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On joint modelling of random uncertainty and fuzzy imprecision

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Chapter 1
On joint modelling of random uncertainty and fuzzy imprecision

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Abstract The paper deals with the problem of the mathematical description of uncertainties of different type. It has been demonstrated by many authors that the theory of probability is not always suitable for the description of uncertainty related to vagueness. We briefly present some of the most promising theories which have been recently proposed for coping with this problem. Then, we concentrate our attention on the application of fuzzy random variables which seem to be very useful for the joint modelling of random uncertainty and fuzzy imprecision, and for the statistical analysis of imprecise data. The application of the statistical methodology for fuzzy data, called fuzzy statistics, is illustrated with a practical example, typical for the problems of making decsions using small amount of available data.

1.1 Introduction

Coping with uncertainty is an important problem in many areas of science, but in systems analysis and decision sciences it becomes a really crucial one. In both these branches of science uncertainty is always present, as systems analysts and decision makers have never full information about past, current and future "states of the world". Thus, they do not have precise and fully reliable information about consequences of proposed by them actions. They can never present descriptions of processes of their interest, as - for example - physicists and astronomers could do. Therefore, they need to use a formal language that could be used for sufficiently precise description of uncertain events, actions, etc. For many decades it appeared to the majority of scientists that the theory of probability and mathematical statistics is the only methodology that should be used for the formal description of uncertainty. However, during last few decades many scientists working in such areas like psychology, economic sciences, quantum physics, artificial intelligence. etc. have

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raised questions about possible inadequacy of the classical (Kolmogorov) theory of probability when applied for solving their particular problems.

This situation is far from being unexpected. If we look at any good dictionary of e.g. English we can find that the word 'probable', which definitely describes uncertainty, has many synonyms and other related words. For example, such words like 'possible', 'plausible', and expressions like 'likely to be true', 'hopeful', 'to be expected'(and their antonyms) are used for the description of a state on uncertainty. One may expect, that they are used for expressing slightly different types of uncertainty, and are not fully exchangeable with the word 'probable'. The differences between their meanings raised doubts among philosophers and mathematicians about the role of the classical theory of probability as the sole mathematical language for the description of uncertainty.

Basic problems with the applicability of the classical theory of probability inspired mathematicians who proposed other, not necessarily equivalent, theories of probability. Some of them are described in a classical book of Fine [22]. Other doubts were raised by the founder of the theory of fuzzy sets L.A. Zadeh who claimed that the classical theory of probability cannot describe uncertainty realated to innate imprecision of many notions and ideas expressed in a plain human language. In his seminal paper [83] Zadeh proposed to use the formalism of fuzzy sets as the formal language of the theory of possibility. Another criticism of the classical probability came from economists and psychologists. The Nobel Prize winner in economics H.Simon in his book [67] noticed that people do not make their decisions according to the principle of expected utility which is based on the classical theory of probability. Another types of doubts were raised by other Nobel Prize winners Tversky and Kahneman who noticed in their works (see, for example, [73]) that probabilities evaluated by humans are not necessarily additive, as it is assummed in the classical theory of probability. An interesting description of theoretical and practical problems with the applicability of the classical theory of probability can be found in the paper by Bordley [5] who also noticed that this theory is in a certain sense incompatible with the quantum physics.

The existence of many extensions and modifications of the classical theory of probability creates problems for systems analysts and decision makers who are expected to model systems in presence of uncertainties of different types. In the second section of this paper we present a very brief description of several generalizations of classical probability. This has to be done in order to set borders between those areas where classical probability is still the best (and probably the only)mathematical model of uncertainty and the areas where its generalizations are needed. We claim that in the majority of practical cases the combination of classical probability and Zadeh's theory of possibility is sufficient for the description of complex systems and making decisions. We describe this methodology in the third section of the paper. An example of the apllication of fuzzy random models and fuzzy statistics is given in the fourth section of the paper. We show that in case of information that has both random and imprecise nature some additional indices, like possibility and necessity measures, are indispensable for a correct description of decision making problems.

1.2 Generalizations of classical probability and their applications in decision making

1.2.1 Measures of uncertainty and criteria of their evaluation

Close analysis of the human perception of uncertainty reveals that this concept does not have one, unanimously approved, interpretation. Zimmermann [84] notes that any definition of uncertainty has to be to some extent arbitrary and subjective. He proposes the following one:

Uncertainty implies that in a certain situation a person does not dispose about information which quantitatively and qualitatively is appropriate to describe, prescribe or predict deterministically and numerically a system, its behavior or other characteristica.

This definition is definitely technology-oriented. One may say that everything what prevents us to describe reality in a deterministic way may be considered as a facet of uncertainty. Thus, there exist many different causes of uncertainty. Zimmermann [84] lists the following: lack of information (quantitative or qualitative), abundance of information, conflicting evidence, ambiguity, measurement, and belief. It is not surprising that he does not believe that the general theory of uncertainty that is able to describe these completely different sources of uncertainty exists, and appropriate mathematical models should be context-dependent. They could be formulated either as different generalizations of probability, or may be formulated in another way, such as Pawlak's rough set theory (see, e.g. [59]) or convex modeling proposed by Ben-Haim and Elishakoff [3].

If we look at different theories of uncertainty we can notice that they can be divided into two general groups: those based on methods of mathematical logics (such as the rough sets theory) or those based on the theory of probability and its generalizations. In this paper we restrict our interest only to the second one. This restriction arises from a fact that in the majority of practical cases the classical theory of probability is sufficient for the mathematical description of uncertainty. This popularity of classical probabilistic models of uncertainty makes many specialists to believe that classical probability is the only mathematical theory that is sufficient for the formal description of uncertainty. Insufficiency of this approach was noticed only recently, mainly by specialists in decision-making or expert systems. Peter Walley, who is the one of the most prominent persons representing that group of scientists, presents the list of pertinent mathematical models, in order of their generality [77] (in parentheses, there are given the most important, according to Walley, references, and indicated by him typical areas of application).

- Possibility measures and necessity measures ([17],[83], vague judgements of uncertainty in natural language).
- Belief functions and plausibility functions([11],[66], multivalued mappings and non-specific information).
- Choquet capacities of order 2 ([8], [12], [42], some types of statistical neghbourhood in robustness studies, and various economic applications).

- Coherent upper and lower probabilities ([42], [52], [68], personal betting rates, and upper and lower bounds for probabilities)
- Coherent upper and lower previsions ([75], [76], [81], buying and selling prices for gambles, upper and lower bounds for expectations, and envelopes of expert opinions).
- Sets of probability measures ([4], [34], [54], partial information about an unknown probability measure, and robust statistical models).
- Sets of desirable gambles([75], [80], [82], preference judgements in decision making)
- Partial preference orderings ([32], [75], preference judgements in decision making).

Walley [77] also notices

- partial comparative probability orderings ([23], [43], [45], qualitative judgements of uncertainty).

In order to evaluate and compare all competing theories of uncertainty (including classical probability) Peter Walley [76] proposes to take into account the following criteria:

a) interpretation,
b) imprecision,
c) calculus,
d) consistency,
e) assessment,
f) computation.

The proposed measure of uncertainty has to be sufficiently easy to understand by its users. For example, conclusions inferred from the application of the theory should be clear enough to be useful for making decisions. It should be able to model partial or complete ignorance, reflected, for example, in imprecision of statements of natural language. There should be rules for merging uncertainties, updating, and using them in inferential processes. There should be methods for the evaluation of coherence off all assessments formulated using the theory and its assumptions. A useful theory of uncertainty should provide guidances how to make assessments about uncertain events and handle imprecise judgements of different types. Finally, it should be computationally feasible. More comprehensive interpretation of the criteria presented above and their practical and theoretical importance can be found in [76].

All these requirements may have different importance in different applications, and none of the existing theories and measures of uncertainty fullfils them sufficiently well. For example, the classical theory of probability does not meet sufficiently well criteria b) and e), and for this reason philosophers, mathematicians, economists, psychologists, and specialists in expert systems have been making a lot of efforts in order to introduce more general, and more useful in specific applications, theories of uncertainty. In the following subsections of this section we present very brief description of some of these theories. This presentation is needed, in our opinion, for the understanding of limits which still exist if we try to cope with uncertainty inherent in the analysis of complex systems.

1.2.2 Probability

The theory of probability is the best known, and the most frequently used in practice, theory of uncertainty. Despite its four hundred years lasting history its fundamentals are still subject to different interpretations and controverses. In general, there exist two different interpretations of the classical probability: an objective 'frequentist' interpretation based on the analysis of empirical observations of series of events, and subjective 'Bayesian' approach, based of subjective assessments of probabilities of events. It is interesting that even in the 1960s the second approach was dismissed as 'non-scientific' by the majority of statisticians, and not present in nearly all popular textbooks. On the other hand, the supporters of the Bayesian approach presented in books of Savage [64] and de Finetti [24], [25] pointed out apparent incoherences inherent for the frequntist approach (see, for example, an excellent monograph by Lindley [55]).

The basics of the theory of probability are well described in all textbooks on probability and statistics. Therefore, there is no need to present them in details in this paper. However, we are going to point out those assumptions of this theory which are criticised by some authors who see them as main reasons of discrepancies between theory and practice of coping with uncertainties.

According to Kolmogorov's theory of probability there exists a sample space Ω consisting of *disjoint* elements, called elementary states, or simply states. These states need not be necessarily observable. Then, Kolmogorov postulates a Borel-field set (an algebra) B consisting of some, but not necessarily all, subsets of Ω. The elements of B are called events and represent observable outcomes of actual or hypothetical experiments. Probabilities are assigned only to elements of B, so they are not asigned to those states of Ω that do not belong to B. The consequence of this assumptions is far-reaching. It means that every event can be *precisely* described using the elements of Ω. Another consequence refers to the feature which Walley [76] calls *'Bayes dogma of precision'*. When probabilities are assessed by frequencies of observed precisely defined events there are no fundamental problems with the accuracy of their evaluation. However, when they are assessed subjectively (and we have to remember that according to the followers of the Bayesian probability and statistics it is the only *coherent* way of doing this) it is assumed that they may be interpreted as precisely defined fair betting rates. The behavioral interpretation of probabilities in terms of fair betting rates was originally introduced by de Finetti (see [25]) who has shown that betting in favor of an event A against its complement A^C will not lead to sure loss only if betting odds are $P(A)$ to $1 - P(A)$, where $P(A)$ is equal to the probability of event A. Moreover, from the postulate of fair betting rates and some additional coherence requirements one can derive that probability is nonnegative, normalised, and finitely-additive set function. In addition, Savage [64] proved that the theory of subjective probabilities constitutes tha basis for an axiomatized and coherent theory of decision-making. Despite of all these unquestionable advantages, empirical observations show however, that in presence of partial or full ignorance about events of interest such precise assessments of probability cannot be made. Moreover, the actual behaviour of decision-makers differs from that prescribed by the theory based on classical precise probabilities. This leads us (and many other researches working in the area of decision sciences) to the conclusion that in case of imprecisely defined states and events it is not possible to obtain precise values of their probabilities, and hence to make precise prescriptions in decision-making processes. A counterargument to this opinion presented by rather dogmatic followers of the classical Bayesian approach to probability is the following: their theory shall be considered as the normative one, and all the differences between the theory and the actual human behaviour are always due to human weakness and shall be overcome by using more precise measurements and precise problem formulation. In many practical cases this is definitely true. However, even in principle this standpoint can be questioned using the results from quantum physics. Bordley [5] shows that as the consequence of the Heisenberg Uncertainty Principle some events cannot be precisely observed, and in such a case precise probability statements are simply impossible. Therefore, generalizations of classical probability are necessary if we want to deal with imprecisely defined events and with partial information about probabilities of their occurence.

1.2.3 Dempster-Shafer theory of evidence and possibility theory

The notion of *'possibility'* attracted attention of philosophers, economists, logicians etc. Dubois and Prade [19] notice that first attempts to formalize the concept of possibility were made in the late 1940s by the economist Shackle [65] who proposed a calculus of "potential surprise" as the base for decision-making. The works of many authors, who have noticed the defficiency of the theory of probability in dealing with many practical problems have led to a more or less independent formalizations of two similar theories of uncertainty: Dempster-Shafer theory of evidence and possibility theory.

The concept of possibility can be understood either as an objective notion or as an epistemic and subjective one. For example, Zadeh [83] understands possibility as objective feasibility; an objective measure of physical easiness to achieve a certain goal. By his famous example of a possibilistic statement, 'it is possible for Hans to eat six eggs for breakfast', he shows an exemplary information which is difficult, or even hardly possible, to be formalized using theory of probability. This type of interpretation of possibility is closely related to the idea of preference. Alternatives that are more easily achieved (more feasible) are usually more preferred. This relation has been described in details in the paper by Dubois, Fargier, and Prade [13]. Second interpretation of possibility is an epistemic one, and is given in terms of plausibility.

An event is fully plausible when its occurence does not create any surprise. This type of interpretation has subjectivistic nature and means that possibility may represent consistency of the observed event with available knowledge. Possibility, understood as plausibility of an event, may also have an objectivistic interpretation, and can be evaluated from the observations of upper bounds of frequency of its occurence [16]. There also exists a deontologic interpretation of possibility (something is possible when it is allowed by law), but satisfactory formal description of this type of possibility has not been proposed yet.

The basic notion of the possibility theory is a *possibility distribution function* $\pi_X(\omega)$, defined on the possibility space Ω (a frame of discernment). The value of $0 \leq \pi_X(\omega) \leq 1$ represents the measure of possibility of the element ω of the set Ω. It is usually assumed that $sup\{\pi_X(\omega) : \omega \in \Omega\} = 1$. A possibility measure of a subset A of Ω is defined as $\pi(A) = sup\{\pi_X(\omega) : \omega \in A\}$. There exist many versions (extensions) of the possibility theory, but in all of them the axiom of finite additivity, characteristic for the probability theory, has been replaced by the axiom of *maxitivity*. Let A and B be two events, and $\Pi(A)$ and $\Pi(B)$ be, respectively, their possibilities. Then,

$$\Pi(A \cup B) = max(\Pi(A), \Pi(B)). \tag{1.1}$$

In his seminal paper [83] Lotfi A. Zadeh proposed to use the formalism of the fuzzy sets theory as the mathematical formalism of the possibility theory. According to this proposal the possibility distribution function that assigns measures of possibility to elements of a certain set (or equivalently, to values of a certain numerical variable) may be interpreted as the membership function assigned to that set. This interpretation allows to use a well developed formal mechanism of the fuzzy sets theory in many different applications. The book by Dubois and Prade [17] describes the links between these two theories, and presents methods for the calculation of numerical values of possibility and necessity measures, both typical for the possibility theory. Moreover, Dubois and Prade [17] pointed out possible links between possibility and probability. The mutual relation between these two major theories of uncertaintity have been later explained and clarified using the Dempster-Shafer theory of evidence. The recent results published in papers of Walley and de Cooman [78], and de Cooman [9] show that measures of possibility are the special case of imprecise probabilities, and thus have a well defined behavioural interpretation.

The original motivation for the development of the possibility theory was to describe imprecise notions or imprecise pieces of information given as statements of a natural language such as, e.g. 'costs are high', 'time to failure is about 5 hours', etc. As a matter of fact, the founders of the possibility theory saw this theory as fundamentaly different from the probability theory. They considered the possibility theory as the formalism for the description of uncertain events or uncertain (partial) information in cases where the probability theory failed to provide satisfactory description. On the other hand, the theory of evidence (also known as the theory of belief functions) proposed originaly by Dempster [11] and further developed by Glenn Shafer [66] aimed at the generalization of the probability theory for dealing with such problems. The basic assumptions of the Dempster-Shafer theory of evidence look similar to the basic assumptions of the theory of probability. It is assumed that there exists a certain possibility space Ω, but probability measures, called in this theory *'probability mass assingments'*, in contrast to the probability theory, are defined on its whole power set 2^{Ω}. This allows to assign probability to an event formed by combinations of the elements of the possibility space Ω who in the case of this particular event are indistingushable, as this is typical for imprecisely described notions.

In the Dempster-Shafer theory of evidence uncertainty is measured using belief functions. A belief function *Bel*, defined on all subsets of the possibility space Ω, is written in the form

$$Bel(A) = \sum_{B \subseteq A} m(B), \tag{1.2}$$

where m is a probability mass assingment function on all subsets of Ω, such that $m(\emptyset) = 0$, $m(B) \geq 0$ for all $B \subseteq \Omega$ and $\sum_{B \subseteq \Omega} m(B) = 1$. In the Dempster-Shafer theory of evidence there exists also a notion of plausibility which is conjugate to the notion of belief. The conjugate funtion to the belief function is called the plausibility function *Pl*, and is defined by

$$Pl(A) = 1 - Bel(A^C) = \sum_{B \cap A \neq \emptyset} m(B). \tag{1.3}$$

It has been shown that there exists a close relationship betweeen the Dempster-Shafer theory of evidence and the possibility theory. When all elements of the set Ω form a nested set then there exists a direct relationship between the mass probability asingments of the Dempster-Shafer theory and the possibility distribution defined as a membership function of a certain fuzzy set (see [18] for more general results). Thus, the possibility theory (for one of its possible interpretations) may be regarded as a special case of the more general Dempster-Shafer theory of evidence. This relationship was used by several authors, who proposed methods for making probability - possibility transformations. As probability and possibility are not the same measures of uncertainty, unique one-to-one transformation does not exist. Thus, any transformation of this type can lead to loss of information (especially from less precise possibility to more precise probability), and transformation methods proposed by some authors differ in the methodology used to decrease that loss. One of these methods, proposed by Klir [44], [27] is based on the principle of information invariance. The other approach, based on the optimization of information content, has been proposed by Dubois et al. [20]. More information on the problem of probability - possibility transformation can be also found in [19].

Despite their very close relations, the possibility theory and the Dempster-Shafer theory of evidence are used in different areas of application. The Dempster-Shafer theory is mainly used in building computer expert systems or, in a more general setting, in computerized decision support systems. It is not used, however, in data analysis and in those instances of decision-making processes where statistical data (precise or imprecise) have to be merged with subjective information. In this particular domain of application the possibility theory is used in practice much more frequently. We will discuss those applications of the possibility theory in the next section of this paper.

1.2.4 Imprecise probabilities and their generalizations

Possibility theory and its main tool, the possibility distribution, have been found very useful for the formal description of imprecise information. However, claims - expressed, for example, by Zadeh - that it is also useful for the formalization of imprecise descriptions of probabilities (in statements like 'event A is much more probable than event B ') have been questionned by Peter Walley [75]. In one of his nice toy examples Walley considers the following information about possible outcomes (win(W), draw (D) or loss(L) of a football game [75], [76]:

a) probably *not* W,
b) W is more probable than D,
c) D is more probable than L.

Walley convincingly explains that information of this type cannot be used, without making some arbitrary assumptions, for the precise evaluation of the probability of e.g. the win $P(W)$. The only consistent evaluation can be done in terms of imprecise probabilities: lower probability $\underline{P}(W)$ and upper probability $\bar{P}(W)$.

Imprecise probabilities have been introduced independently by many authors under different names, such as *interval probabilities* or *non-additive probabilities*. The most comprehensive theory of imprecise probabilities, defined as lower and upper probabilities, was proposed by Peter Walley [75] who introduced the notion of coherent lower (upper) probability. Coherent lower probability may be interpreted as a lower envelope of a set of probability measures that fulfils certain coherence requirements. The similar interpretation exists for the upper probability. The behavioural interpretation of lower probabilities was proposed by Walley [75]. This interpretation is based on the generalization of a similar interpretation of subjective probabilities introduced by de Finetti. According to Walley (see also [76]) the lower (or upper) probability of an event A can be interpreted by specifying acceptable betting rates for betting on (or against) A. If the betting odds on A are x to $1 - x$ one will bet *on* A if $x \leq \underline{P}(A)$ and *against* A if $x \geq \bar{P}(A)$. The choice is not determined if x is between $\underline{P}(A)$ and $\bar{P}(A)$. The basic properties of lower and upper probabilities can be summarized as follows:

a) $\underline{P}(\emptyset) = \bar{P}(\emptyset) = 0$,
b) $\underline{P}(\Omega) = \bar{P}(\Omega) = 1$,
c) $\bar{P}(A) = 1 - \underline{P}(A^C)$,
d) $0 \leq \underline{P}(A) \leq \bar{P}(A) \leq 1$,
e) $\underline{P}(A) = \underline{P}(B) \leq \underline{P}(A \cup B) \leq \underline{P}(A) + \bar{P}(B) \leq \bar{P}(A \cup B) \leq \bar{P}(A) + \bar{P}(B,)$,
for disjoint events A and B

f) $\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B)$, for all events A and B.

It is worth to note, that precise probabilities are the special case of lower and upper probabilities such that $\underline{P}(A) = \bar{P}(A)$. Moreover, posibility and necessity measures of the possibility theory and belief and plausibility measures of the Dempster-Shafer theory of evidence are lower and upper probabilities. On the other hand, as it was noted by Walley (see e.g. [76] and [77]) not all lower and upper probabilities can be interpreted as possibility measures or measures of the Dempster-Shafer theory.

In certain problems of decision-making lower and upper probabilities are not sufficient for dealing with imprecise information. Their further generalization was proposed by Walley (see [75], [77]) in his theory of lower and upper previsions. Below, we present the definition of these measures of uncertainty, as it was given in [77].

Definition 1 (Walley [77]). A bounded mapping from Ω to $\mathbb{R}$ (the real numbers) is called a *gamble*. Let $\mathcal{K}$ be a nonempty set of gambles. A mapping $\underline{P} : \mathcal{K} \to \mathbb{R}$ is called a *lower prevision* or *lower expectation*. A lower prevision is said to be *coherent* when it is the lowest envelope of some set of linear expectations, i.e. when there is a nonempty set of probability measures, $\mathcal{M}$, such that $\underline{P}(X) = min\{E_P(X) : P \in \mathcal{M}\}$ for all $X \in \mathcal{K}$, where $E_P(X)$ denotes the expectation of X with respect to P. The conjugate upper prevision is determined by $\bar{P}(X) = -\underline{P}(-X) = max\{E_P(X) : P \in \mathcal{M}\}$.

The lower (upper) previsions seem to be general enough for the description of subjectively perceived uncertainty. Classical (Bayesian) probabilities, measures of possibility, and Dempster-Shafer measures of evidence can be interpreted as special cases of lower (upper) previsions. From a theoretical point of view this theory is sufficiently well developed. However, there exist some basic problems which require further investigations. For example, the problem of updating the values of imprecise probabilities when new pieces of information are available (i.e. the problem of conditioning) still needs some investigations, as a single generalization of the Bayes updating rule has not been proposed yet. The existing problems with updating procedures are related, for example, to the problems of dealing with observations whose prior probabilities are equal to zero. Other problems arise in relation to concepts of independence or conditional independence. There exist also problems with modelling (in terms of uncertainty) the concepts of preference and weak preference (for example, lower previsions cannot distinguish preference from weak preference). All these problems (for more information, see [77]) motivate researchers to look for more general mathematical models of uncertainty. Some of these models have been indicated in the first subsection of this section. It is interesting that the need to develop more general models of uncertainty has been also recognized in the community of classical Bayesians. The concept of robust Bayesian inference (see, e.g. the paper by Berger [4]) seems to be closely related to the problems presented in this paper. Moreover, some concepts of frequency-based robust statistics (the notion of ϵ-contamination) can be interpreted using the language of the theory of imprecise probabilities and its generalizations.

1.3 Fuzzy random variables and fuzzy statistics

The brief description of the existing main theories of uncertainty presented in the preceding section shows that neither of them is fully sufficient to cope with real problems where statistical data are both random and imprecise. The attempts to propose such a theory resulted with the introduction of the notion of a fuzzy random variable. This notion has been defined by many authors. Historicaly the first widely accepted definition was proposed by Kwakernaak [50], [51]. Kruse [48] proposed an interpretation of this notion, and according to this interpretation a fuzzy random variable $\widetilde{Z}$ may be considered as a perception of an unknown usual random variable $Z : \Omega \rightarrow \mathcal{R}$, called an original of $\widetilde{Z}$. Below, we present another, slightly modified, version of this definition presented in Grzegorzewski [35].

Let $\widetilde{X}$ be a fuzzy number, i.e. $\widetilde{X}$ is a normal, fuzzy convex and bounded fuzzy subset of the real line $\mathcal{R}$ with an upper semicontinuous membership function $\mu_X : \mathcal{R} \rightarrow [0,1]$ (see, e.g., Dubois and Prade [14]). A space of all fuzzy numbers will be denoted by $\mathcal{FN}(\mathcal{R})$. We have $\mathcal{FN}(\mathcal{R}) \subset \mathcal{F}(\mathcal{R})$, where $\mathcal{F}(\mathcal{R})$ denotes a space of all fuzzy sets on the real line. Fuzzy numbers are completely defined by their $\alpha-$cuts. The $\alpha-$cut, $\alpha \in (0,1]$, of a fuzzy number $\widetilde{X}$ with its membership function μ_X is a closed crisp set defined as

$$X_\alpha = \{t \in \mathcal{R} : \mu_X(t) \geq \alpha\}. \tag{1.4}$$

In order to describe α-cuts let us use the following notation: $X_\alpha = [X_\alpha^L, X_\alpha^U]$, where

$$X_\alpha^L = \inf\{t \in \mathcal{R} : \mu_X(t) \geq \alpha\}, \tag{1.5}$$
$$X_\alpha^U = \sup\{t \in \mathcal{R} : \mu_X(t) \geq \alpha\}. \tag{1.6}$$

Definition 2 (Grzegorzewski [35]). Let $(\Omega, \mathcal{A}, P)$ be a probability space, where Ω is a set of all possible outcomes of the random experiment, $\mathcal{A}$ is a σ-algebra of subsets of Ω (the set of all possible events), and P is a probability measure.

A mapping $\widetilde{X} : \Omega \rightarrow \mathcal{FN}(\mathcal{R})$, where $\mathcal{FN}(\mathcal{R})$ is the space of all fuzzy numbers, is called a fuzzy random variable if it satisfies the following properties:

1. $\{X_\alpha(\omega) : \alpha \in [0,1]\}$ is a set representation of $\widetilde{X}(\omega)$ for all $\omega \in \Omega$,
2. for each $\alpha \in [0,1]$ both $X_\alpha^L = X_\alpha^L(\omega) = \inf X_\alpha(\omega)$ and $X_\alpha^U = X_\alpha^U(\omega) = \sup X_\alpha(\omega)$, are usual real-valued random variables on $(\Omega, \mathcal{A}, P)$.

There exists also another popular definition of a fuzzy random variable proposed by Puri and Ralescu [60] and based on the notion of set-valued mapping and random sets. Below, we present this definition in a form given in [30].

Definition 3 (Gil et al. [30]). Let $\mathcal{FN}(\mathcal{R})$ be the space of all fuzzy numbers. Given a probability space $(\Omega, \mathcal{A}, P)$, a mapping $\widetilde{X} : \Omega \rightarrow \mathcal{FN}(\mathbb{R}^p)$ is said to be a *fuzzy random variable* (also called *fuzzy random set*) if for all $\alpha \in [0,1]$ the set-valued mappings $X_\alpha : \Omega \rightarrow \mathcal{K}(\mathbb{R}^p)$, where $\mathcal{K}$ is the class of the non-empty subsets of $\mathbb{R}^p$, defined so that for all $\omega \in \Omega$ $X_\alpha(\omega) = (X(\omega))_\alpha$, are random sets (that is, Borel-measurable mapping with the Borel σ-field generated by the topology associated with the Haussdorf metric on $\mathcal{K}(\mathbb{R}^p)$).

Fuzzy random variables may be used to model random and imprecise measurements. First statistical methods for the analysis of such imprecise fuzzy data were developed in the 1980s. Kruse and Meyer [49] proposed a general methodology for dealing with fuzzy random data. In case of their methodology fuzzy random data are described by fuzzy random variables defined according to Definition 2. This assumption has very important practical consequences. First of all it means that there exists an underlying non-fuzzy probability distribution that governs the origins of the observed imprecise fuzzy data. The parameters of this distribution have non-fuzzy values, but because of the fuzziness of observed data they cannot be precisely estimated. Their fuzziness comes directly from the fuzziness of statistical data and disappears when statistical data are precise. Therefore, fuzzy statistical methods developed according to the methodology proposed by Kruse and Meyer shall be regarded as straightforward generalization of classical (non-fuzzy) statistical methods. Using the aforementioned methodology Kruse and Meyer [49] proposed methods for the construction of estimators and confidence intervals for the parameters of the probability distributions of fuzzy random variables. According to the methodology proposed in [49] the estimators of the parameters of the probability distributions of fuzzy random variables are fuzzy. The same methodology may be applied to the estimators of the limits of confidence intervals that are also represented by fuzzy numbers. The most important practical consequence of the adoption of the Kruse and Meyer's methodology is that all relevant formulae for fuzzy estimators and other fuzzy statistics can be obtained by fuzzification of well known formulae of classical non-fuzzy statistics.

Despite the fact that the generalization of well known statistical methods to the fuzzy case is relatively straightforward, the construction of fuzzy statistical tests and making statistical decisions is far from being trivial. Fuzzy statistical tests may be developed for testing both non-fuzzy (precise) and fuzzy (imprecise) statistical hypotheses, and for fuzzy (imprecise) and non-fuzzy (precise) statistical data. For example, statistical methods for testing fuzzy hypotheses have been considered in the papers by Saade and Schwarzlander [63], Saade [62], Watanabe and Imaizumi [79], Arnold [2], Taheri and Behboodian [70], and Grzegorzewski and Hryniewicz [37]. When the data are also fuzzy interesting solutions have been proposed in the papers by Arnold [1], Casals et al. [6], Kruse and Meyer [49], Saade [62], Saade and Schwarzlander [63], Son et al. [69], Watanabe and Imaizumi [79], Römer and Kandel [61], and Montenegro et al. [57]. Grzegorzewski [35] has proposed a unified approach for testing statistical hypotheses with vague data which is a direct generalisation of the classical approach. Below, we present his definition of the fuzzy statistical test.

Let $\widetilde{Z}_1, \ldots, \widetilde{Z}_n$ denote a fuzzy sample, i.e. a fuzzy perception of the usual random sample $Z_1, \ldots, Z_n$ from the population described by the probability distribution P_Θ, and let δ be a given number from the interval $(0, 1)$. Grzegorzewski [35] has defined a fuzzy test as follows:

Definition 4 (Grzegorzewski [35]). A function $\varphi : (\mathcal{FN}(\mathcal{R}))^n \rightarrow \mathcal{F}(\{0,1\})$, where $\mathcal{F}(\{0,1\})$ is the set of possible decisions, is called a fuzzy test for the hypothesis H, on the significance level δ, if

$$\sup_{\alpha \in [0,1]} P\left\{\omega \in \Omega : \varphi_\alpha\left(\widetilde{Z}_1(\omega), \ldots, \widetilde{Z}_n(\omega)\right) \subseteq \{0\} | H\right\} \leq \delta$$

where φ_α is the α-level set (α-cut) of $\varphi\left(\widetilde{Z}_1, \ldots, \widetilde{Z}_n\right)$.

When we test statistical hypotheses about the values of the parameters of probability distributions we utilize a well known equivalence between the set of values of the considered probability distribution parameter for which the null hypothesis is accepted and a certain confidence interval for this parameter. The same equivalence exists in the case of statistical tests with fuzzy data.

When statistical data are precise (crisp), for testing the null hypothesis $H : \theta \leq \theta_0$, and the alternative hypothesis $K : \theta > \theta_0$ we use a one-to-one correspondence between the acceptance region for this test on the significance level δ and the one-sided confidence interval $[\underline{\pi}_l, +\infty)$ for the parameter θ on the confidence level $1-\delta$, where $\underline{\pi}_l = \underline{\pi}_l(Z_1, \ldots, Z_n; \delta)$. This equvalence was utilized by Kruse and Meyer [49] who introduced the notion of a fuzzy confidence interval for the unknown parameter θ, when the data are fuzzy. In the considered case, a fuzzy equivalent of $[\underline{\pi}_l, +\infty)$ can be defined by the following α-cuts (for all $\alpha \in (0,1]$):

$$\begin{aligned} \underline{\Pi}_\alpha^L &= \underline{\Pi}_\alpha^L\left(\widetilde{Z}_1, \ldots, \widetilde{Z}_n; \delta\right) \\ &= \inf\left\{t \in \mathcal{R} : \forall i \in \{1, \ldots n\} \, \exists z_i \in \left(\widetilde{Z}_i\right)_\alpha \right. \\ &\quad \text{such that } \underline{\pi}_l(z_1, \ldots, z_n; \delta) \leq t\} \end{aligned} \tag{1.7}$$

Similarily, we can define a fuzzy equivalent of the one-sided confidence interval $(-\infty, \overline{\pi}_u]$, as given in Grzegorzewski [35]:

$$\begin{aligned} \overline{\Pi}_\alpha^U &= \overline{\Pi}_\alpha^U\left(\widetilde{Z}_1, \ldots, \widetilde{Z}_n; \delta\right) \\ &= \sup\left\{t \in \mathcal{R} : \forall i \in \{1, \ldots n\} \, \exists z_i \in \left(\widetilde{Z}_i\right)_\alpha \right. \\ &\quad \text{such that } \overline{\pi}_u(z_1, \ldots, z_n; \delta) \geq t\} \end{aligned} \tag{1.8}$$

where $\overline{\pi}_u(z_1, \ldots, z_n; \delta) = \underline{\pi}_l(z_1, \ldots, z_n; 1-\delta)$.

The notion of the one-sided fuzzy interval can be used to define a fuzzy test. In the considered case of one-sided statistical hypothesis, a function $\varphi : (\mathcal{FN}(\mathcal{R}))^n \rightarrow \mathcal{F}(\{0,1\})$ with the following α-cuts:

$$\varphi_\alpha\left(\widetilde{Z}_1,\ldots,\widetilde{Z}_n\right)=\begin{cases}\{1\} & if\ \theta_0\in\left(\underline{\Pi}_\alpha\setminus(\neg\underline{\Pi})_\alpha\right),\\ \{0\} & if\ \theta_0\in\left((\neg\underline{\Pi})_\alpha\setminus\underline{\Pi}_\alpha\right),\\ \{0,1\} & if\ \theta_0\in\left(\underline{\Pi}_\alpha\bigcap(\neg\underline{\Pi})_\alpha\right),\\ \emptyset & if\ \theta_0\notin\left(\underline{\Pi}_\alpha\bigcup(\neg\underline{\Pi})_\alpha\right)\end{cases}\tag{1.9}$$

is a fuzzy test for $H:\theta\leq\theta_0$, against $K:\theta>\theta_0$, on the significance level δ (Grzegorzewski [35]). In a similar way, we can define fuzzy tests for testing other one-sided hypotheses such as $H:\theta\geq\theta_0$, against $K:\theta<\theta_0$, and for testing two-sided hypotheses about θ.

It is worthy to note that in certain cases the application of the fuzzy test defined above does not lead to a clearly indicated decision. This feature is far from being unexpected because, unless we make some additional assumptions, we should not expect precise answers to questions presented in a form of statistical hypotheses if we infer these answers from the analysis of imprecise data. Let us note, however, that we face the similar situation when we use classical statistical methods. In that case a decision cannot be made without setting in advance an appropriate significance level of test. In case of fuzzy statistical data the knowledge of the significance level is not enough, so we have to use some additional indicators that would be helpful in making decisions. There exist several approaches that are suitable for solving this problem. One of these approaches which is formulated in the language of the possibility theory has been proposed by Hryniewicz [40].

In order to present a poosibilistic approach to the problem of statistical testing when both data and statistical hypotheses are imprecise let us consider a fuzzy equivalent of testing the hypothesis $H:\vartheta\leq\vartheta_H$ when we observe a random sample $(X_1,\ldots,X_n)$. In case of precisely formulated hypotheses and precise statistical data we can use a well known method (for reference, see, e.g., the book of Lehmann [53]) and calculate a one-sided confidence interval on a confidence level $1-\delta$ from the formula $[\pi_L(X_1,\ldots,X_n;1-\delta),\infty)$. We reject the null hypothesis on the significance level δ if the observed value of $\pi_L(X_1,\ldots,X_n;1-\delta)$ is larger than ϑ_H, i.e. when the inequality $\vartheta_H<\pi_L(x_1,\ldots,x_n;1-\delta)$ holds. Similarly, we reject the hypothesis $H:\vartheta\geq\vartheta_H$ on the significance level δ when the inequality $\vartheta_H>\pi_U(x_1,\ldots,x_n;1-\delta)$ holds, where $\pi_U(x_1,\ldots,x_n;1-\delta)$ is the observed value of the upper limit of the one-sided confidence interval $(-\infty,\pi_U(X_1,\ldots,X_n;1-\delta)]$ on a confidence level $1-\delta$. When we test the hypothesis $H:\vartheta=\vartheta_H$ on the significance level δ we reject it if either $\vartheta_H<\pi_L(x_1,\ldots,x_n;1-\delta/2)$ or $\vartheta_H>\pi_U(x_1,\ldots,x_n;1-\delta/2)$ holds, where $\pi_L(x_1,\ldots,x_n;1-\delta/2)$ is the observed in the sample value of the lower limit of the two-sided confidence interval $\pi_L(X_1,\ldots,X_n;1-\delta/2)$ on a confidence level $1-\delta$, and the observed value of its upper limit $\pi_U(X_1,\ldots,X_n;1-\delta/2)$ is given by $\pi_U(x_1,\ldots,x_n;1-\delta/2)$. Thus, when we test a hypothesis about the value of the parameter ϑ we find a respective confidence interval, and compare it to the hypothetical value.

Dubois et al. [21] proposed to use statistical confidence intervals of parameters of probability distributions for the construction of possibility distributions of these parameters in a fully objective way. According to their approach, the family of two-sided confidence intervals

$$[\pi_L(x_1,\ldots,x_n;1-\delta/2),\pi_U(x_1,\ldots,x_n;1-\delta/2)],\delta\in(0,1) \tag{1.10}$$

forms the *possibility distribution* $\tilde{\vartheta}$ of the estimated value of the unknown parameter ϑ. In a similar way it is possible to construct one-sided possibility distributions based on one-sided nested confidence intervals. Hryniewicz [40] proposed to compare this possibility distribution with a hypothetical value of the tested parameter. For this purpose he proposed to use the necessity of strict dominance measure introduced by Dubois and Prade [15] for measuring the necessity of the strict dominance relation $\tilde{A} \succ \tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are fuzzy sets. This measure, called the *Necessity of Strict Dominance* index (NSD), is defined as

$$NSD = Ness\left(\tilde{A} \succ \tilde{B}\right) = 1 - \sup_{x,y;x\leq y} \min\{\mu_A(x),\mu_B(y)\}. \tag{1.11}$$

Hryniewicz [40] has shown that in the classical case of precise statistical data and precisely defined statistical hypotheses the value of the NSD index is equal to the p-value of the test.

In case of fuzzy data the confidence intervals used for the construction of the possibility distribution of the estimated parameter θ can be replaced by their fuzzy equivalents, calculated according to the methodology proposed by Kruse and Meyer [49]. In his paper Hryniewicz [40] assumes that the value of the significance level of the corresponding statistical test δ is equal to the possibility degree α that defines the respective α-cut of the possibility distribution of $\tilde{\vartheta}$. He also assumes that in the possibilistic analysis of statistical tests on the significance level δ we should take into account only those possible values of the fuzzy sample whose possibility is not smaller than δ. Thus, the α-cuts of the membership function $\mu_F(\vartheta)$ denoted by $\left[\mu_{F,L}^{(\alpha)},\mu_{F,U}^{(\alpha)}\right]$ are equivalent to the α-cuts of the respective fuzzy confidence intervals on a confidence level $1-\alpha$.

In order to consider the most general case let us also assume that the hypothetical value of the tested parameter may be also imprecisely defined by a fuzzy number $\widetilde{\theta}_H$ described by the membership function $\mu_H(\theta)$. Possibilistic evaluation of the results of statistical fuzzy test consists now in the comparison of the possibility distribution of the estimated parameter θ, and the possibility distribution of the hypothetical value of this parameter. Let us illustrate this procedure by assuming that our fuzzy hypothesis is given as $H:\theta\leq\widetilde{\theta}_H$. In the case of crisp data we compare the lower limit of the one-sided confidence interval on a given confidence level $1-\delta$ with the respective α-cut of the membership function that describes $\widetilde{\theta}_H$. In the possibilistic framework described in [40] it means that we compare the possibility distribution of $\tilde{\vartheta}_L$ based on the left-hand sides of the confidence intervals with the fuzzy value of $\widetilde{\theta}_H$. This distribution is defined by the membership function

$$\mu_{F,L}^{(\alpha)} = \inf_{\gamma\geq\alpha} \Pi_L^{\gamma}\left(\widetilde{X}_1,\ldots,\widetilde{X}_n;1-\delta\right), \tag{1.12}$$

where

$$\begin{aligned} \Pi_L^{\gamma} &= \Pi_L^{\alpha}\left(\widetilde{X}_1, \ldots, \widetilde{X}_n; 1-\delta\right) \\ &= \inf\left\{t \in \mathcal{R} : \forall i \in \{1, \ldots n\}\, \exists x_i \in \left(\widetilde{X}_i\right)_{\gamma} \right. \\ &\quad \left. \text{such that } \pi_L\left(x_1, \ldots, x_n; 1-\delta\right) \leq t\right\}, \end{aligned} \tag{1.13}$$

and $\pi_L(x_1, \ldots, x_n; 1-\delta)$ is the left-hand side limit of the classical confidence interval. In a similar way we can define a fuzzy equivalent of the upper limit of the one-sided confidence interval $(-\infty, \pi_U(X_1, \ldots, X_n; 1-\delta)]$.

In the presence of fuzzy data we have to compare the possibility distribution $\tilde{\vartheta}_{F,L}$ of the estimated value of the unknown parameter θ represented by its α-cuts given by (1.12) with the fuzzy value of $\widetilde{\theta}_H$. In such a case we have to find the intersection point of the membership function $\mu_{F,L}(\vartheta)$ and the left-hand side of $\mu_H(\theta)$. The NSD index of the relation $\tilde{\vartheta}_{F,L} \succ \tilde{\vartheta}_H$ is equal to one minus the ordinate of this point, i.e.

$$Ness\left(\tilde{\vartheta}_{F,L} \succ \widetilde{\theta}_H\right) = 1 - \sup\min\left(\mu_{F,L}(\vartheta), \mu_H(\theta)\right). \tag{1.14}$$

The NSD index defined by (1.14) can be regarded as the generalisation of the observed test size p (also known as *p-value* or *significance*) for the case of imprecisely defined statistical hypotheses and vague statistical data. In exactly the same way we can find the NSD index for other one-sided and two-sided statistical hypotheses.

Statistical analysis of fuzzy random data can be also done in the Bayesian framework. First results presenting the Bayesian decision analysis for imprecise data were given in papers by Casals et al. [6], [7], and Gil [28]. In these papers the authors described fuzzy observations using the notion of the fuzzy information system by Zadeh [83] and Tanaka et al. [72]. As this approach seems to be not very effective in practical applications, other approaches have been proposed by such authors as Viertl [74], Frühwirth-Schnatter [26], and Taheri and Behboodian [71]. Comprehensive Bayesian model comprising fuzzy data, fuzzy hypotheses, and fuzzy utility function has been proposed in the paper by Hryniewicz [39].

When we interprete fuzzy random variables according to the definition proposed by Puri and Ralescu (see Definition 3 above) the statistical analysis of fuzzy data is unfortunately not so simple. The reasons for this difficulty stem from the fact that in that case the underlying classical probability distribution does not exist anymore. For example, it is difficult to formulate fuzzy equivalents of the Central Limit Theorem, as the concept of asymptotic normal distribution cannot be directly applied. Thus, the statistical procedures have to be constructed on a different theoretical basis. Some authors, see [30] for an overview of the problem, define statistical tests in terms of distances, in different metrics, between the observed fuzzy value of a test statistics and the hypothetical value of a certain characteristic of the fuzzy random variable, e.g. its fuzzy expected value. From among few papers devoted to this problem one can mention the papers by Körner [46], Körner and Näther [47], and Montenegro et al. [57]. The problem of lacking underlying probability distribution can be overcome by using a bootstrap methodology, as it has been recently proposed, in the papers by Gil et .al [31], González-Rodríguez et al. [33], and Montenegro et al. [58]. In the case of the Bayesian analysis of fuzzy random variables interesting results have been proposed in the paper by Gil and López-Díaz [29].

1.4 Applications of fuzzy statistics in systems analysis

Systems analysis is oriented on solving complex problems where precise mathematical models are used for a simplified (and sometimes even oversimplified) description of reality. The main problem of every researcher who has to apply the methods of systems analysis in real applications is related to coping with uncertainties of different kinds. What is important, not all of these uncertainties can be described by well developed methods like theory of probability and mathematical statistics. The methodology of fuzzy statistics, presented in the previous section, gives possibility to describe phenomena where probabilistic randomness is merged with possibilistic imprecision (fuzziness). In this section we present an application of this methodology which seems to be useful in solving real problems. The character of this paper does not allow us to present too many details that may be necessary to fully understand this application. The details will be presented in forthcoming papers dedicated to particular problems.

As a possible application of the fuzzy statistical methodology in the systems analysis we may consider the problem of testing a hypothesis about the mean value of a random variable described by a normal distribution when sampling costs are high, and we are forced to observe as few sample items as possible. This situation often happens when we have to control costs of a large project consisted of many individually assessed partial costs. If the number of partial costs is relatively large, we can assume - following the Central Limit Theorem - that the observed total cost, say X is distributed according to the normal distribution $N(\mu, \sigma)$. Suppose now that we are interested in keeping the total costs constant for a certain period of time at a level μ_0. A simple, and the most effective statistical test for verification of the statistical hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_1 : \mu = \mu_1$ is the sequential probability ratio test proposed originally by Wald (see the book by Lehmann [53] for more information) for the case of the known value of the standard deviation σ. The test statistic, based on a sample $(x_1 \ldots x_n)$ is, in that case, a simple sum of re-scaled observations

$$S_n = \sum_{i=1}^{n} (x_i/\sigma). \tag{1.15}$$

Let α be the probability of the type-I error (probability of an erroneous rejection of H_0), and β be the probability of the type-II error (probability of an erroneous acceptance of H_1). We accept the null hypothesis H_0 if

$$S_n \geq \frac{A}{\mu_1 - \mu_0} + \frac{n}{2}(\mu_1 + \mu_0), \tag{1.16}$$

where $A = \beta/(1-\alpha)$. We reject H_0 in favour of H_1 if

$$S_n \leq \frac{B}{\mu_1 - \mu_0} + \frac{n}{2}(\mu_1 + \mu_0), \tag{1.17}$$

where $B = (1-\beta)/\alpha$. If neither of these inequality holds we have to increase the sample size by one, and repeat the same procedure.

In the sequential test described above we have assummed that the value of σ is known. In practice, we never know this value in advance, but when the amount of historical data is large enough we can estimate σ, and take this estimated value as the known one. However, when the available amount of data is scarce, as it is usually the case in the analysis of large systems, we cannot proceed this way. A possible way out is to use a procedure proposed by Hryniewicz [41] for the analysis of reliability data.

Having some historical data we can estimate the value of σ. When the available sample size is equal to m, and the estimated value is given by σ_m^*, we can calculate the two-sided confidence interval for the unknown value of σ:

$$\left(\sigma_L(\gamma) = \sqrt{\frac{(m-1)(\sigma_m^*)^2}{\chi^2_{m-1}((1+\gamma)/2)}}, \sigma_R(\gamma) = \sqrt{\frac{(m-1)(\sigma_m^*)^2}{\chi^2_{m-1}((1-\gamma)/2)}}\right), \tag{1.18}$$

where $\chi^2_{m-1}(\gamma)$ is the quantile of order γ of the chi-square distribution with $m-1$ degrees of freedom. The confidence intervals defined by (1.18) can be used for the construction of the *possibility distribution* of σ defined by its δ-cuts (we use here the symbol of δ because the symbol α, traditionally used in this context, has been used for the description of the type-I error of the sequential test). The left-hand side limits of this possibility distribution are given by

$$\mu_L^\delta(\sigma) = \begin{cases} \sigma_L(1-\delta) & \text{if } \delta \geq \delta_0 \\ \sigma_L(1-\delta_0) & \text{if } \delta < \delta_0 \end{cases}, \tag{1.19}$$

where δ_0 is a small number close to 0 (e.g. 0.01). Similarily, the rigt-hand limits are given by

$$\mu_R^\delta(\sigma) = \begin{cases} \sigma_R(1-\delta) & \text{if } \delta \geq \delta_0 \\ \sigma_R(1-\delta_0) & \text{if } \delta < \delta_0 \end{cases}. \tag{1.20}$$

If we assume that σ is a fuzzy number defined by this possibility distribution we immediately find that the test statistic S_n becomes also fuzzy. Thus, we cannot directly verify if inequalities (1.16) and (1.17) are fulfilled. However, we may assume that they are fulfilled if the NSD index for respective fuzzy relations is greater than a prescribed value.

In this example of the application of fuzzy statistics we haven't used any subjective imprecise fuzzy information. The fuzziness has been introduced in a purely objective way using some historical statistical data. We can also generalize this problem by allowing imprecise hypotheses about the values of μ_0 and μ_1. The methodology for dealing with such a problem is the same, but the necessary calculations become much more complicated.

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