced by the papers that are listed in the bibliography to this paper.

2 Evaluation of reliability in case of imprecise probabilities

The methodology for the evaluation of reliability of systems characterized by binary states of its elements and binary states of the whole system was proposed in the early 1960s. Its detailed description can be found in fundamental books by Barlow and Proschan [1],[2]. We recall now only some basic notions of this theory.

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ be a vector that describes the state of n elements of the system such that

$$x_i = \begin{cases} 1 \text{ if the element } i \text{ is functioning} \\ 0 \text{ if the element } i \text{ is failed} \end{cases}, \quad i = 1, ..., n,$$

and ϕ describes a binary state of the whole system, i.e.

$$\phi = \begin{cases} 1 \text{ if the system is functioning} \\ 0 \text{ if the system is failed} \end{cases}.$$

We assume that the state of the whole system is completely determined by the states of its elements, i.e. $\phi = \phi(x_1, x_2, ..., x_n)$. Function $\phi(x_1, x_2, ..., x_n)$ is called the structure function of the system, or, simply, the structure. It is possible to show that every structure can be expressed by the following general formula:

$$\phi(\mathbf{x}) = \sum_{\mathbf{y}} \prod_{j=1}^{n} x_j^{y_j} (1 - x_j)^{1-y_j} \phi(\mathbf{y}) \qquad (1)$$

where the summmation is taken over all n-dimensional binary vectors $\mathbf{y}$ $(0^0 \equiv 1)$. Hence, every structure can be expressed as a polynomial of binary functions x_i that describe elements of the system.

Now, let us introduce the following notation:

$$\begin{aligned} (1_i, \mathbf{x}) &\equiv (x_1, ...x_{i-1}, 1, x_{i+1}, ..., x_n) \\ (0_i, \mathbf{x}) &\equiv (x_1, ...x_{i-1}, 0, x_{i+1}, ..., x_n) \\ (*_i, \mathbf{x}) &\equiv (x_1, ...x_{i-1}, *, x_{i+1}, ..., x_n) \end{aligned}$$

The ith element of the system is irrelevant if $\phi(1_i, \mathbf{x}) = \phi(0_i, \mathbf{x})$ for all $(*_i, \mathbf{x})$; otherwise such element is relevant. The system is called *coherent* if (a) its structure functon ϕ is increasing in every component, and (b) all its elements are relevant. For the coherent systems there exist many algorithms for efficient calculations of their reliability defined as the probability that the system is functioning.

One of the fundamental concepts of reliability of systems is the notion of a minimal path. A minimal path is a subset of system's elements such that if all these elements work, the whole system works. A dual concept to the minimal path is that of a minimal cut. A minimal set of system's elements is called a minimal cut if the failures of all its elements cause the failure of the whole system. Suppose that the considered system has n_c minimal cuts, and n_p minimal paths. Denote by $C_s, s \in \{1, ..., n_c\}$ a minimal cut of a system, and by $P_r, r \in \{1, ..., n_p\}$ its minimal path. According to the fundamental result of Birnbaum et al. [3] the structure of any binary system can be decomposed using either minimal paths or minimal cuts, and the following formula holds:

$$\phi(x_1, x_2, ..., x_n) = \bigvee_{1 \le r \le n_p} \bigwedge_{i \in P_r} x_i = \bigwedge_{1 \le s \le n_c} \bigvee_{i \in C_s} x_i \tag{2}$$

Therefore, the knowledge of all minimal cuts and/or minimal paths is sufficient for the full reliability description of a system.

Let us now recall basic results that are used in the calculation of reliability of a system. The reliability state of a system X_s and of each of its elements(X_i, $i = 1, ..., n$) is a random variable distributed according to a two-point probability distribution. Let q_i, $i = 1, ..., n$ be the reliability of the ith element of a system, and q_s the reliability of the whole system. Then, the following general expression holds:

$$q_s = E(X_s) = h(q_1, q_2, ..., q_n) \tag{3}$$

When the system is coherent and failures of its elements are statistically independent, then $h(q_1, q_2, ..., q_n)$, technically, is constructed by replacing $x_1, x_2, ..., x_n$ in (2) with $q_1, q_2, ..., q_n$; next by changing $\bigvee$ to a product operator on $[0, 1]$, and $\bigwedge$ to a probabilistic sum on $[0, 1]$, and finally replacing the powers like $q_k^m, m \ge 2$ (if exist) with respective values of q_k. Thus, the knowledge of $\phi(\mathbf{x})$ and the values of $q_1, q_2, ..., q_n$, in case of coherent binary structures and independent failures of elements, is fully sufficient for the calculation of the reliability of the whole system.

Reliability analysis of complex systems can be divided into two phases: determination of the structure function and evaluation of the reliabilities of system's elements. The sets of minimal cuts and minimal paths can be obtained using different methods. However, the most efficient, and thus the most frequently used, method is the fault tree analysis. This method was introduced more than forty years ago, and since that time has been successfuly used in many areas, such as aerospace industry, nuclear power plants, etc. The method consists in defining a structure of physical events related to failures of system's elements. There exist methods for the extraction of minimal paths and minimal cuts from the information contained in a fault tree when its events are precisely defined. However, it is much more difficult to evaluate probabilities of specific failures, and thus the reliabilities of systems components. In a classical approach to a fault tree analysis it is assummed that all these probabilities are precisely known. However, in many practical situation, especially in case of reliable components, the knowledge of probabilities of failures (or reliabilities) is hardly precise. Even if we use statistical data for the evaluation of those probabilities, we cannot be sure that these data have been obtained in exactly same conditions. Usually, we use data from reliability tests of similar objects conducted in similar conditions, but very often our data come from tests conducted in completely different conditions, e.g. from accelerated life tests. In all these cases there is a need to recalculate the results of reliability tests to the case of the considered system. Such recalculation very often needs opinions of experts, and these opinions are usually expressed in a natural language using vague and imprecise expressions. The formal description of this lack of precision is one of the most important practical problems of reliability analysis. Some researchers claim that the language of the probability theory is the only one that can be used for the description of uncertainty. However, there exist multitude counterexamples that indicate a necessity to apply other approaches. Moreover, the application of the theory of probability for the description of all imprecise information in the case of the reliability analysis of complex systems will make this analysis impossible to do due to an extremely high complexity of necessary computations. Therefore, the theory of fuzzy sets introduced by Lotfi A. Zadeh seems to be much better suited for this purpose.

In this paper we assume that the theory of fuzzy sets gives us tools appropriate for modeling and handling vague data such as imprecisely defined probabilities of failures. In the theory of fuzzy sets all objects of interest (events, numbers, etc.) have associated values of the so called *membership function* μ. The value of the membership function can be interpreted in different ways depending on the context. In the context of the evaluation of imprecise probabilities the value of the membership function $\mu(p)$ can be inerpreted a a *possibility* that the unknown probability adopts the value of p.

Let us now recall some basic notions of the theory of fuzzy sets that will be used in this paper. We start with the definition of a *fuzzy number.*

Definition 1. *The fuzzy subset A of the real line $\mathcal{R}$, with the membership function $\mu_A : \mathcal{R} \rightarrow [0, 1]$, is a fuzzy number iff*

(a) A is normal, i.e. there exists an element x_0 such that $\mu_A(x_0) = 1$;

(b) A is fuzzy convex, i.e. $\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2)$,

$\forall x_1, x_2 \in \mathcal{R}, \quad \forall \lambda \in [0, 1]$;

(c) μ_A is upper semicontinuous;

(d) suppA is bounded.

This definition is due to Dubois and Prade (see [13]). It is easily seen from this definition that if A is a fuzzy number then its membership function has the following general form:

$$\mu_A(x) = \begin{cases} 0 & \text{dla } x < a_1 \\ r_l(x) & \text{dla } a_1 \leq x < a_2 \\ 1 & \text{dla } a_2 \leq x \leq a_3 \\ r_u(x) & \text{dla } a_3 < x \leq a_4 \\ 0 & \text{dla } x > a_4, \end{cases} \tag{4}$$

where $a_1, a_2, a_3, a_4 \in \mathcal{R}$, $a_1 \le a_2 \le a_3 \le a_4$, $r_l : [a_1, a_2] \to [0, 1]$ is a nondecreasing upper semicontinuous and $r_u : [a_3, a_4] \to [0, 1]$ – is a nonincreasing upper semicontinuous function. Functions r_l and r_u are called sometimes the left and the right arms (or sides) of the fuzzy number, respectively.

By analogy to classical arithmetic we can add, subtract, multiply and divide fuzzy numbers (for more details we refer the reader to [13] or [35]). In a general case all these operations become rather complicated, especially if the sides of fuzzy numbers are not described by simple functions. Thus, only simple fuzzy numbers - e.g. with linear or piecewise linear sides - are preferred in practice. Such fuzzy numbers with simple membership functions have more natural interpretation. Therefore the most often used fuzzy numbers are *trapezoidal fuzzy numbers*, i.e. fuzzy numbers whose both sides are linear. Trapezoidal fuzzy numbers can be used for the representation of such expressions as, e.g., "more or less between 6 and 7", "approximately between 12 and 14", etc. Trapezoidal fuzzy numbers with $a_2 = a_3$ are called *triangular fuzzy numbers* and are often used for modeling such expressions as, e.g., "about 5", "more or less 8", etc. Triangular fuzzy numbers with only one side may be useful for the description of opinions like "just before 50" ($a_2 = a_3 = a_4$) or "just after 30" ($a_1 = a_2 = a_3$). If $a_1 = a_2$ and $a_3 = a_4$ then we get, so called, *rectangular fuzzy numbers* which may represent such expressions as, e.g., "between 20 and 25". It is easy to notice that rectangular fuzzy numbers are equivalent to well known *interval numbers.* In a special case of $a_1 = a_2 = a_3 = a_4 = a$ we get a crisp (non-fuzzy) number, i.e. a number which is no longer vague but represents precise value and can be identified with the proper real number a.

An useful tool for dealing with fuzzy numbers is the concept of α–cut or α–level set . The α–cut of a fuzzy number A is a nonfuzzy set defined as

$$A_\alpha = \{x \in R : \mu_A(x) \ge \alpha\}. \tag{5}$$

A family $\{A_\alpha : \alpha \in [0, 1]\}$ is a set representation of the fuzzy number A. Basing on the resolution identity introduced by L.Zadeh, we get:

$$\mu_A(x) = \sup\{\alpha I_{A_\alpha}(x) : \alpha \in [0, 1]\}, \tag{6}$$

where $I_{A_\alpha}(x)$ denotes the characteristic function of A_α. From Definition 1 we can see that every α–cut of a fuzzy number is a closed interval. Hence we have $A_\alpha = [A_\alpha^L, A_\alpha^U]$, where

$$\begin{aligned} A_\alpha^L &= \inf\{x \in \mathcal{R} : \mu_A(x) \ge \alpha\}, \\ A_\alpha^U &= \sup\{x \in \mathcal{R} : \mu_A(x) \ge \alpha\}. \end{aligned} \tag{7}$$

Hence, by (4) we get $A_\alpha^L = r_l^{-1}$, $A_\alpha^U = r_u^{-1}$.

In the analysis of fuzzy numbers and their functions we use the *extension principle* introduced by Zadeh [61], and described by Dubois and Prade [15] as follows:

Definition 2. *Let X be a Cartesian product of universes, $X = X_1 \times X_2 \times \ldots \times X_r$, and $A_1, \ldots, A_r$ be r fuzzy sets in $X_1, \ldots, X_r$, respectively. Let f be a mapping* *from* $X = X_1 \times X_2 \times \ldots \times X_r$ *to a universe* Y *such that* $y = f(x_1, x_2, \ldots, x_r)$. *The extension principle allows us to induce from* r *fuzzy sets* A_i *a fuzzy set* B *on* Y *through* f *such that*

$$\mu_B(y) = \sup_{x_1,\ldots,x_r:\ y=f(x_1,\ldots,x_r)} \min\left[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_r}(x_r)\right] \tag{8}$$

$$\mu_B(y) = 0,\ f^{-1}(y) = \emptyset \tag{9}$$

Using the extension principle we can calculate membership functions of fuzzy sets that are defined as functions of other fuzzy sets.

In their pioneering work Tanaka et al. [51] used the concept of fuzzy numbers for the description of imprecise probabilities in the context of fault tree analysis. They assumed that probabilities of events of a fault tree are described by the mentioned above trapezoidal fuzzy numbers. In such a case it is easy to show that the fuzzy probability of the failure (or fuzzy reliability) of a whole system is also a fuzzy number, but its membership function does not preserve trapezoidal shape. However, we can use the concept of α-cuts for relatively simple computations.

Let us assume that the reliabilities of systems components are described by fuzzy numbers defined by their α-cuts: $(q^{\alpha}_{i,L}, q^{\alpha}_{i,U})$, $i = 1, \ldots, n$. Then, the α-cut $(q^{\alpha}_{s,L}, q^{\alpha}_{s,U})$ for a coherent system can be calculated from (3) as follows:

$$q_{s,L} = h(q^{\alpha}_{1,L}, q^{\alpha}_{2,L}, \ldots, q^{\alpha}_{n,L}) \tag{10}$$

$$q_{s,U} = h(q^{\alpha}_{1,U}, q^{\alpha}_{2,U}, \ldots, q^{\alpha}_{n,U}) \tag{11}$$

This relatively simple way of calculations can be used only in the case of a known function $h(\star, \ldots, \star)$. Formal description of the general procedure for the calculation of fuzzy system reliability can be also found in Wu [58]. However, when the calculations have to be made using directly the information from a fault tree, the methodology proposed in [51] has some drawbacks as it cannot be used for the fault trees with repeated events, and fault trees that contain events and their complementary events at the same tree. These drawbacks have been resolved by Misra and Soman who in [34] proposed a more general methodology for dealing with multistate systems and vectors of dependent fuzzy probabilities.

The general methodology described above is valid for any fuzzy description of fuzzy reliabilities $\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_n$. However, for practical calculations it is recommended to select several values of α, and to calculate α-cuts of the fuzzy reliability of the system $\tilde{q}_s$ for these values of α. Then, the membership function of $\tilde{q}_s$ may be approximated by a piecewise linear function that connects the ends of consecutive α-cuts. More precise results can be obtained if for the description of imprecise probabilities we use the so called L-R fuzzy numbers introduced by Dubois and Prade [14]. For this case Singer [49] has presented recursive formulae that can be used for the calculation of the fuzzy reliability of a system.

Interesting application of fuzzy sets in the analysis of fault trees can be found in the paper by Lin and Wang [31], who considered the problem of elicitating fuzzy probabilities of events using imprecise linguistic assessments for human performance and vague events. Fuzzy measures of importance of the elements of a fault tree described by fuzzy probabilities were considered in the paper by Suresh et al. [50]. Practical example of the fault tree analysis with fuzzy failure rates can be found in the paper by Huang et al. [24].

The general approach presented in this section can be used for solving any well defined problem of reliability analysis with imprecisely defined parmeters. For example, Cheng [9] used fuzzy sets to describe reliability of repairable systems using a fuzzy GERT methodology. In all such cases the extension principle and the concept of α-cuts is quite sufficient for making necessary computations. However, if in these computations non-monotonic functions are involved, then it may be necessary to solve non-linear programming problems in order to arrive at required solutions.

3 Possibilistic approach to the evaluation of reliability

In the previous section we have described the results of research in the area of system reliability for the case of imprecise (linguistic) description of probabilities of failures (or probabilities of survival, i.e. reliabilities). In all these papers life times were assumed to have probabilistic nature, but their distribution were imprecisely defined, resulting with imprecise probabilities of failures. Imprecise values in these models were described by fuzzy sets, and this description was often interpreted in terms of the theory of possibility introduced by L.A.Zadeh [62].

Zadeh [62] introduced the notion of *possibility* for the description of vaguely defined events whose interpretation in terms of probabilities is at least questionable. He introduced the notion of the *possibility distribution*, and showed that it can be formally described by fuzzy sets. This theory was further developed by many authors in the framework of the theory of fuzzy sets, and in the late 1980s found its applications in the area of reliability. The distinctive feature of the theory of possibility is not the way it describes vaguely defined concepts, but how it is used for merging uncertainties of possibilistic nature. In this respect it is basically different from the theory of probability, as it is not additive, and is governed by fuzzy logic.

For the readers who are not familiar with fuzzy logic we recall now two its most important features. Suppose we have two fuzzy sets $\tilde{A}$ and $\tilde{B}$ described by the membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Then, the membership function of the logical sum (union) of $\tilde{A}$ and $\tilde{B}$ is given by

$$\mu_{A\bigcup B}(x) = max(\mu_A(x), \mu_B(x)), \tag{12}$$

and the the membership function of the logical product (intersection) of $\tilde{A}$ and $\tilde{B}$ is given by

$$\mu_{A\bigcap B}(x) = min(\mu_A(x), \mu_B(x)). \tag{13}$$

Thus, possibility measures are rather 'maxitive' in contrast to the 'additivity' of their probabilistic counterparts.

Possibilistic approach to reliability was introduced in works of Cai and his collaborators (for references see [4], [6], [5]) and Onisawa(see [41], [42]). Cai in his papers has given practical examples which let him conclude that in many cases life times have no probabilistic meaning but should be described by possibilistic (fuzzy) variables. The rationale behind that reasoning was the following: in many cases failures such as, e.g. software failures, cannot happen more than once. In such cases, Cai claims, probabilistic approach with its interpretation in terms of frequencies is not appropriate. Thus, times to such singular failures should be rather described by possibility distributions than by probability distributions. Introduction of possibilistic models of reliability from a purely mathematical point of view can be found in [8] and [12].

The agreement to possibilistic assumptions has many far reaching consequences for the analysis of system reliability. Let us define the system (or its component) life time X as a *fuzzy variable* [6]:

$$X = u : \mathcal{U}_X = u : \pi_X(u), \ u \in \mathcal{R}^+ = [0, +\infty), \tag{14}$$

where $\pi_X(u)$ is the possibility distribution of X. In such a case possibilistic reliability ('posbist' reliability in Cai's terminology) is defined as the possibility that for given conditions the system performs its assigned functions, and is calculated from the following formula [6]:

$$R(t) = \sigma(X > t) = \sup_{u>t} \mathcal{U}_X(u), \tag{15}$$

where σ is a possibility measure.

Now, let us present two important theorems (formal definitions of some concepts used in these theorems are given in [6]).

Theorem 1. (Cai et al. [6]) *Suppose a series system has two components. Let X_1, X_2 be the component lifetimes, respectively. Further we assume X_1, X_2 are both normed unrelated fuzzy variables, defined on $(\Gamma, \mathcal{G}, \sigma)$, with continuous possibility distribution functions and induce strictly convex fuzzy sets, $X_1 = u : \mathcal{U}_{X_1}(u), X_2 = u : \mathcal{U}_{X_2}(u)$. Let X be the system lifetime. Then there exists a unique pair (a_1, a_2), $a_1, a_2 \in \mathcal{R}^+$, such that the possibility distrbution function of X, denoted by $\mathcal{U}_X(x)$, is given by*

$$\mathcal{U}_X(x) = \begin{cases} \max(\mathcal{U}_{X_1}(u), X_2 = u : \mathcal{U}_{X_2}(u)) & \text{if } x \le a_1 \le a_2 \\ \mathcal{U}_{X_1}(u) & \text{if } a_1 < x \le a_2 \\ \min(\mathcal{U}_{X_1}(u), X_2 = u : \mathcal{U}_{X_2}(u)) & \text{if } a_1 \le a_2 < x \end{cases} \tag{16}$$

Theorem 2. (Cai et al. [6]) *Suppose a parallel system has two components. Let X_1, X_2 be the component lifetimes, respectively. Further we assume X_1, X_2 are both normed unrelated fuzzy variables, defined on $(\Gamma, \mathcal{G}, \sigma)$, with continuous possibility distribution functions and induce strictly convex fuzzy sets, $X_1 = u : \mathcal{U}_{X_1}(u), X_2 = u : \mathcal{U}_{X_2}(u)$. Let X be the system lifetime. Then there exists a unique pair (a_1, a_2), $a_1, a_2 \in \mathcal{R}^+$, such that the possibility distrbution function* *of X, denoted by $\mathcal{U}_X(x)$, is given by*

$$\mathcal{U}_X(x) = \begin{cases} \min(\mathcal{U}_{X_1}(u), X_2 = u : \mathcal{U}_{X_2}(u)) & \textit{if } x \leq a_1 \leq a_2 \\ \mathcal{U}_{X_2}(u) & \textit{if } a_1 < x \leq a_2 \\ \max(\mathcal{U}_{X_1}(u), X_2 = u : \mathcal{U}_{X_2}(u)) & \textit{if } a_1 \leq a_2 < x \end{cases} \quad (17)$$

Similar results have been also given in [6] for other reliability systems like a k-out-of-n system, and for the most general case of a binary coherent system.

The cosequences of both theorems (and their extensions) are somewhat strange. Cai et al. [6] already noticed: "the reliability of a parallel system with an arbitrary number of unrelated components coincides with the reliability of a series system with another arbitrary number of unrelated components, provided that all of the components contained in the systems are identical". This feature, in our opinion, indicates that the notion of the possibilistic reliability of systems should be used very cautiously.

In the possibilistic model described above it has been assumed that reliability states of the system and its components are binary. However, in many real cases, especially for large and complex systems, this assumption is not true. In the classical (probabilistic) theory of reliability the notion of 'multistate systems' is used in order to cope with this problem. Unfortunately, the existing reliability data is usually not sufficient for the proper identification of such systems. Moreover, for multistate components and systems it is usually very difficult to define precisely the failures, especially in the case of failures made by human (operator) errors. Therefore, some researchers proposed to use fuzzy sets for the description of vaguely defined failures.

The importance of the problem of vaguely defined failures was recognized for the first time in the papers by Nowakowski [39], Nishiwaki [38], Nishiwaki and Onisawa [44], and Onisawa [40], [42] devoted to the problem of reliability analysis of man-machine systems. Interesting approach to that problem, both from probabilistic and fuzzy point of view, was also proposed by Rotshtein [48]. Similar problems have been also noticed in the analysis of fault trees constructed for complex systems. Fault trees, or more general event trees, are used for the description of the relationships betwwen physical states of a system and its reliability states. In the classical case of binary systems this relationship is well defined, and described using logical gates AND, OR, and NOT. However, in many practical cases we do not have enough information to establish sure links between particular physical states of a system and its particular failures.

Different approaches have been used to model imprecise relationships between physical and reliability states of a system. Pan and Yun [45] proposed to use fuzzy gates with outputs described by triangular fuzzy numbers instead of crisp values 0 or 1. Another generalization of fault tree gates was proposed by Onisawa (see [42]) who considered parametric operations called Dombi t-norm and Dombi t-conorm instead of AND and OR operators, respectively. Full application of the theory of possiblity in the analysis of fault trees has been proposed by Nahman [37] and Huang et al. [25] who used possibility measures for the description of transition between states of a fault tree, and fuzzy logic for the description of its gates.

One of the most challenging problems of the reliability of complex systems is the multistate nature of their behaviour. Structure function describing the behaviour of systems composed of multistate elements could be extremely difficult to find and very often even impossible to be precisely identified. An attempt to describe such complex situation with the usage of fuzzy sets has been proposed by Montero et al. [36] and Cutello et al. [11].

Possibilistic approach to reliability has been also used for the analysis of repairable systems. Utkin and Gurov [52], [53] presented a mathematical model for the description of exploitation processes of systems using functional equations that describe transition processes between different states of a system. In a probability context these equations describe a stochastic process of the random behaviour of the system. However, the same equations can be used for that description in the possibilistic context. The resulting formulae look very awkwardly, but rather surprisingly they are easier to solve.

4 Statistical inference with imprecise reliability data

4.1 Fuzzy estimation of reliability characteristics

In the previous sections we have assumed that all probabilities, crisp or fuzzy, that are necessary for the computations of reliability are known. However, in practice they have to be estimated from statistical data. One of the most important problem of reliability analysis is the estimation of the *mean lifet ime* of the item under study (system or component). In technical applications this parameter is also called *mean time to failure* ($MTTF$) and is often included in a technical specification of a product. For example, producers are interested whether this time is sufficiently large, as large $MTTF$ allows them to extend a warranty time. Classical estimators require precise data obtained from strictly controlled reliability tests (for example, those performed by a producer at his laboratory). In such a case a failure should be precisely defined, and all tested items should be continuously monitored. However, in real situation these requirements might not be fulfilled. In the extreme case, the reliability data come from users whose reports are expressed in a vague way. The vagueness of the data has many different sources: it might be caused by subjective and imprecise perception of failures by a user, by imprecise records of reliability data, by imprecise records of the rate of usage, etc. The discussion concerning different sources of vagueness of reliability data can be found in Grzegorzewski and Hryniewicz [18]. Therefore we require different tools appropriate for modeling vague data and suitable statistical methodology to handle these data as well.

To cope with the formal description of data that are both random and imprecise (fuzzy) it is convenient to use the notion of a *fuzzy random variable*. It was introduced by Kwakernaak [30]. There exist also definitions of fuzzy random variables that have been proposed by other authors, for example by Kruse [27] or by Puri and Ralescu [47]. The definition, we present below, was proposed in [19], and is similar to those of Kwakernaak and Kruse (see [17]). Suppose that a random experiment is described as usual by a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, where

Ω is a set of all possible outcomes of the experiment, $\mathcal{A}$ is a $\sigma-$algebra of subsets of Ω (the set of all possible events) and P is a probability measure

Definition 3. *A mapping $X : \Omega \to \mathcal{FN}$ is called a fuzzy random variable if it satisfies the following properties:*

(a) $\{X_\alpha(\omega) : \alpha \in [0,1]\}$ *is a set representation of $X(\omega)$ for all $\omega \in \Omega$,*

(b) *for each $\alpha \in [0,1]$ both $X^L_\alpha = X^L_\alpha(\omega) = \inf X_\alpha(\omega)$ and $X^U_\alpha = X^U_\alpha(\omega) =$*

$= \sup X_\alpha(\omega)$, *are usual real-valued random variables on $(\Omega, \mathcal{A}, \mathcal{P})$.*

Thus a fuzzy random variable X is considered as a perception of an unknown usual random variable $V : \Omega \to \mathcal{R}$, called an *original* of X. Let $\mathcal{V}$ denote a set of all possible originals of X. If only vague data are available, it is of course impossible to show which of the possible originals is the true one. Therefore, we can define a fuzzy set on $\mathcal{V}$, with a membership function $\iota : \mathcal{V} \to [0,1]$ given as follows:

$$\iota(V) = \inf\{\mu_{X(\omega)}(V(\omega)) : \omega \in \Omega\}, \tag{18}$$

which corresponds to the grade of acceptability that a fixed random variable V is the original of the fuzzy random variable in question (see Kruse and Meyer [28]).

Similarly $n-$dimensional fuzzy random sample $X_1, \ldots, X_n$ may be considered as a fuzzy perception of the usual random sample $V_1, \ldots, V_n$ (where $V_1, \ldots, V_n$ are independent and identically distributed crisp random variables). A set $\mathcal{V}^n$ of all possible originals of that fuzzy random sample is, in fact, a fuzzy set with a membership function

$$\iota(V_1, \ldots, V_n) = \min_{i=1,\ldots,n} \inf\{\mu_{X_i(\omega)}(V_i(\omega)) : \omega \in \Omega\}. \tag{19}$$

Random variables are completely characterized by their probability distributions. However, in many practical cases we are interested only in some parameters of a probability distribution, such as expected value or standard deviation. Let $\theta = \theta(V)$ be a parameter of a random variable V. This parameter may be viewed as an image of a mapping $\Gamma : \mathcal{P} \to \mathcal{R}$, which assigns each random variable V having distribution $P_\theta \in \mathcal{P}$ the considered parameter θ, where $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ is a family of distributions. However, in case of fuzzy random variables we cannot observe parameter θ but only its vague image. Using this reasoning together with Zadeh's extension principle Kruse and Meyer [28] introduced the notion of a *fuzzy parameter of a fuzzy random variable* which may be considered as a *fuzzy perception* of the unknown parameter θ. It is defined as a fuzzy set with the following membership function:

$$\mu_{\Lambda(\theta)}(t) = \sup\{\iota(V) : V \in \mathcal{V}, \theta(V) = t\}, \quad t \in \mathcal{R}, \tag{20}$$

where $\iota(V)$ is given by (18). This notion is well defined because if our data are crisp, i.e. $X = V$, we get $\Lambda(\theta) = \theta$. Similarly, for a random sample of size n we get

$$\mu_{\Lambda(\theta)}(t) = \sup\left\{\iota(V_1,\ldots,V_n) : (V_1,\ldots,V_n) \in \mathcal{V}^n, \theta(V_1) = t\right\}, \quad t \in \mathcal{R}. \qquad (21)$$

One can easily obtain α−cuts of $\Lambda(\theta)$:

$$\begin{aligned}\Lambda_\alpha(\theta) = \{t \in \mathcal{R} : \exists (V_1,\ldots,V_n) \in \mathcal{V}^n, \theta(V_1) = t, \text{ such that} \\ V_i(\omega) \in (X_i(\omega))_\alpha \text{ for } \omega \in \Omega \text{ and for } i = 1,\ldots,n\}.\end{aligned} \qquad (22)$$

For more information we refer the reader to Kruse, Meyer [28].

First papers devoted to the analysis of fuzzy reliability data did not use explicitly the concept of a fuzzy random variable. Pioneering works in this field can be attributed to Viertl [54],[55], who found appropriate formulae for important reliability characteristics by fuzzifying formulae well known from classical statistics of reliability data. The results of those and other works have been presented in the paper by Viertl and Gurker [56], who considered such problems as estimation of the mean life-time, estimation of the reliability function, and estimation in the accelerated life testing (with a fuzzy acceleration factor). Original approach has been proposed in Hryniewicz [21] who did not model fuzzy time to failures, but fuzzy survival times. In his models only the right-hand side of the fuzzy numbers has been considered, but this approach let him consider in a one mathematical model such phenomena like censored life times and partial failures.

One of the first attempts to propose a comprehensive mathematical model of fuzzy life times as fuzzy random variables was given in Grzegorzewski and Hryniewicz [18]. Grzegorzewski and Hryniewicz considered the case of exponentially distributed fuzzy life time data, and proposed the methodology for point estimation, interval estimation, and statistical hypothesis testing for the fuzzy mean life time. These results have been further extended in [19] where they also considered the case of vague censoring times and vague failures. In the case of vague failures the number of failures observed during the life time test is also fuzzy. The methodology for the description of a fuzzy number of failures in the context of the life time estimation was considered in [16].

Let us now present a summary of the results given in [19]. To begin with, let us recall some basic results from a classical theory of the statistical analysis of life time data. The mean lifetime may be efficiently estimated by the sample average from the sample of the times to failure $W_1,\ldots,W_n$ of n tested items, i.e.

$$MTTF = \frac{W_1 + \cdots + W_n}{n} \qquad (23)$$

However, in the majority of practical cases the lifetimes of all tested items are not known, as the test is usually terminated before the failure of all items. It means that exact lifetimes are known for only a portion of the items under study, while remaining life times are known only to exceed certain values. This feature of lifetime data is called *censoring*. More formally, a fixed censoring time

$Z_i > 0$, $i = 1, \ldots, n$ is associated with each item. We observe W_i only if $W_i \leq Z_i$. Therefore our lifetime data consist of pairs $(T_1, Y_1), \ldots, (T_n, Y_n)$, where

$$T_i = \min\{W_i, Z_i\}, \tag{24}$$

$$Y_i = \begin{cases} 1, \text{ if } & W_i = T_i \\ 0, \text{ if } & W_i = Z_i. \end{cases} \tag{25}$$

There are many probability distributions that are used in the lifetime data analysis. In [19] the exponential distribution has been used for modeling the lifetime T. The probability density function in this case is given by

$$f(t) = \begin{cases} \frac{1}{\theta} e^{-\frac{t}{\theta}}, \text{ if } & t > 0 \\ 0, \quad\quad \text{ if } & t \leq 0. \end{cases} \tag{26}$$

where $\theta > 0$ is the mean lifetime. Let

$$T = \sum_{i=1}^{n} T_i = \sum_{i \in D} W_i + \sum_{i \in C} Z_i \tag{27}$$

be the total survival time (sometimes called a total time on test), where D and C denote the sets of items for whom exact life times and censoring times are observed, respectively. Moreover, let

$$r = \sum_{i=1}^{n} Y_i \tag{28}$$

be the number of observed failures. In the considered exponential model the statistic (r, T) is minimally sufficient statistic for θ and the maximum likelihood estimator of the mean lifetime θ is (assuming $r > 0$)

$$\widehat{\theta} = \frac{T}{r}. \tag{29}$$

Now suppose that the life times (times to failure) and censoring times may be imprecisely reported. In case of precisely known failures we assume that the values of the indicators $Y_1, Y_2, \ldots, Y_n$ defined above are either equal to 0 or equal to 1, i.e. in every case we know if the test has been terminated by censoring or as a result of failure. In order to describe the vagueness of life data we use the previously defined notion of a fuzzy number.

Now we consider fuzzy life times $\widetilde{T}_1, \ldots, \widetilde{T}_n$ described by their membership functions $\mu_1(t), \ldots, \mu_n(t) \in \mathcal{NFN}$. Thus applying the extension principle to (27) we get a fuzzy total survival lifetime $\widetilde{T}$ (which is also a fuzzy number)

$$\widetilde{T} = \sum_{i=1}^{n} \widetilde{T}_i, \tag{30}$$

with a membership function

$$\mu_{\widetilde{T}}(t) = \sup_{t_1, \ldots, t_n \in \mathcal{R}^+ :\, t_1 + \ldots + t_n = t} \{\mu_1(t_1) \wedge \ldots \wedge \mu_n(t_n)\}. \tag{31}$$

Using operations on α-cuts we may find a set representation of $\widetilde{T}$ given as follows

$$\begin{aligned}\widetilde{T}_\alpha &= (T_1)_\alpha + \ldots + (T_n)_\alpha = \\ &= \{t \in \mathcal{R}^+ : t = t_1 + \ldots + t_n, \text{ where } t_i \in (T_i)_\alpha,\ i = 1, \ldots, n\},\end{aligned} \tag{32}$$

where $\alpha \in (0, 1]$.

In the special case of trapezoidal fuzzy numbers that describe both life times and censoring times the total time on test calculated according to (30) is also trapezoidal.

If the number of observed failures r is known we can use the extension principle once more, and define a fuzzy estimator of the mean lifetime $\widehat{\Theta}$ in the presence of vague life times as

$$\widetilde{\theta} = \frac{\widetilde{T}}{r}. \tag{33}$$

Since $r \in \mathcal{N}$ we can easily find the following set representation of $\widehat{\Theta}$:

$$\widetilde{\theta}_\alpha = \left\{ t \in \mathcal{R}^+ : t = \frac{x}{r}, \text{ where } x \in \widetilde{T}_\alpha \right\}. \tag{34}$$

For more details and the discussion on fuzzy confidence intervals we refer the reader to [18].

However, in many practical situations the number of failures r cannot be precisely defined. Especially in case of non-critical failures the lifetime data may not be reported in a precise way. In order to take into account such non-critical failures Grzegorzewski and Hryniewicz [19] consider the state of each observed item at the time Z_i. Let G denote a set of all items which are functioning at their censoring times Z_i. Therefore we can assign to each item $i = 1, \ldots, n$ its degree of belongingness $g_i = \mu_G(i)$ to G, where $g_i \in [0, 1]$. When the item hasn't failed before the censoring time Z_i, i.e. it works perfectly at Z_i, we set $g_i = 1$. On the other hand, if a precisely defined failure has occured before or exactly at time moment Z_i, we set $g_i = 0$. If $g_i \in (0, 1)$ then the item under study neither works perfectly nor is completely failed. This situation we may consider as a *partial failure* of the considered item. Let us notice that in the described above case G can be considered as a fuzzy set with a finite support.

There are different ways to define the values of g_i depending upon considered applications. However, in the majority of practical situations we may describe partial failures linguistically using such notions as, e.g. "slightly possible", "highly possible", "nearly sure", etc. In such a case we may assign arbitrary weights $g_i \in (0, 1)$ to such imprecise expressions. Alternatively, one can consider a set D of faulty items and, in the simplest case, the degree of belongingness to D equals $d_i = \mu_D(i) = 1 - g_i$. Further on we'll call g_i and d_i as degrees of the up state and down state, respectively. Having observed the degrees of down states it is possible to count the number of failures with the degrees of down states exceeding certain rejection limit. Hence, we get a following (fuzzy) number of failures:

$$\widetilde{r}^{f}_{opt} = |D|_f\,, \tag{35}$$

where $|D|_f$ denotes fuzzy cardinality of fuzzy set D. We may also start from up states. Therefore

$$\tilde{r}^f_{pes} = n - |G|_f \,, \tag{36}$$

where $|G|_f$ denotes fuzzy cardinality of fuzzy set G. However, contrary to the crisp counting $|D|_f \neq n - |G|_f$. It is seen that such fuzzy number of observed failures is a finite fuzzy set. Moreover, if we assume that at least one crisp failure is observed it is also a normal fuzzy set.

Using the extension principle, we may define a fuzzy estimator of the mean lifet ime $\tilde{\tilde{\theta}}$ in the presence of fuzzy life times and vague number of failures. Namely, for crisp failure counting methods we get a following formula

$$\tilde{\tilde{\theta}} = \frac{\tilde{T}}{\tilde{r}}, \tag{37}$$

where $\tilde{T}$ is the fuzzy total survival time and $\tilde{r}$ denotes the number of vaguely defined failures. Actually (37) provides a family of estimators that depend on the choice of $\tilde{r}$. However, in the case of a fuzzy failure number we have

$$\tilde{\tilde{\theta}} = \frac{\tilde{T}}{conv(\tilde{r})}, \tag{38}$$

where $conv(\tilde{r})$ is the convex hull of the fuzzy set $\tilde{r}$ and is defined as follows

$$conv(\tilde{r}) = \inf\{A \in \mathcal{NFN} : \tilde{r} \subseteq A\}. \tag{39}$$

Since now the denominator of (38) is a fuzzy number, our estimator of the mean life time is a fuzzy numbers whose membership function can be calculated using the extension principle.

First look at the results presented above gives impression that even in the simplest case of the estimation of the mean life time for the exponential distribution the analysis of fuzzy data is not simple. It becomes much more complicated in the case of other life time distributions, such as the Weibull distribution, and in the case of such characteristics like the reliability function. Further complications will be encountered if we have to evaluate the reliability of a system using fuzzy data obtained for its components. In these and similar cases there is an urgent need to find approximate solutions that will be useful for practitioners. An example of such attempt can be found in the paper by Hryniewicz [23]. In this paper Hryniewicz considers the problem of the estimation of reliability of a coherent system $R_s(t)$ consisted of independent components having exponenentialy distributed life times when available observed life times for components are fuzzy. He assumes that observed life times (and censoring times) of inividual components are described by trapezoidal fuzzy numbers. Then, he finds the membership function for the probability of failure

$$P(t) = 1 - e^{-t/\theta}, \; t > 0. \tag{40}$$

The obtained formulae are too complicated for the further usage in the calculation of the reliability of complex systems. Therefore, Hryniewicz [23] proposes to approximate fuzzy total time on test by shadowed sets intoduced by Pedrycz [46] who proposes to approximate a fuzzy number by a set defined by four parameters: a_1, a_2, a_3, a_4. The interpretation of the shadowed set is the following: for values of the fuzzy number that are smaller than a_1 and greater than a_4 the value of the membership function is reduced to zero, in the interval (a_2, a_3) this value is elevated to 1, and in the remaining intervals, i.e. (a_1, a_2) and (a_3, a_4) the value of the membership function is not defined. It is easy to see that all arithmetic operations on so defined shadowed sets are simple operations on intervals, and their result is also a shadowed set. Thus, calculations of imprecise reliability of a system using (3) is quite straightforward.

4.2 Fuzzy Bayes estimation of reliability characteristics

In statistical analysis of reliability data the amount of information from life tests and field data is usually not sufficient for precise evaluation of reliability. Therefore, there is a need to merge existing information from different sources in order to obtain plausible results. Bayesian methods, such as Bayes estimators and Bayes statistical tests, provide a mathematical framework for processing information of a different kind. Thus, they are frequently used in the reliability analysis, especially in such fields as reliability and safety analysis of nuclear power plants and reliability evaluation of the products of an aerospace industry. There are two main sources of imprecise information in the Bayesian approach to reliability. First source is related to imprecise reliability data, and second is connected with imprecise formulation of prior information. First papers on the application of fuzzy methodology in the Bayesian analysis of reliability can be traced to the middle of 1980s. For example, Hryniewicz [20] used the concept of a fuzzy set to model the prior distribution of the failure risk in the Bayes estimation of reliability characteristics in the exponential model. He proposed a method for building a membership function using experts opinions. However, in his model the membership function is interpreted as a kind of an improper prior probability distribution. Thus he finally arrived at non-fuzzy Bayes point estimators. At the same time Viertl (see [56] and [57]) used fuzzy numbers in order to model imprecise life times in the context of Bayes estimators.

Despite the significant progress in the development of fuzzy Bayesian methodology important practical results in reliability applications have been published only recently. Wu [59] considered Bayes estimators of different reliability characteristics. For example, he found Bayes estimators of the survival probability (reliability) using the results of binomial sampling and Pascal sampling experiments. In the Binomial sampling experiment n items are tested, and the number of survivors x is recorded. Reparameterized beta distribution is then used for the description of the prior information about the estimated survival probability (reliability) q. The parameters of the prior distribution have the following interpretation: n_0 is a 'pseudo' sample size, and x_0 is a 'pseudo' number of survivors in an imaginary experiment whose results subsume our prior information about

q. Then, the Bayes point estimator of q is given by

$$\hat{q}_B = \frac{x + x_0}{n + n_0}. \tag{41}$$

Wu [59] considers now the situation when the parameter x_0 is known imprecisely, and is described by a fuzzy number. Straightforward application of the extension principle leads to formulae for the limits of α-cuts of $\hat{q}_B$:

$$\hat{q}^{\alpha}_{B,L} = \frac{x + x^{\alpha}_{0,L}}{n + n_0} \tag{42}$$

and

$$\hat{q}^{\alpha}_{B,U} = \frac{x + x^{\alpha}_{0,U}}{n + n_0}. \tag{43}$$

In the case of Pascal sampling the number of failures s is fixed, and the number of tested items N is a random variable. The parameters of the prior distribution of q have the same interpretation as in the case of Binomial sampling. Then, the Bayes point estimator of q is given by

$$\hat{q}_B = \frac{n + x_0 - s}{n + n_0}, \tag{44}$$

where n is the observed value of N. When x_0 is known imprecisely, and described by a fuzzy number, the limits of α-cuts of $\hat{q}_B$ are given by [59]:

$$\hat{q}^{\alpha}_{B,L} = \frac{n - s + x^{\alpha}_{0,L}}{n + n_0} \tag{45}$$

and

$$\hat{q}^{\alpha}_{B,U} = \frac{n - s + x^{\alpha}_{0,U}}{n + n_0}. \tag{46}$$

Wu [59] presents also Bayes estimators for the failure rate λ and reliability function $e^{-\lambda t}$ in the exponential model. In his paper Wu [59] also proposes an algorithm for the calculation of the membership value $\mu(q)$ of $\hat{q}$.

The Bayes estimator of λ has been independently investigated by Hryniewicz [22] who considered the case of the crisp number of observed failures d, the fuzzy total time on test $\tilde{T}$, and gamma prior distribution of λ reparameterized in such a way that one of its parameters (scale) had the interpretation either of the expected value of λ, denoted by E_λ, or its mode, denoted by D_λ . He also assumed that the shape parameter δ of the prior gamma distribution is known, but the values of E_λ (D_λ) are fuzzy. Now the fuzzy Bayes estimators of λ are given by the following formulae [22]:

$$\hat{\lambda}_E = \frac{d + \delta}{\tilde{T} + \delta/\tilde{E}_\lambda}, \tag{47}$$

and

$$\hat{\lambda}_D = \frac{d + \delta}{\tilde{T} + (\delta - 1)/\tilde{D}_\lambda}, \ \delta > 1. \tag{48}$$

The α-cuts for these estimators can be calculated straightforwardly using the extension principle.

In his recent paper Wu [60] considers the case of Bayes estimators for the reliability of series, parallel, and k-out-of-n reliability systems in the case of the available results or reliability tests conducted according to the binomial sampling scheme. By applying the Mellin transform he finds the posterior distribution for the system reliability, and then fuzzifies its expected value arriving at the fuzzy Bayes point estimators.

5 Conclusions

Evaluation of reliability of complex systems seems to be much more difficult than it appeared to be even twenty years ago. At that time probabilistic models developed by mathematicians and statisticians were offered with the aim to solve all important problems. However, reliability practitioners asked questions that could have not beeen successfuly answered using the probabilistic paradigm. The usage of fuzzy sets in the description of reliability of complex systems opened areas of research in that field. This work has not been completed yet. In this paper we have presented only some results that seem to be important both from a theoretical and practical points of view. We focused our attention on probabilistic–possibilistic models whose aim is to combine probabilistic uncertainty (risk) with possibilistic lack of precision (vagueness). We believe that this approach is the most promising for solving complex practical problems. It has to be stressed, however, that we have not presented all the applications of fuzzy sets to reliability. For example, we have not presented interesting applications of fuzzy sets for the strength–stress reliability analysis or for a more general problem the reliability analysis of structural systems. The readers are encouraged to look for the references to papers devoted to these problems in the papers by Jiang and Chen [26] and Liu et al. [32]. Another important problem that has not been considered in this paper is the construction of possibility measures from the information given by experts. Interesting practical example of the application of fuzzy 'IF-THEN' rules for the solution of this problem has been presented by Cizelj et al. [10]. To sum up the presentation of the application of fuzzy sets in reliability we have to conclude that the problem of the appropriate description and analysis of complex reliability systems is still far from being solved.

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