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Intuitionistic Fuzzy Numbers

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Abstract

A definition of intuitionistic fuzzy numbers is suggested. The notion of expected interval of an intuitionistic fuzzy number is proposed. Then two families of metrics in space of intuitionistic fuzzy numbers are considered and a method of ranking intuitionistic fuzzy numbers based on these metrics is also suggested and investigated.

Keywords: intuitionistic fuzzy sets, intuitionistic fuzzy numbers, metrics, ranking methods.

1 Introduction

A membership function of a standard fuzzy set assigns to each element of the universe of discourse a number from the unit interval to indicate the degree of belongingness to the set under consideration. The degree of nonbelongingness is just automatically the complement to 1 of the membership degree. However, a human being who expresses the degree of membership of given element in a fuzzy set very often does not express corresponding degree of nonmembership as the complement to 1. This reflects a well known psychological fact that the linguistic negation not always identifies with logical negation.

Thus Atanassov [1] introduced the concept of an intuitionistic fuzzy set which is characterized by two functions expressing the degree of belongingness and the degree of nonbelongingness, respectively. This idea, which is a natural generalization of a standard fuzzy set, seems to be useful in modelling many real life situations, like negotiation processes, decision making, etc.

The most important and probably the most often explored subfamily of fuzzy sets are fuzzy numbers. In this paper we introduce the definition of intuitionistic fuzzy numbers and we suggest how to generalize the concept of expected interval defined on a family of fuzzy numbers into a family of all intuitionistic fuzzy numbers.

Ranking fuzzy numbers is one of the fundamental problems of fuzzy arithmetic and fuzzy decision making. It is due to the fact that fuzzy numbers are not linearly ordered. This problem is also important in the case of intuitionistic fuzzy numbers. In this paper we propose and investigate two families of metrics in space of intuitionistic fuzzy numbers. Then we suggest a method of ranking intuitionistic fuzzy numbers based on these metrics.

2 Fuzzy numbers

Let X denote a universe of discourse. Then a fuzzy set A in X is defined as a set of ordered pairs

$$A = \{\langle x, \mu_A(x)\rangle : x \in X\}, \tag{1}$$

where $\mu_A : X \to [0,1]$ is the membership function of A and $\mu_A(x)$ is the grade of belongingness of x into A (see [12]). We will denote a family of fuzzy sets in X by $FS(X)$.

The most important subfamily of all fuzzy sets are fuzzy numbers. It is not surprising since the predominant carrier of information are numbers. The notion of a fuzzy number was introduced by Dubois and Prade [4]. Let us recall that definition and some basic concepts related to this notion as a starting point for the generalization given in Sec. 3.

Definition 1
A fuzzy subset A of the real line $\mathcal{R}$ with membership function $\mu_A : \mathcal{R} \to [0,1]$ is called a fuzzy number if
(a) A is normal, i.e. there exist an element x_0 such that $\mu_A(x_0) = 1$
(b) A is fuzzy convex, i.e. $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2)$ $\forall x_1, x_2 \in \mathcal{R}$, $\forall \lambda \in [0,1]$
(c) μ_A is upper semicontinuous
(d) $suppA$ is bounded, where $suppA = cl(\{x \in \mathcal{R} : \mu_A(x) > 0\})$, and cl is the closure operator.

A space of all fuzzy numbers will be denoted by FN. It is known that for any fuzzy number A there exist four numbers $a_1, a_2, a_3, a_4 \in \mathcal{R}$ and two functions $f_A, g_A : \mathcal{R} \to [0,1]$, where f_A is nondecreasing and g_A is nonincreasing, such that we can describe a membership function μ_A in a following manner

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1 \\ f_A(x) & \text{if} \quad a_1 \leq x < a_2 \\ 1 & \text{if} \quad a_2 \leq x \leq a_3 \\ g_A(x) & \text{if} \quad a_3 < x \leq a_4 \\ 0 & \text{if} \quad a_4 < x. \end{cases} \tag{2}$$

Functions f_A and g_A are called the left side and the right side of a fuzzy number A, respectively.

Some authors using this concept in their papers do not quote requirement (d) given above. They just adopt more general assumption

(d′) $\int_{-\infty}^{+\infty} \mu_A(x)dx < +\infty$.

However, others – especially practitioners – argue that (d) is more natural than (d′) since it means that real numbers less than a_1 or greater than a_4 surely do not belong to A. Hence in our paper we adopt (d), although – from the mathematical point of view – (d′) is enough.

A useful tool for dealing with fuzzy numbers are their $\alpha-$cuts. The $\alpha-$cut of a fuzzy number A is a nonfuzzy set defined as

$$A_\alpha = \{x \in R : \mu_A(x) \geq \alpha\}. \tag{3}$$

A family $\{A_\alpha : \alpha \in (0,1]\}$ is a set representation of the fuzzy number A. According to the definition of a fuzzy number it is seen at once that every $\alpha-$cut of a fuzzy number is a closed interval. Hence we have $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where

$$A_L(\alpha) = \inf\{x \in \mathcal{R} : \mu_A(x) \geq \alpha\}, \tag{4}$$

$$A_U(\alpha) = \sup\{x \in \mathcal{R} : \mu_A(x) \geq \alpha\}. \tag{5}$$

If the sides of the fuzzy number A are strictly monotone then by (2) one can see easily that $A_L(\alpha)$ and $A_U(\alpha)$ are inverse functions of f_A and g_A, respectively. In general, we may adopt the convention that $f_A(x)^{-1} = \inf\{x \in \mathcal{R} : \mu_A(x) \geq \alpha\} = A_L(\alpha)$ and $g_A(x)^{-1} = \sup\{x \in \mathcal{R} : \mu_A(x) \geq \alpha\} = A_U(\alpha)$.

Another important notion connected with fuzzy numbers is an expected interval $EI(A)$ of a fuzzy number A, introduced independently by Dubois and Prade [5] and Heilpern [10]. Is is given by

$$EI(A) = [E_*(A), E^*(A)] = \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha\right]. \tag{6}$$

It can be shown that if A is a fuzzy number with continuous and strictly monotone sides f_A and g_A then

$$E_*(A) = a_2 - \int_{a_1}^{a_2} f_A(x)dx, \tag{7}$$

$$E^*(A) = a_3 + \int_{a_3}^{a_4} g_A(x)dx. \tag{8}$$

Let us also recall that a value w_A given by

$$w_A = \int_{-\infty}^{\infty} \mu_A(x)dx, \tag{9}$$

is called the width of the fuzzy number A. It can be shown (see [3]) that

$$W_A = E^*(A) - E_*(A). \tag{10}$$

3 Intuitionistic fuzzy numbers

According to (1) $\mu_A(x)$ denotes the grade of belongingness of x into A. Thus automatically the grade of nonbelongingness of x into A is equal to $1 - \mu_A(x)$. However, in real life the linguistic negation not always identifies with logical negation. This situation is very common in natural language processing, computing with words, etc. Therefore Atanassov [1–2] suggested a generalization of a standard fuzzy set, called an intuitionistic fuzzy set.

An intuitionistic fuzzy set C in X is given by a set of ordered triples

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}, \tag{11}$$

where $\mu_A, \nu_A : X \to [0,1]$ are functions such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \qquad \forall x \in X. \tag{12}$$

For each x the numbers $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and degree of nonmembership of the element $x \in X$ to $A \subset X$, respectively. It is easily seen that an intuitionistic fuzzy set $\{\langle x, \mu_C(x),\ 1 - \mu_C(x)\rangle : x \in X\}$ is equivalent to (1), i.e. each fuzzy set is a particular case of the intuitionistic fuzzy set.

We will denote a family of all intuitionistic fuzzy sets in X by $FS(X)$. For each element $x \in X$ we can compute, so called, the intuitionistic fuzzy index of x in A defined as follows

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \tag{13}$$

which measures the degree of hesitation of whether x belongs to A. Thus function π_A is sometimes also called the hesitation margin of the intuitionistic set A. It is seen immediately that $\pi_A(x) \in [0,1]\ \forall x \in X$. If $A \in FS(X)$ then $\pi_A(x) = 0\ \forall x \in X$.

Atanassov has also defined two kinds of α–cuts for intuitionistic fuzzy sets. Namely

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}, \tag{14}$$

$$A^\alpha = \{x \in X : \nu_A(x) \leq \alpha\}. \tag{15}$$

Since an intuitionistic fuzzy set is a natural generalization of a standard fuzzy set, therefore a definition of intuitionistic fuzzy number should be a direct generalization of a fuzzy number given in Sec. 2. Therefore, from now on our universe of discourse would be the real line, i.e. $X = \mathcal{R}$. However, let us firstly propose some introductory definitions.

Definition 2
An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is if-normal if there exist at least two points $x_0, x_1 \in X$ such that $\mu_A(x_0) = 1$ and $\nu_A(x_1) = 1$).

It is easily seen that given intuitionistic fuzzy set A is if-normal if there is at least one point that surely belongs to A and at least one point which does not belong to A.

Definition 3
An intuitionistic fuzzy subset of the real line $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ is if-convex if $\forall x_1, x_2 \in \mathcal{R},\ \forall\lambda \in [0,1]$

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2) \tag{16}$$

$$\nu_A(\lambda x_1 + (1-\lambda)x_2) \leq \nu_A(x_1) \vee \nu_A(x_2). \tag{17}$$

Thus A is if-convex if its membership function μ is fuzzy convex and its nonmembership function is fuzzy concave.

Definition 4
A support of an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ *is a crisp set* $suppA = cl\,(\{x \in X : \nu_A(x) < 1\})$.

Having in mind all the concepts discussed above we get

Definition 5
An intuitionistic fuzzy subset $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ *of the real line is called an intuitionistic fuzzy number if*
(a) A *is if-normal,*
(b) A *is if-convex,*
(c) μ_A *is upper semicontinuous and* ν_A *is lower semicontinuous,*
(d) $suppA$ *is bounded.*

From the definition given above we get at once that for any intuitionistic fuzzy number A there exist eight numbers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathcal{R}$ such that $b_1 \leq a_1 \leq b_2 \leq a_2 \leq a_3 \leq b_3 \leq a_4 \leq b_4$ and four functions $f_A, g_A, h_A, k_A : \mathcal{R} \to [0,1]$, called the sides of a fuzzy number, where f_A and k_A are nondecreasing and g_A and h_A are nonincreasing, such that we can describe a membership function μ_A in a form

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1 \\ f_A(x) & \text{if} \quad a_1 \leq x < a_2 \\ 1 & \text{if} \quad a_2 \leq x \leq a_3 \\ g_A(x) & \text{if} \quad a_3 < x \leq a_4 \\ 0 & \text{if} \quad a_4 < x. \end{cases}, \tag{18}$$

while a nonmembership function ν_A has a following form

$$\nu_A(x) = \begin{cases} 1 & \text{if} \quad x < b_1 \\ h_A(x) & \text{if} \quad b_1 \leq x < b_2 \\ 0 & \text{if} \quad b_2 \leq x \leq b_3 \\ k_A(x) & \text{if} \quad b_3 < x \leq b_4 \\ 1 & \text{if} \quad b_4 < x. \end{cases} \tag{19}$$

It is worth noting that each intuitionistic fuzzy number $A = \{\langle x, \mu_A(x),\ \nu_A(x)\rangle : x \in \mathcal{R}\}$ is a conjunction of two fuzzy numbers: A^+ with a membership function $\mu_{A^+}(x) = \mu_A(x)$ and A^- with a membership function $\mu_{A^-}(x) = 1 - \nu_A(x)$. It is seen that $suppA^+ \subseteq suppA^-$. Moreover, $EI(A^+) \subseteq EI(A^-)$, where $EI(A^+) = [E_*(A^+), E^*(A^+)]$, $EI(A^-) = [E_*(A^-), E^*(A^-)]$ and $E_*(A^+)$, $E^*(A^+)$ are given by (8), while

$$E_*(A^-) = b_2 - \int_{b_1}^{b_2} (1 - h_A(x))dx = b_1 + \int_{b_1}^{b_2} h_A(x)dx, \tag{20}$$

$$E^*(A^-) = b_3 + \int_{b_3}^{b_4} (1 - k_A(x))dx = b_4 - \int_{b_3}^{b_4} k_A(x)dx. \tag{21}$$

Hence we get

Definition 6
An expected interval of an intuitionistic fuzzy number $A = \{\langle x, \mu_A(x), \ \nu_A(x)\rangle : x \in \mathcal{R}\}$ is a crisp interval $\widetilde{EI}(A)$ given by

$$\begin{aligned} \widetilde{EI}(A) &= \left[\widetilde{E_*}(A), \widetilde{E^*}(A)\right] \qquad (22) \\ &= \left[\frac{E_*(A^-) + E_*(A^+)}{2}, \frac{E^*(A^-) + E^*(A^+)}{2}\right] \end{aligned}$$

One may compute that

$$\widetilde{E_*}(A) = \frac{b_1 + a_2}{2} + \frac{1}{2}\int_{b_1}^{b_2} h_A(x)dx - \frac{1}{2}\int_{a_1}^{a_2} f_A(x)dx, \tag{23}$$

$$\widetilde{E^*}(A) = \frac{a_3 + b_4}{2} + \frac{1}{2}\int_{a_3}^{a_4} g_A(x)dx - \frac{1}{2}\int_{b_3}^{b_4} k_A(x)dx. \tag{24}$$

Let us also adopt a generalization of the width, given by (9).

Definition 7
A width of an intuitionistic fuzzy number $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ is a real number given by

$$w_A = \frac{1}{2}\left[\int_{-\infty}^{\infty} \mu_A(x)dx + \int_{-\infty}^{\infty} (1 - \nu_A(x))dx\right]. \tag{25}$$

One may easily seen that

$$w_A = \frac{w_{A^+} + w_{A^-}}{2}. \tag{26}$$

It can be shown, that similarly as in the case of fuzzy numbers, the width of intuitionistic fuzzy number is equal to the length of the expected interval corresponding to this intuitionistic fuzzy number. Namely

Lemma 8
Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ be an intuitionistic fuzzy number. Then

$$w_A = \widetilde{E^*}(A) - \widetilde{E_*}(A). \tag{27}$$

Proof:
Substituting (18) and (19) to (25) we get

$$\begin{aligned}
w_A &= \frac{1}{2}\left[\int_{-\infty}^{\infty} \mu_A(x)dx + \int_{-\infty}^{\infty} (1-\nu_A(x))dx\right] \\
&= \frac{1}{2}\left[\int_{a_1}^{a_2} f_A(x)dx + a_3 - a_2 + \int_{a_3}^{a_4} g_A(x)dx \right.\\
&\quad \left. + \int_{b_1}^{b_2}(1-h_A(x))dx + b_3 - b_2 + \int_{b_3}^{b_4}(1-k_A(x))dx\right] \\
&= \frac{a_3+b_4}{2} + \frac{1}{2}\int_{a_3}^{a_4} g_A(x)dx - \frac{1}{2}\int_{b_3}^{b_4} k_A(x)dx \\
&\quad - \left[\frac{b_1+a_2}{2} + \frac{1}{2}\int_{b_1}^{b_2} h_A(x)dx - \frac{1}{2}\int_{a_1}^{a_2} f_A(x)dx\right] \\
&= \widetilde{E^*}(A) - \widetilde{E_*}(A),
\end{aligned}$$

which establishes the formula. ■

As it was mentioned above, a useful tool for dealing with fuzzy numbers are their $\alpha-$cuts. In the case of intuitionistic fuzzy numbers it is convenient to distinguish following $\alpha-$cuts: $(A^+)_\alpha$ and $(A^-)_\alpha$. It is easily seen that

$$(A^+)_\alpha = \{x \in X : \mu_A(x) \geq \alpha\} = A_\alpha, \tag{28}$$

$$\begin{aligned}(A^-)_\alpha &= \{x \in X : 1-\nu_A(x) \geq \alpha\} \\ &= \{x \in X : \nu_A(x) \leq 1-\alpha\} = A^{1-\alpha}.\end{aligned} \tag{29}$$

According to the definition it is seen at once that every $\alpha-$cut $(A^+)_\alpha$ or $(A^-)_\alpha$ is a closed interval. Hence we have $(A^+)_\alpha = [A_L^+(\alpha), A_U^+(\alpha)]$ and $(A^-)_\alpha = [A_L^-(\alpha), A_U^-(\alpha)]$, respectively, where

$$A_L^+(\alpha) = \inf\{x \in \mathcal{R} : \mu_A(x) \geq \alpha\}, \tag{30}$$

$$A_U^+(\alpha) = \sup\{x \in \mathcal{R} : \mu_A(x) \geq \alpha\}, \tag{31}$$

$$A_L^-(\alpha) = \inf\{x \in \mathcal{R} : \nu_A(x) \leq 1-\alpha\}, \tag{32}$$

$$A_U^-(\alpha) = \sup\{x \in \mathcal{R} : \nu_A(x) \leq 1-\alpha\}. \tag{33}$$

If the sides of the fuzzy number A are strictly monotone then by (18) and (19) one can see easily that $A_L^+(\alpha)$, $A_U^+(\alpha)$, $A_L^-(\alpha)$ and $A_U^-(\alpha)$ are inverse functions of f_A, g_A, h_A and k_A, respectively. In general, we may adopt the convention that $f_A^{-1}(\alpha) = A_L^+(\alpha)$, $g_A^{-1}(\alpha) = A_U^+(\alpha)$, $h_A^{-1}(\alpha) = A_L^-(\alpha)$ and $k_A^{-1}(\alpha) = A_U^-(\alpha)$.

4 Distances between intuitionistic fuzzy numbers

Various methods for measuring distances between intuitionistic fuzzy sets are considered in the literature (see, e.g., [2], [7], [8], [11]). Below we suggest two families of metrics that seem to be useful for measuring distances between intuitionistic fuzzy numbers (see also [8]). We get

Definition 9
The $d_p(A,B)$ distance, indexed by a parameter $1 \le p \le \infty$, for any two intuitionistic fuzzy numbers $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in \mathcal{R}\}$ is given by

$$d_p(A,B) = \left(\frac{1}{4}\int_0^1 \left|A_L^+(\alpha) - B_L^+(\alpha)\right|^p d\alpha + \frac{1}{4}\int_0^1 \left|A_U^+(\alpha) - B_U^+(\alpha)\right|^p d\alpha + \frac{1}{4}\int_0^1 \left|A_L^-(\alpha) - B_L^-(\alpha)\right|^p d\alpha + \frac{1}{4}\int_0^1 \left|A_U^-(\alpha) - B_U^-(\alpha)\right|^p d\alpha\right)^{1/p} \quad (34)$$

for $1 \le p < \infty$ and

$$d_p(A,B) = \frac{1}{4}\sup_{0<\alpha\le 1}\left|A_L^+(\alpha) - B_L^+(\alpha)\right| + \frac{1}{4}\sup_{0<\alpha\le 1}\left|A_U^+(\alpha) - B_U^+(\alpha)\right| + \frac{1}{4}\sup_{0<\alpha\le 1}\left|A_L^-(\alpha) - B_L^-(\alpha)\right| + \frac{1}{4}\sup_{0<\alpha\le 1}\left|A_U^-(\alpha) - B_U^-(\alpha)\right| \quad (35)$$

for $p = \infty$.

Definition 10
The $\rho_p(A,B)$ distance, indexed by a parameter $1 \le p \le \infty$, for any two intuitionistic fuzzy numbers $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in \mathcal{R}\}$ is given by

$$\rho_p(A,B) = \max\left\{\sqrt[p]{\int_0^1 \left|A_L^+(\alpha) - B_L^+(\alpha)\right|^p d\alpha}, \sqrt[p]{\int_0^1 \left|A_U^+(\alpha) - B_U^+(\alpha)\right|^p d\alpha}, \sqrt[p]{\int_0^1 \left|A_L^-(\alpha) - B_L^-(\alpha)\right|^p d\alpha}, \sqrt[p]{\int_0^1 \left|A_U^-(\alpha) - B_U^-(\alpha)\right|^p d\alpha}\right\} \quad (36)$$

for $1 \le p < \infty$ and

$$\rho_p(A,B) = \max\left\{\sup_{0<\alpha\le 1}\left|A_L^+(\alpha) - B_L^+(\alpha)\right|, \sup_{0<\alpha\le 1}\left|A_U^+(\alpha) - B_U^+(\alpha)\right|, \sup_{0<\alpha\le 1}\left|A_L^-(\alpha) - B_L^-(\alpha)\right|, \sup_{0<\alpha\le 1}\left|A_U^-(\alpha) - B_U^-(\alpha)\right|\right\} \quad (37)$$

for $p = \infty$.

Let IFN' denote a space of all intuitionistic fuzzy numbers. We may partition IFN' into disjoint sets in such a way that two intuitionistic fuzzy numbers A and B belong to the same set if and only if the corresponding functions $A_L^+(\alpha)$, $A_U^+(\alpha)$, $A_L^-(\alpha)$, $A_U^-(\alpha)$ and $B_L^+(\alpha)$, $B_U^+(\alpha)$, $B_L^-(\alpha)$, $B_U^-(\alpha)$ differ only on a set of measure zero. This way we obtain a space IFN of equivalence classes. It is not misleading that we regard elements of the space IFN as intuitionistic fuzzy numbers and by integrals in (34) and (36) we mean the integrals of arbitrary representative of the class containing A.

A following theorem holds

Theorem 11
Spaces (IFN, d_p) and (IFN, ρ_p) for $1 \leq p \leq \infty$ are metric spaces.

The proof is standard.

5 Ranking intuitionistic fuzzy numbers

It is known that there is no unique linear ordering in a family of fuzzy numbers. Thus ranking fuzzy numbers is one of the fundamental problems of fuzzy arithmetic. The same is true in the case of intuitionistic fuzzy numbers. Below we suggest a method of ranking intuitionistic fuzzy numbers based on metrics introduced in the previous section. This method is a direct generalization of the method for ranking classical fuzzy numbers presented in [6].

Let us start from some definitions. Suppose $\mathcal{A} \subset IFN$ is a subfamily of all intuitionistic fuzzy numbers.

Definition 12
An intuitionistic fuzzy number $L(\mathcal{A})$ is called the lower horizon of a given subfamily $\mathcal{A}$ if $\sup(suppL(\mathcal{A})) \leq \inf(suppA)$ for any $A \in \mathcal{A}$. Similarly, an intuitionistic fuzzy number $U(\mathcal{A})$ is called the upper horizon of a given subfamily $\mathcal{A}$ if $\inf(suppU(\mathcal{A})) \geq \sup(suppA)$ for any $A \in \mathcal{A}$.

It is obvious that $\mathcal{A}$ may have one or more horizons. For a fixed intuitionistic fuzzy number we may consider following subfamilies of IFN:

Definition 13
Let $H \in IFN$. A subfamily $\mathcal{H}_L(H) \subset IFN$ of a form

$$\mathcal{H}_L(H) = \{A \in IFN : \sup(suppH) \leq \inf(suppA)\} \tag{38}$$

is said to be lower-dominated by the intuitionistic fuzzy number H. Similarly, a subfamily $\mathcal{H}_U(H) \subset IFN$ of a form

$$\mathcal{H}_U(H) = \{A \in IFN : \inf(suppH) \geq \sup(suppA)\} \tag{39}$$

is said to be upper-dominated by the intuitionistic fuzzy number H.

The proof of the following lemma is straightforward.

Theorem 14
Let $H \in IFN$. Then $H = L(\mathcal{H}_L(H))$ and $H = U(\mathcal{H}_U(H))$.

Now we may propose two following orders:

Definition 15
Let $A, B \in \mathcal{A} \subset IFN$. Moreover, let $H = L(\mathcal{A})$ and let d be a metric in IFN. The relation $\succ_L$ in $\mathcal{A} \times \mathcal{A}$ given by

$$A \succ_L B \Longleftrightarrow d(A, H) \geq d(B, H) \tag{40}$$

is called the order respect to the lower horizon H.

Definition 16
Let $A, B \in \mathcal{A} \subset IFN$. Moreover, let $H = U(\mathcal{A})$ and let d be a metric in IFN. The relation $\succ_U$ in $\mathcal{A} \times \mathcal{A}$ given by

$$A \succ_U B \Longleftrightarrow d(A, H) \leq d(B, H) \tag{41}$$

is called the order respect to the upper horizon H.

Of course, using different metrics, e.g. d_p or ρ_p given above, we may obtain different orders. Anyway, the following theorem holds

Theorem 17
Let $H \in IFN$. Then $\mathcal{H}_L(H)$ is quasi-ordered by the relation $\succ_L$, while $\mathcal{H}_U(H)$ is quasi-ordered by the relation $\succ_U$. Moreover, both relations $\succ_L$ and $\succ_U$ are connected.

Proof:
Let $A, B \in \mathcal{H}_L(H)$. By Theorem 14 $H = L(\mathcal{H}_L(H))$. Thus by Definition 15

$$A \succ_L B \Longleftrightarrow d(A, H) \geq d(B, H),$$

where d is a given metric. We need to show that the relation is reflexive and transitive.

(a) Reflexivity: $A \succ_L B \quad \forall A \in \mathcal{H}_L(H)$, because $d(A, H) \geq d(A, H)$;

(b) Transitivity: $\forall A, B, C \in \mathcal{H}_L(H)$

$$\begin{aligned} & [(A \succ_L B \;\; \& \;\; B \succ_L C) \Rightarrow A \succ_L C] \\ \Longleftrightarrow \quad & [(d(A, H) \geq d(B, H) \;\; \& \;\; d(B, H) \geq d(C, H)) \Rightarrow d(A, H) \geq d(C, H)], \end{aligned}$$

which holds, because d is transitive as a metric. The relation $\succ_L$ is reflexive and transitive, thus $\succ_L$ quasi-orders the set $\mathcal{H}_L(H)$.

This relation is also connected: $\forall A, B \in \mathcal{H}_L(H)$

$$(A \succ_L B \text{ or } B \succ_L A) \Longleftrightarrow [d(A, H) \geq d(B, H) \text{ or } d(B, H) \geq d(A, H)],$$

which is true, because the relation $\geq$ in real numbers is connected.

In the same way we can show that the relation $\succ_U$ is a connected quasi-ordering in the set $\mathcal{H}_U(H)$. Hence the proof is completed. ■

It should be noticed that both relations $\succ_L$ and $\succ_U$ are not antisymmetric and hence they are only quasi-ordering relations, not ordering relations. However, every quasi-ordering determines an equivalence relation and an ordering relation (on equivalence classes) in a natural way. Therefore the last theorem is of great importance since it makes possible to rank any subfamily of intuitionistic fuzzy numbers which is lower-dominated or upper-dominated that is very common in practical applications.

Now we show some properties of the proposed quasi-orderings based on the metrics introduced in Sec. 4. Namely

Theorem 18

The quasi-order $\succ_L$ with respect to the lower horizon, based on the metric d_1 (i.e. d_p for $p=1$), does not depend on the choice of the lower horizon. Similarly, the quasi-order $\succ_U$ with respect to the upper horizon, based on the metric d_1, does not depend on the choice of the upper horizon.

Proof:

It suffices to show that the quasi-order $\succ_L$ based on the metric d_1 does not depend on the choice of the lower horizon H. The proof for $\succ_U$ is analogous.

Let $\mathcal{A}$ denote a subset of a family of intuitionistic fuzzy numbers. Let $A, B \in \mathcal{A}$ and let $\mathcal{A}_L$ denote the set of all lower horizons of $\mathcal{A}$. Our objective is to show that if there exist such $H \in \mathcal{A}_L$ that $A \succ_L B$ then $A \succ_L B$ for all $H \in \mathcal{A}_L$.

Suppose $A \succ_L B$ for a fixed $H \in \mathcal{A}_L$. Thus we have

$$\begin{aligned} A \succ_L B &\iff d_1(A,H) \geq d_1(B,H) \\ &\iff d_1(A,H) - d_1(B,H) = \Delta(H) \geq 0. \end{aligned}$$

Since $A, B \in \mathcal{H}_L(H)$ thus by (34) we get

$$\begin{aligned} 4\Delta(H) &= \int_0^1 \left|A_L^+(\alpha) - H_L^+(\alpha)\right| d\alpha + \int_0^1 \left|A_U^+(\alpha) - H_U^+(\alpha)\right| d\alpha \\ &+ \int_0^1 \left|A_L^-(\alpha) - H_L^-(\alpha)\right| d\alpha + \int_0^1 \left|A_U^-(\alpha) - H_U^-(\alpha)\right| d\alpha \\ &- \left[\int_0^1 \left|B_L^+(\alpha) - H_L^+(\alpha)\right| d\alpha + \int_0^1 \left|B_U^+(\alpha) - H_U^+(\alpha)\right| d\alpha \right. \\ &\left. \int_0^1 \left|B_L^-(\alpha) - H_L^-(\alpha)\right| d\alpha + \int_0^1 \left|B_U^-(\alpha) - H_U^-(\alpha)\right| d\alpha \right] \end{aligned}$$

$$
\begin{aligned}
= & \int_0^1 (A_L^+(\alpha) - H_L^+(\alpha))d\alpha + \int_0^1 (A_U^+(\alpha) - H_U^+(\alpha))d\alpha \\
& + \int_0^1 (A_L^-(\alpha) - H_L^-(\alpha))d\alpha + \int_0^1 (A_U^-(\alpha) - H_U^-(\alpha))d\alpha \\
& - \left[\int_0^1 (B_L^+(\alpha) - H_L^+(\alpha))d\alpha + \int_0^1 (B_U^+(\alpha) - H_U^+(\alpha))d\alpha \right. \\
& \left. \int_0^1 (B_L^-(\alpha) - H_L^-(\alpha))d\alpha + \int_0^1 (B_U^-(\alpha) - H_U^-(\alpha))d\alpha \right]
\end{aligned}
$$

$$
\begin{aligned}
= & \int_0^1 (A_L^+(\alpha) - B_L^+(\alpha))d\alpha + \int_0^1 (A_U^+(\alpha) - B_U^+(\alpha))d\alpha \qquad (42) \\
& + \int_0^1 (A_L^-(\alpha) - B_L^-(\alpha))d\alpha + \int_0^1 (A_U^-(\alpha) - B_U^-(\alpha))d\alpha
\end{aligned}
$$

so $\Delta(H)$ does not depend on H. Hence $A \succ_L B$ for all $H \in \mathcal{A}_L$. If there is no such H that $A \succ_L B$, we have $B \succ_L A \;\; \forall H \in \mathcal{A}_L$, because $\succ_L$ is a connected quasi-order (Th. 17), which completes the proof. ■

Theorem 19

The quasi-orders $\succ_L$ with respect to the lower horizon, based on the metric d_∞, ρ_1 and ρ_∞, do not depend on the choice of the lower horizon, provided that the horizon is a crisp number. Similarly, the quasi-orders $\succ_U$ with respect to the upper horizon, based on the metric d_∞, ρ_1 and ρ_∞, do not depend on the choice of the upper horizon, provided that the horizon is a crisp number.

Proof:

Similarly as before, we deal only with the lower horizon H.

Let $\mathcal{A}$ denote a subset of intuitionistic fuzzy numbers. Let $A, B \in \mathcal{A}$ and let $\mathcal{A}_L$ denote the set of all lower horizons of $\mathcal{A}$. Let $\mathcal{C}_L \subset \mathcal{A}_L$ denote a subset of all lower horizons which are crisp numbers, i.e. $\mathcal{C}_L = \{H \in \mathcal{A}_L : H_L^+(\alpha) = H_U^+(\alpha) = h = const., \; H_L^-(\alpha) = H_U^-(\alpha) = h = const.\}$. Our goal is to show, that if $\succ_L$ is the quasi-order based on a metric d, where $d \in \{d_\infty, \rho_1, \rho_\infty\}$, and if there exist such $H \in \mathcal{C}_L$ that $A \succ_L B$ then $A \succ_L B$ for all $H \in \mathcal{C}_L$.

Suppose $A \succ_L B$ for a fixed $H \in \mathcal{C}_L$, where $\succ_L$ is the quasi-order based on a metric d. Thus we have

$$
\begin{aligned}
A \succ_L B & \iff d(A,H) \geq d(B,H) \\
& \iff d(A,H) - d(B,H) = \Delta(H) \geq 0.
\end{aligned}
$$

(a) Let $d = d_\infty$. If $H \in \mathcal{C}_L$ then by (35) we get

$$\begin{aligned}
4\Delta(H) &= d_\infty(A,H) - d_\infty(B,H) \\
&= \sup_{0<\alpha\leq 1} \left|A_L^+(\alpha) - h\right| + \sup_{0<\alpha\leq 1} \left|A_U^+(\alpha) - h\right| + \sup_{0<\alpha\leq 1} \left|A_L^-(\alpha) - h\right| \\
&\quad + \sup_{0<\alpha\leq 1} \left|A_U^-(\alpha) - h\right| - \left[\sup_{0<\alpha\leq 1} \left|B_L^+(\alpha) - h\right| + \sup_{0<\alpha\leq 1} \left|B_U^+(\alpha) - h\right|\right. \\
&\quad \left.\sup_{0<\alpha\leq 1} \left|B_L^-(\alpha) - h\right| + \sup_{0<\alpha\leq 1} \left|B_U^-(\alpha) - h\right|\right] \\
&= \left|A_L^+(0) - h\right| + \left|A_U^+(0) - h\right| + \left|A_L^-(1) - h\right| + \left|A_U^-(1) - h\right| \\
&\quad - \left[\left|B_L^+(0) - h\right| + \left|B_U^+(0) - h\right| + \left|B_L^-(1) - h\right| + \left|B_U^-(1) - h\right|\right] \\
&= (A_L^+(0) - h) + (A_U^+(0) - h) + (A_L^-(1) - h) + (A_U^-(1) - h) \\
&\quad - \left[(B_L^+(0) - h) + (B_U^+(0) - h) + (B_L^-(1) - h) + (B_U^-(1) - h)\right] \\
&= (A_L^+(0) - B_L^+(0)) + (A_U^+(0) - B_U^+(0)) \\
&\quad + (A_L^-(1) - B_L^-(1)) + (A_U^-(1) - B_U^-(1))
\end{aligned}$$

so $\Delta(H)$ does not depend on H.

(b) Let $d = \rho_1$. If $H \in \mathcal{C}_L$ then by (36) we get

$$\begin{aligned}
\rho_1(A,H) &= \max\left\{\int_0^1 \left|A_L^+(\alpha) - h\right| d\alpha, \int_0^1 \left|A_U^+(\alpha) - h\right| d\alpha,\right. \\
&\quad \left.\int_0^1 \left|A_L^-(\alpha) - h\right| d\alpha, \int_0^1 \left|A_U^-(\alpha) - h\right| d\alpha\right\}
\end{aligned}$$

However, it is obvious that for $H \in \mathcal{C}_L$

$$\begin{aligned}
\int_0^1 \left|A_L^+(\alpha) - h\right| d\alpha &\leq \int_0^1 \left|A_U^+(\alpha) - h\right| d\alpha, \\
\int_0^1 \left|A_L^-(\alpha) - h\right| d\alpha &\leq \int_0^1 \left|A_U^-(\alpha) - h\right| d\alpha.
\end{aligned}$$

Moreover, since $\mu_A(x) \leq 1 - \nu_A(x)$ hence $A_U^+(1-\alpha) \leq A_U^-(\alpha)$ and therefore

$$\int_0^1 \left|A_U^+(\alpha) - h\right| d\alpha \leq \int_0^1 \left|A_U^-(\alpha) - h\right| d\alpha.$$

Thus finally

$$\rho_1(A,H) = \int_0^1 (A_U^-(\alpha) - h) d\alpha.$$

So we have

$$\begin{aligned}\Delta(H) &= \rho_1(A,H)-\rho_1(B,H)\\ &= \int_0^1 (A_U^-(\alpha)-h)d\alpha-\int_0^1 (B_U^-(\alpha)-h)d\alpha\\ &= \int_0^1 (A_U^-(\alpha)-B_U^-(\alpha))d\alpha\end{aligned}$$

and $\Delta(H)$ does not depend on H again.

(c) Now let $d=\rho_\infty$. If $H\in\mathcal{C}_L$ then by (37) we get

$$\begin{aligned}\rho_\infty(A,H) &= \max\Big\{\sup_{0<\alpha\le 1}\left|A_L^+(\alpha)-h\right|, \sup_{0<\alpha\le 1}\left|A_U^+(\alpha)-h\right|,\\ &\qquad \sup_{0<\alpha\le 1}\left|A_L^-(\alpha)-h\right|, \sup_{0<\alpha\le 1}\left|A_U^-(\alpha)-h\right|\Big\}\\ &= \max\left\{\left|A_L^+(0)-h\right|,\left|A_U^+(0)-h\right|,\left|A_L^-(1)-h\right|,\left|A_U^-(1)-h\right|\right\}\\ &= \max\left\{A_L^+(0)-h,\ A_U^+(0)-h,\ A_L^-(1)-h,\ A_U^-(1)-h\right\}.\end{aligned}$$

However, it is obvious that

$$\begin{aligned}A_L^+(0) &\le A_U^+(0),\\ A_L^-(1) &\le A_U^-(1).\end{aligned}$$

Moreover, since $\mu_A(x)\le 1-\nu_A(x)$ hence $A_U^+(1-\alpha)\le A_U^-(\alpha)$ and therefore

$$A_U^+(0)\le A_U^-(1).$$

Thus finally

$$\rho_\infty(A,H)=A_U^-(1)-h.$$

Hence

$$\begin{aligned}\Delta(H) &= \rho_\infty(A,H)-\rho_\infty(B,H)\\ &= A_U^-(1)-h-\left[B_U^-(1)-h\right]\\ &= A_U^-(1)-B_U^-(1)\end{aligned}$$

and $\Delta(H)$ does not depend on H.

Thus it is easily seen that $A\succ_L B$ $\forall H\in\mathcal{C}_L$ for any quasi-order $\succ_L$ based on $d\in\{d_\infty,\rho_1,\rho_\infty\}$. Since $\succ_L$ is a connected quasi-order, if there is no such H that $A\succ_L B$, we have $B\succ_L A$ $\forall H\in\mathcal{C}_L$ and the proof is completed. ■

In Sec. 3 we have defined the notion of the expected interval of an intuitionistic fuzzy number. As in the case of the standard fuzzy numbers (compare [5], [10]) we may also define the expected value of intuitionistic fuzzy number:

Definition 20
The expected value $\widetilde{EV}(A)$ *of an intuitionistic fuzzy number* $A = \{\langle x, \ \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ *is the center of the expected interval of that intuitionistic fuzzy number, i.e.*

$$\widetilde{EV}(A) = \frac{\widetilde{E_*}(A) + \widetilde{E^*}(A)}{2}. \tag{43}$$

The expected value of an intuitionistic fuzzy number might be used in ordering intuitionistic fuzzy numbers. Namely, a following theorem holds

Theorem 21
Let $\succ_L$ *and* $\succ_U$ *denote the quasi-order with respect to the lower and upper horizon, respectively, based on the metric* d_1 *(i.e.* d_p *for* $p = 1$*). Then for any* $A, B \in IFN$ *we get*

$$A \succ_L B \Longleftrightarrow \widetilde{EV}(A) \geq \widetilde{EV}(B) \tag{44}$$

and

$$A \succ_U B \Longleftrightarrow \widetilde{EV}(A) \geq \widetilde{EV}(B). \tag{45}$$

Proof:
By (23) and (24) we see that

$$\widetilde{E_*}(A) = \frac{b_1 + a_2}{2} + \frac{1}{2}\int_{b_1}^{b_2} h_A(x)dx - \frac{1}{2}\int_{a_1}^{a_2} f_A(x)dx \tag{46}$$

$$= \frac{1}{2}\int_0^1 (A_L^+(\alpha) + A_L^-(\alpha))d\alpha, \tag{47}$$

$$\widetilde{E^*}(A) = \frac{a_3 + b_4}{2} + \frac{1}{2}\int_{a_3}^{a_4} g_A(x)dx - \frac{1}{2}\int_{b_3}^{b_4} k_A(x)dx \tag{48}$$

$$= \frac{1}{2}\int_0^1 (A_U^+(\alpha) + A_U^-(\alpha))d\alpha, \tag{49}$$

and finally

$$\widetilde{EV}(A) = \frac{1}{2}\int_0^1 (A_L^+(\alpha) + A_L^-(\alpha) + A_U^+(\alpha) + A_U^-(\alpha))d\alpha.$$

By Def. 15

$$A \succ_L B \Longleftrightarrow d_1(A, H) \geq d_1(B, H),$$

where H is a fixed lower horizon. However, Th. 18 shows that the quasi-order $\succ_L$ based on metric d_1 does not depend on H and by (42) our condition is equivalent to the following:

$$A \succ_L B \iff \int_0^1 (A_L^+(\alpha) - B_L^+(\alpha))d\alpha + \int_0^1 (A_U^+(\alpha) - B_U^+(\alpha))d\alpha$$
$$+ \int_0^1 (A_L^-(\alpha) - B_L^-(\alpha))d\alpha + \int_0^1 (A_U^-(\alpha) - B_U^-(\alpha))d\alpha \geq 0$$

$$\iff \int_0^1 (A_L^+(\alpha) + A_U^+(\alpha) + A_L^-(\alpha) + A_U^-(\alpha))d\alpha$$
$$- \int_0^1 (B_L^+(\alpha) + B_U^+(\alpha) + B_L^-(\alpha) + B_U^-(\alpha))d\alpha \geq 0$$
$$\iff 2\left(\widetilde{EV}(A) - \widetilde{EV}(B)\right) \geq 0,$$

which is a desired conclusion.

Now, by Def. 16

$$A \succ_U B \iff d_1(A, H) \leq d_1(B, H),$$

where H is a fixed upper horizon. By Th. 18 we know that the quasi-order $\succ_U$ based on metric d_1 does not depend on H and thus we get:

$$A \succ_U B \iff \int_0^1 \left|A_L^+(\alpha) - H_L^+(\alpha)\right| d\alpha + \int_0^1 \left|A_U^+(\alpha) - H_U^+(\alpha)\right| d\alpha$$
$$+ \int_0^1 \left|A_L^-(\alpha) - H_L^-(\alpha)\right| d\alpha + \int_0^1 \left|A_U^-(\alpha) - H_U^-(\alpha)\right| d\alpha$$
$$- \left[\int_0^1 \left|B_L^+(\alpha) - H_L^+(\alpha)\right| d\alpha + \int_0^1 \left|B_U^+(\alpha) - H_U^+(\alpha)\right| d\alpha\right.$$
$$\left.\int_0^1 \left|B_L^-(\alpha) - H_L^-(\alpha)\right| d\alpha + \int_0^1 \left|B_U^-(\alpha) - H_U^-(\alpha)\right| d\alpha\right] \leq 0$$

$$\begin{aligned}
\iff\ & \int_0^1 (H_L^+(\alpha) - A_L^+(\alpha))d\alpha + \int_0^1 (H_U^+(\alpha) - A_U^+(\alpha))d\alpha \\
& + \int_0^1 (H_L^-(\alpha) - A_L^-(\alpha))d\alpha + \int_0^1 (H_U^-(\alpha) - A_U^-(\alpha))d\alpha \\
& - \left[\int_0^1 (H_L^+(\alpha) - B_L^+(\alpha))d\alpha + \int_0^1 (H_U^+(\alpha) - B_U^+(\alpha))d\alpha \right. \\
& \left. \int_0^1 (H_L^-(\alpha) - B_L^-(\alpha))d\alpha + \int_0^1 (H_U^-(\alpha) - B_U^-(\alpha))d\alpha \right] \le 0 \\
\iff\ & \int_0^1 (A_L^+(\alpha) - B_L^+(\alpha))d\alpha + \int_0^1 (A_U^+(\alpha) - B_U^+(\alpha))d\alpha \\
& + \int_0^1 (A_L^-(\alpha) - B_L^-(\alpha))d\alpha + \int_0^1 (A_U^-(\alpha) - B_U^-(\alpha))d\alpha \ge 0 \\
\iff\ & \int_0^1 (A_L^+(\alpha) + A_U^+(\alpha) + A_L^-(\alpha) + A_U^-(\alpha))d\alpha \\
& - \int_0^1 (B_L^+(\alpha) + B_U^+(\alpha) + B_L^-(\alpha) + B_U^-(\alpha))d\alpha \ge 0 \\
\iff\ & 2\left(\widetilde{EV}(A) - \widetilde{EV}(B)\right) \ge 0,
\end{aligned}$$

which is a desired conclusion and therefore the whole theorem is proved. ■

As a natural consequence of Th. 18 and Th. 21 we get

Corollary 22
Quasi-order $\succ_L$ and $\succ_U$ with respect to the lower and upper horizon, respectively, based on the metric d_1, do not depend on the choice of any horizon. Moreover, these two quasi-orders are equivalent, i.e.

$$A \succ_L B \iff A \succ_U B. \tag{50}$$

6 Conclusions

In this paper we have defined intuitionistic fuzzy numbers and we have suggested how to generalize the concept of expected interval defined on a family of fuzzy numbers into a family of all intuitionistic fuzzy numbers. This notion seems to be very important in many applications of fuzzy numbers, like in ordering fuzzy numbers, measuring distances between fuzzy numbers, measuring correlation between fuzzy numbers, etc.

Moreover, we have introduced two families of metrics in a space of intuitionistic fuzzy numbers. We have also proposed a method of ranking intuitionistic fuzzy numbers based on the suggested metrics and considered some properties of these methods. Quasi-orderings based on the metric d_1 are of a special interest because of the close relationship with the concept of expected value of an intuitionistic fuzzy number. It is worth noting that these results are direct generalizations of the results obtained for the classical fuzzy numbers.

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