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Improved trapezoidal approximations of fuzzy numbers

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Abstract

Fuzzy number approximation by trapezoidal fuzzy numbers which preserves expected interval is discussed. The coorected formulae for the approximation operator discussed in the previous papers is given.

Key words: fuzzy numbers, approximation of fuzzy numbers, expected interval.

1 Introduction

In [9] we have formulated a list of desirable criteria which trapezoidal approximation operators should possess. We have also suggested a new approach to trapezoidal approximation of fuzzy numbers that lead to, so called, the nearest trapezoidal approximation operator preserving expected interval.

Unfortunately, we have not noticed that in some situations our operator may fail to lead a trapezoidal fuzzy number. Allahviranloo and Firozja [1] and Ban [3] showed some examples illustrating such situations. Thus in the present paper we propose a corrected version of our trapezoidal approximation operator. Actually, we obtain four approximation operators. Which one should be used depends on the shape of a fuzzy number to be approximated.

2 Fuzzy numbers

Let us consider a fuzzy number A, i.e. such fuzzy subset A of the real line $\mathbb{R}$ with membership function $\mu_A : \mathbb{R} \rightarrow [0, 1]$ which is (see [6]): normal (i.e. there exist an element x_0 such that $\mu_A(x_0) = 1$), fuzzy convex (i.e. $\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2) \quad \forall x_1, x_2 \in \mathbb{R}, \forall \lambda \in [0, 1]$), μ_A is upper semicontinuous, supp$A$ is bounded, where supp$A = cl(\{x \in \mathbb{R} : \mu_A(x) > 0\})$, and cl is the closure operator. A space of all fuzzy numbers will be denoted by $\mathbb{F}(\mathbb{R})$.

Moreover, let $A_\alpha = \{x \in R : \mu_A(x) \geq \alpha\}$ denote an α-cut of a fuzzy number A. As it is known, every α-cut of a fuzzy number is a closed interval, i.e. $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where $A_L(\alpha) = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$ and $A_U(\alpha) = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$.

Let us also recall that an expected interval $EI(A)$ of a fuzzy number A is given by (see [7], [11])

$$EI(A) = \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha\right]. \tag{1}$$

In [9] we have discussed the problem of the trapezoidal approximation of fuzzy numbers. Roughly speaking we have shown how to substitute arbitrary fuzzy number by so called, *trapezoidal fuzzy number*, i.e. by a fuzzy number with linear sides and the membership function having a following form:

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ \frac{x-t_1}{t_2-t_1} & \text{if} \quad t_1 \leq x < t_2, \\ 1 & \text{if} \quad t_2 \leq x \leq t_3, \\ \frac{t_4-x}{t_4-t_3} & \text{if} \quad t_3 < x \leq t_4, \\ 0 & \text{if} \quad t_4 < x. \end{cases} \tag{2}$$

Since the trapezoidal fuzzy number is completely characterized by four real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$ it is often denoted in brief as $A(t_1, t_2, t_3, t_4)$. A family of all trapezoidal fuzzy number will be denoted by $\mathbb{F}^T(\mathbb{R})$ (of course, $\mathbb{F}^T(\mathbb{R}) \subset \mathbb{F}(\mathbb{R})$). By (1) and (2) the expected interval of the trapezoidal fuzzy number is given by

$$EI(B) = \left[\frac{t_1+t_2}{2}, \frac{t_3+t_4}{2}\right]. \tag{3}$$

For two fuzzy numbers A and B with α-cuts $[A_L(\alpha), A_U(\alpha)]$ and $[B_L(\alpha), B_U(\alpha)]$, respectively, the quantity

$$d(A,B) = \sqrt{\int_0^1 (A_L(\alpha) - B_L(\alpha))^2 d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 d\alpha} \tag{4}$$

is the distance between A and B (for more details we refer the reader to [8]).

To simplify the representation of fuzzy numbers Delgado et al. [5] suggested two parameters – value and ambiguity – which represent some basic features of fuzzy numbers and hence they were called a canonical representation of fuzzy numbers.

Let $s : [0,1] \to [0,1]$ denote so-called reducing function. Then

$$Val_s(A) = \int_0^1 s(\alpha)[A_L(\alpha) + A_U(\alpha)]d\alpha \tag{5}$$

is called the *value* of fuzzy number A. Index $Val_s(A)$ may be seen as a point that corresponds to the typical value of the magnitude that the fuzzy number A represents. The next index, called the *ambiguity* is given by

$$Amb_s(A) = \int_0^1 s(\alpha)[A_U(\alpha) - A_L(\alpha)]d\alpha, \tag{6}$$

and it characterizes the vagueness of fuzzy number A.

These two parameters defined above depend on the choice of the reducing function s. Using, so-called, regular reducing function the value and ambiguity of a fuzzy number A is defined as follows

$$Val(A) = \int_0^1 \alpha(A_L(\alpha) + A_U(\alpha))d\alpha, \quad (7)$$

$$Amb(A) = \int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))d\alpha. \quad (8)$$

Another parameter utilized for representing the typical value of the fuzzy number is the middle of the expected interval of a fuzzy number and is called the expected value of a fuzzy number A, i.e.

$$EV(A) = \frac{1}{2}\left(\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha\right) \quad (9)$$

(see [7], [11]). Sometimes its generalization, called weighted expected value, might be interesting. It is defined as

$$EV_q(A) = (1-q)\int_0^1 A_L(\alpha)d\alpha + q\int_0^1 A_U(\alpha)d\alpha, \quad (10)$$

where $q \in [0,1]$ (see [8]).

Finally, the width of a fuzzy number (see [4]), defined by

$$w(A) = \int_{-\infty}^{\infty} \mu_A(x)dx = \int_0^1 (A_U(\alpha) - A_L(\alpha))d\alpha. \quad (11)$$

is also an useful parameter characterizing the nonspecifity of a fuzzy number. It is worth remembering (see [4]) that

$$w(A) = w(EI(A)).$$

3 Trapezoidal approximation

In this section we propose an approximation operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ which produces a trapezoidal fuzzy number that is the closest to given original fuzzy number among all trapezoidal fuzzy numbers having identical expected interval as the original one. Therefore, this operator will be called *the nearest trapezoidal approximation operator preserving expected interval.*

Suppose A is a fuzzy number and $[A_L(\alpha), A_U(\alpha)]$ is its α-cut. Given A we'll try to find a trapezoidal fuzzy number $T(A)$ which is the nearest to A with respect to metric d (4). Let $[T_L(\alpha), T_U(\alpha)]$ denote the α-cut of $T(A)$. Thus we have to minimize

$$d(A,T(A)) = \sqrt{\int_0^1 (A_L(\alpha) - T_L(\alpha))^2 d\alpha + \int_0^1 (A_U(\alpha) - T_U(\alpha))^2 d\alpha} \quad (12)$$

with respect to $T_L(\alpha)$ and $T_U(\alpha)$. However, since a trapezoidal fuzzy number is completely described by four real numbers that are borders of its support and core, our goal reduces to finding such real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$ that characterize $T(A) = T(t_1, t_2, t_3, t_4)$.

Theorem 1 *The nearest trapezoidal approximation operator preserving expected interval is such operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ which for any fuzzy number A with α-cuts $[A_L(\alpha), A_U(\alpha)]$ assigns a following trapezoidal fuzzy number $T(A) = T(t_1, t_2, t_3, t_4)$:*

(a) *if $Amb(A) \geq \frac{1}{3}w(A)$ then*

$$t_1 = -6\int_0^1 \alpha A_L(\alpha)\, d\alpha + 4\int_0^1 A_L(\alpha)\, d\alpha, \tag{13}$$

$$t_2 = 6\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2\int_0^1 A_L(\alpha)\, d\alpha, \tag{14}$$

$$t_3 = 6\int_0^1 \alpha A_U(\alpha)\, d\alpha - 2\int_0^1 A_U(\alpha)\, d\alpha, \tag{15}$$

$$t_4 = -6\int_0^1 \alpha A_U(\alpha)\, d\alpha + 4\int_0^1 A_U(\alpha)\, d\alpha; \tag{16}$$

(b) *if $Amb(A) < \frac{1}{3}w(A)$ and $EV_{\frac{1}{3}}(A) < Val(A) < EV_{\frac{2}{3}}(A)$ then*

$$t_1 = 3\int_0^1 A_L(\alpha)d\alpha - 3\int_0^1 \alpha A_L(\alpha)\, d\alpha - 3\int_0^1 \alpha A_U(\alpha)\, d\alpha + \int_0^1 A_U(\alpha)d\alpha, \tag{17}$$

$$t_2 = t_3 = 3\int_0^1 \alpha A_L(\alpha)\, d\alpha + 3\int_0^1 \alpha A_U(\alpha)\, d\alpha - \int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha, \tag{18}$$

$$t_4 = 3\int_0^1 A_U(\alpha)d\alpha - 3\int_0^1 \alpha A_L(\alpha)\, d\alpha - 3\int_0^1 \alpha A_U(\alpha)\, d\alpha + \int_0^1 A_L(\alpha)d\alpha; \tag{19}$$

(c) *if $Amb(A) < \frac{1}{3}w(A)$ and $Val(A) \leq EV_{\frac{1}{3}}(A)$ then*

$$t_1 = t_2 = t_3 = \int_0^1 A_L(\alpha)d\alpha, \tag{20}$$

$$t_4 = 2\int_0^1 A_U(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha; \tag{21}$$

(d) *if $Amb(A) < \frac{1}{3}w(A)$ and $Val(A) \geq EV_{\frac{2}{3}}(A)$ then*

$$t_1 = 2\int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha, \tag{22}$$

$$t_2 = t_3 = t_4 = \int_0^1 A_U(\alpha)d\alpha. \tag{23}$$

Proof:
As $T(A) = T(t_1, t_2, t_3, t_4)$ is a trapezoidal fuzzy number, its α-cuts have a following form $[t_1 + (t_2 - t_1)\,\alpha, t_4 - (t_4 - t_3)\alpha]$, where $\alpha \in (0, 1]$. Therefore (12) reduces to

$$d(A, T(A)) = \sqrt{\int_0^1 \left[A_L(\alpha) - (t_1 + (t_2 - t_1)\,\alpha)\right]^2 d\alpha + \int_0^1 \left[A_U(\alpha) - (t_4 - (t_4 - t_3)\alpha)\right]^2 d\alpha}, \tag{24}$$

and our goal is to minimize (24) with respect to t_1, t_2, t_3, t_4. Moreover, since we are looking for an operator which preserves the expected interval of a fuzzy number, a following condition should be fulfilled

$$EI(T(A)) = EI(A). \tag{25}$$

By (1) and (3) we can rewrite (25) as follows

$$\left[\frac{t_1 + t_2}{2}, \frac{t_3 + t_4}{2}\right] = \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha\right]. \tag{26}$$

It is easily seen that in order to minimize $d(A, T(A))$ it suffices to minimize function $f(t_1, t_2, t_3, t_4) = d^2(A, T(A))$ with respect to following conditions:

$$\frac{t_1 + t_2}{2} - \int_0^1 A_L(\alpha)d\alpha = 0, \tag{27}$$

$$\frac{t_3 + t_4}{2} - \int_0^1 A_U(\alpha)d\alpha = 0. \tag{28}$$

Finally, we have to assure that $t_1 \le t_2 \le t_3 \le t_4$, i.e. following inequalities should hold:

$$t_1 - t_2 \le 0, \tag{29}$$

$$t_2 - t_3 \le 0, \tag{30}$$

$$t_3 - t_4 \le 0. \tag{31}$$

Thus, to sum up, we wish to minimize function

$$f(\mathbf{t}) = \int_0^1 \left[A_L(\alpha) - (t_1 + (t_2 - t_1)\,\alpha)\right]^2 d\alpha + \int_0^1 \left[A_U(\alpha) - (t_4 - (t_4 - t_3)\alpha)\right]^2 d\alpha \tag{32}$$

subject to

$$\mathbf{h}(\mathbf{t}) = \left[t_1 + t_2 - 2\int_0^1 A_L(\alpha)d\alpha, t_3 + t_4 - 2\int_0^1 A_U(\alpha)d\alpha\right]^T = \mathbf{0}^T, \tag{33}$$

$$\mathbf{g}(\mathbf{t}) = [t_1 - t_2, t_2 - t_3, t_3 - t_4] \le 0, \tag{34}$$

where $\mathbf{t} \in \mathbb{R}^4$.

By the Karush-Kuhn-Tucker theorem, if $\mathbf{t}^*$ is a local minimizer for the problem of minimizing f subject to $\mathbf{h}(\mathbf{t}) = \mathbf{0}$ and $\mathbf{g}(\mathbf{t}) \leq 0$, then there exist the Lagrange multiplier vector $\boldsymbol{\lambda}$ and the Karush-Kuhn-Tucker multiplier $\boldsymbol{\mu}$ such that

$$\begin{aligned} Df(\mathbf{t}^*) + \boldsymbol{\lambda}^T D\mathbf{h}(\mathbf{t}^*) + \boldsymbol{\mu}^T D\mathbf{g}(\mathbf{t}^*) &= \mathbf{0}^T, & (35)\\ \boldsymbol{\mu}^T \mathbf{g}(\mathbf{t}^*) &= 0, & (36)\\ \boldsymbol{\mu} &\geq 0. & (37) \end{aligned}$$

In our case, after some calculations, we get

$$\begin{aligned} Df(\mathbf{t}^*) = \Big[& \frac{2}{3}t_1 + \frac{1}{3}t_2 + 2\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2\int_0^1 A_L(\alpha)\, d\alpha, & (38)\\ & \frac{1}{3}t_1 + \frac{2}{3}t_2 - 2\int_0^1 \alpha A_L(\alpha)\, d\alpha, \frac{2}{3}t_3 + \frac{1}{3}t_4 - 2\int_0^1 \alpha A_U(\alpha)\, d\alpha, \\ & \frac{1}{3}t_3 + \frac{2}{3}t_4 + 2\int_0^1 \alpha A_U(\alpha)\, d\alpha - 2\int_0^1 A_U(\alpha)\, d\alpha \Big], \end{aligned}$$

$$D\mathbf{h}(\mathbf{t}^*) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad (39)$$

$$D\mathbf{g}(\mathbf{t}^*) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0, & 1, & -1, & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \qquad (40)$$

Therefore, we can rewrite the Karush-Kuhn-Tucker conditions in a following way

$$\begin{aligned} \frac{2}{3}t_1 + \frac{1}{3}t_2 + 2\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2\int_0^1 A_L(\alpha)\, d\alpha + \lambda_1 + \mu_1 &= 0, & (41)\\ \frac{1}{3}t_1 + \frac{2}{3}t_2 - 2\int_0^1 \alpha A_L(\alpha)\, d\alpha + \lambda_1 - \mu_1 + \mu_2 &= 0, & (42)\\ \frac{2}{3}t_3 + \frac{1}{3}t_4 - 2\int_0^1 \alpha A_U(\alpha)\, d\alpha + \lambda_2 - \mu_2 + \mu_3 &= 0, & (43)\\ \frac{1}{3}t_3 + \frac{2}{3}t_4 + 2\int_0^1 \alpha A_U(\alpha)\, d\alpha - 2\int_0^1 A_U(\alpha)\, d\alpha + \lambda_2 - \mu_3 &= 0, & (44)\\ t_1 + t_2 - 2\int_0^1 A_L(\alpha) d\alpha &= 0, & (45)\\ t_3 + t_4 - 2\int_0^1 A_U(\alpha) d\alpha &= 0, & (46)\\ \mu_1(t_1 - t_2) &= 0, & (47)\\ \mu_2(t_2 - t_3) &= 0, & (48)\\ \mu_3(t_3 - t_4) &= 0, & (49)\\ \mu_1 &\geq 0, & (50)\\ \mu_2 &\geq 0, & (51)\\ \mu_3 &\geq 0. & (52) \end{aligned}$$

To find points that satisfy the above conditions, we first try $\mu_1 = \mu_2 = \mu_3 = 0$. Then the system of equations (41)-(52) reduces to following six linear equations

$$
\begin{aligned}
\frac{2}{3}t_1 + \frac{1}{3}t_2 + 2\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2\int_0^1 A_L(\alpha)\, d\alpha + \lambda_1 &= 0, && (53)\\
\frac{1}{3}t_1 + \frac{2}{3}t_2 - 2\int_0^1 \alpha A_L(\alpha)\, d\alpha + \lambda_1 &= 0, && (54)\\
\frac{2}{3}t_3 + \frac{1}{3}t_4 - 2\int_0^1 \alpha A_U(\alpha)\, d\alpha + \lambda_2 &= 0, && (55)\\
\frac{1}{3}t_3 + \frac{2}{3}t_4 + 2\int_0^1 \alpha A_U(\alpha)\, d\alpha - 2\int_0^1 A_U(\alpha)\, d\alpha + \lambda_2 &= 0, && (56)\\
t_1 + t_2 - 2\int_0^1 A_L(\alpha)d\alpha &= 0, && (57)\\
t_3 + t_4 - 2\int_0^1 A_U(\alpha)d\alpha &= 0. && (58)
\end{aligned}
$$

Solving the above equations we obtain

$$
\begin{aligned}
t_1 &= -6\int_0^1 \alpha A_L(\alpha)\, d\alpha + 4\int_0^1 A_L(\alpha)\, d\alpha, && (59)\\
t_2 &= 6\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2\int_0^1 A_L(\alpha)\, d\alpha, && (60)\\
t_3 &= 6\int_0^1 \alpha A_U(\alpha)\, d\alpha - 2\int_0^1 A_U(\alpha)\, d\alpha, && (61)\\
t_4 &= -6\int_0^1 \alpha A_U(\alpha)\, d\alpha + 4\int_0^1 A_U(\alpha)\, d\alpha, && (62)\\
\lambda_1 &= 0, && (63)\\
\lambda_2 &= 0. && (64)
\end{aligned}
$$

One can notice that the solution $\mathbf{t} = (t_1, t_2, t_3, t_4)$, given by (59)-(62), coincides with the solution given in [9].

Now suppose $\mu_2 = \mu_3 = 0$ and $\mu_1 > 0$, which by (47) and (45) implies

$$
t_1 = t_2 = \int_0^1 A_L(\alpha)d\alpha. \qquad (65)
$$

Substituting (65) into (41) and (42) we get

$$
\begin{aligned}
\lambda_1 &= 0, && (66)\\
\mu_1 &= \int_0^1 A_L(\alpha)d\alpha - 2\int_0^1 \alpha A_L(\alpha)\, d\alpha. && (67)
\end{aligned}
$$

However, it is not difficult to see that inequality $\int_0^1 A_L(\alpha)d\alpha - 2\int_0^1 \alpha A_L(\alpha)\, d\alpha > 0$ does not hold in general which contradicts the assumption that $\mu_1 > 0$. Hence, there is no solution for $\mu_2 = \mu_3 = 0$ and $\mu_1 > 0$. In a similar way one may conclude that assuming $\mu_2 = 0$ the solution exists if and only if both $\mu_1 = 0$ and $\mu_3 = 0$.

Now let us suppose that $\mu_2 > 0$. Thus by (48) we get immediately

$$t_2 = t_3. \tag{68}$$

Assume firstly that $\mu_1 = \mu_3 = 0$. The system of equations (41)-(52) reduces to following six linear equations:

$$\begin{aligned}
\frac{2}{3}t_1 + \frac{1}{3}t_2 + 2\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2\int_0^1 A_L(\alpha)\, d\alpha + \lambda_1 &= 0, && (69)\\
\frac{1}{3}t_1 + \frac{2}{3}t_2 - 2\int_0^1 \alpha A_L(\alpha)\, d\alpha + \lambda_1 + \mu_2 &= 0, && (70)\\
\frac{2}{3}t_2 + \frac{1}{3}t_4 - 2\int_0^1 \alpha A_U(\alpha)\, d\alpha + \lambda_2 - \mu_2 &= 0, && (71)\\
\frac{1}{3}t_2 + \frac{2}{3}t_4 + 2\int_0^1 \alpha A_U(\alpha)\, d\alpha - 2\int_0^1 A_U(\alpha)\, d\alpha + \lambda_2 &= 0, && (72)\\
t_1 + t_2 - 2\int_0^1 A_L(\alpha) d\alpha &= 0, && (73)\\
t_2 + t_4 - 2\int_0^1 A_U(\alpha) d\alpha &= 0. && (74)
\end{aligned}$$

Solving the above system of equations we get

$$\begin{aligned}
t_1 &= 3\int_0^1 A_L(\alpha)d\alpha - 3\int_0^1 \alpha A_L(\alpha)\, d\alpha - 3\int_0^1 \alpha A_U(\alpha)\, d\alpha + \int_0^1 A_U(\alpha)d\alpha, && (75)\\
t_2 &= t_3 = 3\int_0^1 \alpha A_L(\alpha)\, d\alpha + 3\int_0^1 \alpha A_U(\alpha)\, d\alpha - \int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha, && (76)\\
t_4 &= 3\int_0^1 A_U(\alpha)d\alpha - 3\int_0^1 \alpha A_L(\alpha)\, d\alpha - 3\int_0^1 \alpha A_U(\alpha)\, d\alpha + \int_0^1 A_L(\alpha)d\alpha, && (77)\\
\lambda_1 &= \frac{1}{3}\int_0^1 A_L(\alpha)d\alpha - \int_0^1 \alpha A_L(\alpha)\, d\alpha + \int_0^1 \alpha A_U(\alpha)\, d\alpha - \frac{1}{3}\int_0^1 A_U(\alpha)d\alpha, && (78)\\
\lambda_2 &= -\frac{1}{3}\int_0^1 A_L(\alpha)d\alpha + \int_0^1 \alpha A_L(\alpha)\, d\alpha - \int_0^1 \alpha A_U(\alpha)\, d\alpha + \frac{1}{3}\int_0^1 A_U(\alpha)d\alpha, && (79)\\
\mu_2 &= 2\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2\int_0^1 \alpha A_U(\alpha)\, d\alpha - \frac{2}{3}\int_0^1 A_L(\alpha)d\alpha + \frac{2}{3}\int_0^1 A_U(\alpha)d\alpha. && (80)
\end{aligned}$$

However, by the assumption that $\mu_2 > 0$, we have a legitimate solution to the Karush-Kuhn-Tucker conditions if and only if

$$\int_0^1 \alpha A_L(\alpha)\, d\alpha - \frac{1}{3}\int_0^1 A_L(\alpha)d\alpha > \int_0^1 \alpha A_U(\alpha)\, d\alpha - \frac{1}{3}\int_0^1 A_U(\alpha)d\alpha \tag{81}$$

or, in other words,

$$\int_0^1 (A_U(\alpha) - A_L(\alpha))\,(\frac{1}{3} - \alpha)\,d\alpha > 0, \tag{82}$$

and then we get a solution $\mathbf{t} = (t_1, t_2, t_3, t_4)$, given by (75)-(77). One may notice that this very solution was also mentioned in [10].

Now let us consider a situation when not only $\mu_2 > 0$ but also $\mu_1 > 0$ and still $\mu_3 = 0$. Then by (45)-(48) we get immediately

$$t_1 = t_2 = t_3 = \int_0^1 A_L(\alpha)d\alpha \tag{83}$$

and

$$t_4 = 2\int_0^1 A_U(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha. \tag{84}$$

Thus we get another solution $\mathbf{t} = (t_1, t_2, t_3, t_4)$, given by (83)-(84), provided $\mu_1 > 0$ and $\mu_2 > 0$, i.e. if inequalities (82) and

$$\int_0^1 A_U(\alpha)(\alpha - \frac{1}{3})d\alpha < \frac{2}{3}\int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha \tag{85}$$

are fulfilled.

We may also consider another situation when $\mu_2 > 0$, $\mu_3 > 0$ and $\mu_1 = 0$ which leads to the fourth solution

$$t_1 = 2\int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha, \tag{86}$$

$$t_2 = t_3 = t_4 = \int_0^1 A_U(\alpha)d\alpha, \tag{87}$$

which holds provided inequalities (82) and

$$\int_0^1 A_L(\alpha)(\alpha - \frac{1}{3})d\alpha > \frac{2}{3}\int_0^1 A_U(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha \tag{88}$$

hold.

Finally one may ask what happen if $\mu_1 > 0$, $\mu_2 > 0$ and still $\mu_3 > 0$. But it is seen immediately that this situation has no sense.

Now we have to verify that all our solutions $\mathbf{t}$ satisfy the second-order sufficient conditions. For this we form a matrix

$$L(\mathbf{t}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = D^2 f(\mathbf{t}) + [\boldsymbol{\lambda} D^2\mathbf{h}(\mathbf{t})] + \boldsymbol{\mu} D^2 g(\mathbf{t}), \tag{89}$$

where $[\boldsymbol{\lambda} D^2\mathbf{h}(\mathbf{t})] = \lambda_1 D^2 h_1(\mathbf{t}) + \lambda_2 D^2 h_2(\mathbf{t})$ and $D^2 h_i(\mathbf{t})$ is the Hessian of $h_i(\mathbf{t})$. One check easily that for our four solutions $\mathbf{t}$ we have $\mathbf{y}^T L(\mathbf{t}, \boldsymbol{\lambda}, \boldsymbol{\mu})\mathbf{y} > 0$ for all vectors $\mathbf{y}$ in the tangent space to the surface defined by active constraints, i.e. $\{\mathbf{y} : D\mathbf{h}(\mathbf{t})\mathbf{y} = 0, D^2 g(\mathbf{t})\mathbf{y} = 0\}$.

Therefore, we conclude that we have received four different solutions which lead to the nearest trapezoidal fuzzy number that preserves the expected value of the original fuzzy number. Theses solutions are the outputs of four different trapezoidal approximation operators: $T_i(A) = T_i(t_1, t_2, t_3, t_4)$, $i = 1, 2, 3, 4$, where T_1 denotes the approximation operator given by equations (13)-(16) that leads to trapezoidal (but not triangular) fuzzy number, T_2 stands for the operator given by (17)-(19) that leads to triangular fuzzy number with two sides, T_3 given by (20)-(21) produces a triangular fuzzy number with the right side only and T_4 given by (22)-(23) produces a triangular fuzzy number with the left side only. Which one should be used in a particular situation depends on a given fuzzy number, i.e. it depends on conditions (82), (85) and (88). To make them more clear and to get a better interpretation of those conditions.

Firstly, let us notice that according to (8) and (11) we can rewrite (82) easily as (see also [10])

$$Amb(A) \geq \frac{1}{3}w(A). \tag{90}$$

It means that we approximate a fuzzy number A by the trapezoidal approximation operator T_1 provided ambiguity of this fuzzy number is greater than one third of the width of that fuzzy number A. For less vague fuzzy numbers, i.e. when

$$Amb(A) < \frac{1}{3}w(A), \tag{91}$$

we will approximate A by a triangular number. Now the proper choice of the operator (T_2, T_3 or T_4) depends also on the location of the typical value of the fuzzy number. If (85) is satisfied then by (7) and (10) for $q = \frac{1}{3}$we get

$$Val(A) \leq EV_{\frac{1}{3}}(A). \tag{92}$$

It might be interpreted in such a way that a fuzzy number with a slight ambiguity and which typical value is located closely to the left border of its support would be approximated by a trapezoidal fuzzy number with the right side only, produced by the operator T_3.

Similarly, by (7) and (10) for $q = \frac{2}{3}$we get

$$Val(A) \geq EV_{\frac{2}{3}}(A), \tag{93}$$

which means that a fuzzy number with a slight ambiguity and which typical value is located closely to the right border of its support would be approximated by a trapezoidal fuzzy number with the left side only, produced by the operator T_4. All other fuzzy numbers with a slight ambiguity, i.e. which satisfy (91), would be approximated by the operator T_2.

This ends the proof. ■

It is worth noting that another interpretation of the condition (82), equivalent to (90), can be found in [10].

Example 1
Suppose a fuzzy number A has a following membership function

$$\mu_A(x) = \begin{cases} 1 - x^2 & \text{if} \quad -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

An α-cut of that fuzzy number is $A_\alpha = [-\sqrt{1-\alpha}, \sqrt{1-\alpha}]$. Hence

$$Amb(A) = \int_0^1 2\alpha\sqrt{1-\alpha}d\alpha = \frac{8}{15}$$

while

$$\frac{1}{3}w(A) = \frac{1}{3}\int_0^1 2\sqrt{1-\alpha}d\alpha = \frac{4}{9}$$

and it is easily seen that condition (90) is fulfilled. Thus we apply a triangular approximation operator $T_1(A)$ given by (13)–(16) and we get a trapezoidal fuzzy number $T_1(A) = T_1(-\frac{16}{15}, -\frac{4}{15}, \frac{4}{15}, \frac{16}{15})$, i.e. a membership function $\mu_{T_1(A)}$ of $T_1(A)$ is

$$\mu_{T_1(A)}(x) = \begin{cases} \frac{4}{3} + \frac{5}{4}x & \text{if} \quad -\frac{16}{15} \le x < -\frac{4}{15}, \\ 1 & \text{if} \quad -\frac{4}{15} \le x \le \frac{4}{15}, \\ \frac{4}{3} - \frac{5}{4}x & \text{if} \quad \frac{4}{15} < x \le \frac{16}{15}, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2
Let us consider a fuzzy number A, discussed in [1], with membership function

$$\mu_A(x) = \begin{cases} (x+1)^2 & \text{if} \quad -1 \le x \le 0, \\ (1-x)^2 & \text{if} \quad 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus an α-cut of that fuzzy number is $A_\alpha = [\sqrt{\alpha} - 1, 1 - \sqrt{\alpha}]$ for $\alpha \in [0,1]$. One can easily see that

$$Amb(A) = \int_0^1 \alpha(2 - 2\sqrt{\alpha})d\alpha = \frac{1}{5}$$

and

$$w(A) = \frac{1}{3}\int_0^1 (2 - 2\sqrt{\alpha})d\alpha = \frac{2}{9}$$

which means that condition (90) does not hold. Simultaneously, $Val(A) = 0$, while

$$EV_{\frac{2}{3}}(A) = \frac{1}{3}\int_0^1 (\sqrt{\alpha} - 1)\, d\alpha + \frac{2}{3}\int_0^1 (1 - \sqrt{\alpha})\, d\alpha = \frac{1}{9}$$

and

$$EV_{\frac{1}{3}}(A) = \frac{2}{3}\int_0^1 (\sqrt{\alpha} - 1)\, d\alpha + \frac{1}{3}\int_0^1 (1 - \sqrt{\alpha})\, d\alpha = -\frac{1}{9}.$$

Therefore $EV_{\frac{1}{3}}(A) < Val(A) < EV_{\frac{2}{3}}(A)$, so the nearest trapezoidal approximation is given by (17)–(19). Finally we obtain a trapezoidal fuzzy number $T_2(A) = T_2(-\frac{2}{3}, 0, 0, \frac{2}{3})$, i.e. a membership function $\mu_{T_2(A)}$ of $T_2(A)$ is

$$\mu_{T_2(A)}(x) = \begin{cases} \frac{3}{2}x + 1 & \text{if } -\frac{2}{3} \le x \le 0, \\ 1 - \frac{3}{2}x & \text{if } 0 < x \le \frac{2}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3

Now let us consider a fuzzy number A, discussed in [3], with membership function

$$\mu_A(x) = \begin{cases} (x-1)^2 & \text{if } 1 \le x \le 2, \\ 1 & \text{if } 2 \le x \le 3, \\ (\frac{30-x}{27})^2 & \text{if } 3 \le x \le 30, \\ 0 & \text{otherwise.} \end{cases}$$

We get following α-cuts of that fuzzy number: $A_\alpha = [1 + \sqrt{\alpha}, 30 - 27\sqrt{\alpha}]$ for $\alpha \in [0, 1]$. Thus

$$Amb(A) = \int_0^1 \alpha(29 - 28\sqrt{\alpha})d\alpha = \frac{33}{10}$$

and

$$\frac{1}{3}w(A) = \frac{1}{3}\int_0^1 (29 - 28\sqrt{\alpha})d\alpha = \frac{31}{9}$$

which means that condition (90) does not hold. We also get

$$Val(A) = \int_0^1 \alpha(31 - 26\sqrt{\alpha})d\alpha = \frac{51}{10}$$

and

$$EV_{\frac{1}{3}}(A) = \frac{2}{3}\int_0^1 (1 + \sqrt{\alpha})\, d\alpha + \frac{1}{3}\int_0^1 (30 - 27\sqrt{\alpha})\, d\alpha = \frac{46}{9}.$$

Thus condition (92) is fulfilled, so the nearest trapezoidal approximation is given by (20)–(21). Finally we obtain a triangular fuzzy number $T_3(A) = T_3(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{67}{3})$, i.e. a membership function $\mu_{T_3(A)}$ of $T_3(A)$ is

$$\mu_{T_3(A)}(x) = \begin{cases} \frac{67-3x}{62} & \text{if } \frac{5}{3} \le x \le \frac{67}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4

Consider also another fuzzy number A, discussed in [3], with membership function

$$\mu_A(x) = \begin{cases} (\frac{x-1}{27})^2 & \text{if } 1 \le x \le 28, \\ 1 & \text{if } 28 \le x \le 29, \\ (30-x)^2 & \text{if } 29 \le x \le 30, \\ 0 & \text{otherwise.} \end{cases}$$

with α-cuts: $A_\alpha = [1+27\sqrt{\alpha}, 30-\sqrt{\alpha}]$ for $\alpha \in [0,1]$. Here we get as before

$$Amb(A) = \int_0^1 \alpha(29 - 28\sqrt{\alpha}) = \frac{33}{10}$$

and

$$\frac{1}{3}w(A) = \frac{1}{3}\int_0^1 (29 - 28\sqrt{\alpha})d\alpha = \frac{31}{9}$$

which means that condition (90) does not hold. But now

$$Val(A) = \int_0^1 \alpha(31 + 26\sqrt{\alpha})d\alpha = \frac{259}{10}$$

and

$$EV_{\frac{2}{3}}(A) = \frac{1}{3}\int_0^1 (1+27\sqrt{\alpha})\,d\alpha + \frac{2}{3}\int_0^1 (30-\sqrt{\alpha})\,d\alpha = \frac{233}{9}.$$

Since condition (93) holds thus the nearest trapezoidal approximation is given by (22)–(23) and we obtain a triangular fuzzy number $T_4(A) = T_4(\frac{26}{3}, \frac{88}{3}, \frac{88}{3}, \frac{88}{3})$ with a membership function

$$\mu_{T_4(A)}(x) = \begin{cases} \frac{3x-26}{62} & \text{if } \frac{26}{3} \le x \le \frac{88}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

4 Properties

Since one can propose many approximation methods for fuzzy numbers the question about the quality of approximation is of importance. A list of criteria that the approximation operator should or just can possess has been suggested in [9]. Then we have proved that the nearest trapezoidal approximation operator preserving expected interval - denoted in this paper by T_1 - is invariant to translations and scale invariant, is monotonic and fulfills identity criterion, preserves the expected interval and fulfills the nearness criterion with respect to metric (4) in subfamily of all trapezoidal fuzzy numbers with fixed expected interval, is continuous and compatible with the extension principle, is order invariant with respect to some preference fuzzy relations and is correlation invariant. Moreover, it preserves the width, value and ambiguity of fuzzy number (for details we refer the reader to [9]).

Unfortunately, if condition (90) is not fulfilled then some of these properties do not hold. Namely, it can be shown that operators T_2, T_3 and T_4 do not preserve the ambiguity of fuzzy number, i.e.

$$Amb(A) < Amb(T_i(A))$$

for $i = 2, 3, 4$ (see also [10]).

Moreover, the identity criterion also does not apply to operators T_2, T_3 and T_4. It is obvious, because these three operators always produce triangular fuzzy numbers (so if the "input" is a trapezoidal but not triangular fuzzy number, the "output" of T_2, T_3 and T_4 is a triangular fuzzy number).

However, operators T_2, T_3 and T_4 still possess many other properties discussed in [9]. It is so because the expected interval invariance criteria hold for them.

5 Conclusion

In the present contribution we have improved two papers [9] and [10] devoted to trapezoidal approximation of fuzzy numbers. We have shown the shape of the nearest trapezoidal approximation operator preserving expected interval depends on the particular shape of the original fuzzy number: whether it is more or less vague and more or less symmetrical. This way we have obtained four approximation operators which possess many desired properties suggested and discussed in [9].

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