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Properties of estimators of the preference relation based on pairwise comparisons – simulation survey

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Abstract

The paper presents results of simulation survey aimed at determination of properties of the estimators of the preference relation, based on pairwise comparisons disturbed by random errors, proposed by the author (see Klukowski [2] – [8]). The properties characterize precision of estimates and cannot be determined in analytical way. The survey confirms excellent efficiency of the estimators and allows determining of appropriate parameters of sample, especially number of comparisons of each pair, for given (or assumed) distributions of comparisons errors.

Keywords: estimation of the preference relation, binary pairwise comparisons, multivalent pairwise comparisons

1 Introduction

The properties of the estimators of three relations (preference, equivalence, tolerance) presented in Klukowski 2010, obtained in analytical way, do not comprise typical measures of precision of estimates, especially: frequency of errorless estimate, average error and distribution of estimation error. These features have been determined for the estimators of the preference relation, based on binary and multivalent comparisons, with the use of simulation survey. The results of the survey allow determining: the speed of convergence of estimates to actual relation form and number of comparisons N, of each pair, guaranteeing necessary precision.

The following results have been obtained in this area. The set of simulations comprises 90 cases: three relations form (determined on the set including nine elements), i.e. $\{x_1\}, ..., \{x_9\}$; $\{x_1\}, \{x_2, x_3\}, \{x_4\}$, $\{x_5, x_6\}, \{x_7\}, \{x_8, x_9\}$; $\{x_1, x_2\}$, $\{x_3, x_4, x_5\}$, $\{x_6, x_7, x_8, x_9\}$, three forms of distributions of comparison errors, two types of comparisons (binary and multivalent), five values of number of comparisons N, i. e. 1, 3, 5, 7, 9. The precision of estimates has been characterized by the measures expressing: fraction of errorless estimates, average errors of estimates and distributions of errors. The conclusions from the survey allow determining the parameters, especially the number of comparisons N, guaranteeing required precision of estimates.

The idea of estimators proposed by the author is based on the concept of nearest adjoining order (see Slater 1961, David 1988, section 2.1). However, the assumptions about distributions of comparisons errors, made by the author, are weaker than those in the papers mentioned.

The paper is organized as follows. The second section comprises definitions and notations. In the third section are presented: the concept of survey (assumptions, purpose) and its parameters. The main results - simulations and their interpretation are presented in fourth section. Last section summarizes the results.

2 Definitions and notations

The problem of estimation of the preference relation, on the basis of pairwise comparisons with random errors, can be formulated in the following way.

Given the set $\mathbf{X} = \{x_1, ..., x_m\}$ ($m \geq 3$); there exists a complete preference relation R in the set $\mathbf{X}$:

$$R = I \cup P, \tag{1}$$

where:

I – equivalence relation (refleksive, transitive, symmetric),

P – strict preference relation (transitive, asymmetric).

The preference relation R generates a family of subsets $\chi_1^*, ..., \chi_n^*$ ($n \geq 2$) with the following properties:

$$\bigcup_{q=1}^{n} \chi_q^* = \mathbf{X}; \qquad \chi_r^* \cap \chi_s^* = \{0\}; \tag{2}$$

$$(x_i \in \chi_r^{(p)*}) \cap (x_j \in \chi_s^{(p)*}) \equiv \text{ an element } x_i \text{ is preferred to an element } x_j \text{ for } r<s, \tag{3}$$

where: $\{0\}$ – the empty set.

The relation $\chi_1^*, ..., \chi_n^*$ can be characterized by the functions $T_\upsilon(x_i, x_j)$ ($\upsilon \in \{h, \mu\}$; $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$) defined as follows:

$$T_b(x_i,x_j)=\begin{cases}0 \text{ if there exists } r \text{ such that } (x_i,x_j)\in\chi_r^*,\\ -1 \text{ if } x_i\in\chi_r^*, x_j\in\chi_s^* \text{ and } r<s;\\ 1 \text{ if } x_i\in\chi_r^*, x_j\in\chi_s^* \text{ and } r>s;\end{cases} \tag{4}$$

$$T_\mu(x_i,x_j)=r-s. \tag{5}$$

The function $T_b(x_i,x_j)$ corresponds to binary (qualitative) comparisons, $T_\mu(x_i,x_j)$ - to multivalent comparisons (differences of ranks).

The relation $\chi_1^*,\dots,\chi_n^*$ has to be estimated on the basis of pairwise comparisons $g_{\upsilon k}(x_i,x_j)$ $(k=1,\dots,N;\ N\geq 1)$, which are evaluations of the values $T_\upsilon(x_i,x_j)$ $((x_i,x_j)\in\mathbf{X}\times\mathbf{X})$, disturbed by random errors. The error means that any difference $g_{\upsilon k}(x_i,x_j)-T_\upsilon(x_i,x_j)$ is a random variable. In the case of binary comparisons the variables have to satisfy the condition:

$$P(T_b(x_i,x_j)=g_{bk}(x_i,x_j))\geq 1-\delta \qquad (\delta\in(0,\tfrac{1}{2})). \tag{6}$$

Multivalent comparisons assume values from the set $\{-(m-1),\dots,0,\dots,m-1\}$, because the number n is assumed unknown; their distributions have to satisfy the conditions:

$$\sum_{l\leq 0}P(g_{\mu k}(x_i,x_j)-T_\mu(x_i,x_j)=l\,|\,T_\mu(x_i,x_j)=\kappa_{\mu ij})>\tfrac{1}{2} \quad (\kappa_{\mu ij}\in\{0,\dots,\pm(m-1)\}), \tag{7}$$

$$\sum_{l\geq 0}P(g_{\mu k}(x_i,x_j)-T_\mu(x_i,x_j)=l\,|\,T_\mu(x_i,x_j)=\kappa_{\mu ij})>\tfrac{1}{2} \quad (\kappa_{\mu ij}\in\{0,\dots,\pm(m-1)\}), \tag{8}$$

$$P(g_{\mu k}(x_i,x_j)-T_\mu(x_i,x_j)=l)\geq P(g_{\mu k}(x_i,x_j)-T_\mu(x_i,x_j)=l+1\,|\, T_\mu(x_i,x_j)=\kappa_{ij}) \quad (\kappa_{\mu ij}\in\{0,\dots,\pm(m-)\},\ l>0), \tag{9}$$

$$P(g_{\mu k}(x_i,x_j)-T_\mu(x_i,x_j)=l)\geq P(g_{\mu k}(x_i,x_j)-T_\mu(x_i,x_j)=l-1\,|\, T_\mu(x_i,x_j)=\kappa_{\mu ij}) \quad (\kappa_{\mu ij}\in\{0,\dots,(m-1)\},\ l<0). \tag{10}$$

Two estimators of the preference relation have been considered in earlier papers of the author (Klukowski [2] – [8]); the first one is based on total sum of differences between relation form, expressed by one of the functions $T_\upsilon(x_i,x_j)$ $(\upsilon\in\{b,\mu\})$, and comparisons, the second – between relation form and medians from comparisons of each pair. The estimates, denoted respectively – $\hat{\chi}_1,\dots,\hat{\chi}_{\hat{n}}$ and $\tilde{\chi}_1,\dots,\tilde{\chi}_{\tilde{n}}$ (or $\hat{T}(x_i,x_j)$ and $\tilde{T}(x_i,x_j)$), results from the optimal solutions of discrete optimization tasks:

$$\min_{\chi_1^{(\iota)},\ldots,\chi_r^{(\iota)} \in F_X} \{ \sum_{<i,j> \in R_m} \sum_{k=1}^{N} \left| t_\upsilon^{(\iota)}(x_i, x_j) - g_{\upsilon k}(x_i, x_j) \right| \}, \tag{11}$$

$$\min_{\chi_1^{(\iota)},\ldots,\chi_r^{(\iota)} \in F_X} \{ \sum_{<i,j> \in R_m} \left| t_\upsilon^{(\iota)}(x_i, x_j) - g_\upsilon^{(me)}(x_i, x_j) \right| \}, \tag{12}$$

where:

F_X - the feasible set (a family of all preference relations in the set **X**),

$\chi_1^{(\iota)}, \ldots, \chi_r^{(\iota)}$ - ι-th element of the set F_X,

R_m - the set of indices $R_m = \{<i,j> \mid 1 \le i,j \le m;\ j > i\}$,

$t_\upsilon^{(\iota)}(x_i, x_j)$ - the function determining the relation $\chi_1^{(\iota)}, \ldots, \chi_r^{(\iota)}$, defined in the same way, as $T_\upsilon(x_i, x_j)$,

$g_\upsilon^{(me)}(x_i, x_j)$ - the median in the set $\{g_1(x_i, x_j), \ldots, g_N(x_i, x_j)\}$.

The solutions of the tasks (11), (12) can be not unique – in such a case each solution can be considered as an estimate or the unique estimate can be selected in random way. The analytical properties of both estimators have been presented concisely in Klukowski 2010.

The estimation errors are multidimensional random variables: $\hat{T}_\upsilon(x_i, x_j) - T_\upsilon(x_i, x_j)$ and $\tilde{T}_\upsilon(x_i, x_j) - T_\upsilon(x_i, x_j)$ $(<i,j> \in R_m)$. They are not useful in analysis and are replaced by one dimension errors:

$$\hat{\Delta}_\upsilon = \sum_{<i,j> \in R_m} \left| \hat{T}_\upsilon(x_i, x_j) - T_\upsilon(x_i, x_j) \right|, \tag{13}$$

$$\tilde{\Delta}_\upsilon = \sum_{<i,j> \in R_m} \left| \tilde{T}_\upsilon(x_i, x_j) - T_\upsilon(x_i, x_j) \right|. \tag{14}$$

3 Parameters of simulation survey

The simulation survey has been aimed at examination of efficiency of the estimators in some situations - typical in practice. It comprises some number of: relation forms, sample size N and probability distributions – binary and multivalent. The multivalent distributions have been assumed in universal – quasi-uniform form.

The following estimation problems have been analyzed.

- The set **X** comprising 9 elements;
- three relation forms:

- nine subsets relation (linear order): $\{x_1\},...,\{x_9\}$ (n=9),
- six subsets relation: $\{x_1\},\{x_2,x_3\},\{x_4\},\{x_5,x_6\},\{x_7\},\{x_8,x_9\}$ (n=6),
- three subsets relation: $\{x_1,x_2\},\{x_3,x_4,x_5\},\{x_6,x_7,x_8,x_9\}$ (n=3);

• the binary comparisons with probability functions:

$P(g_{bk}(x_i,x_j)=T_b(x_i,x_j))=\alpha_{ij}$, $P(g_{bk}(x_i,x_j)\neq T_b(x_i,x_j))=(1-\alpha_{ij})/2$, with three values of α_{ij} ($<i,j>\in R_m$) : 0,85, 0,90, 0,95, (typical levels in statistical tests);

• the multivalent comparisons with the probability functions:

$P(g_{\mu k}(x_i,x_j)=T_\mu(x_i,x_j))=\alpha_{ij}$,

$P(g_{\mu k}(x_i,x_j)-T_\mu(x_i,x_j)=-l)=(1-\alpha_{ij})/2L_{ij}^{(d)}$

$(L_{ij}^{(d)}=T_\mu(x_i,x_j)+(m-1);\ l=-1,...,-L_{ij}^{(d)})$,

$P(g_{\mu k}(x_i,x_j)-T_\mu(x_i,x_j)=l)=(1-\alpha_{ij})/2L_{ij}^{(u)}$,

$(L_{ij}^{(u)}=m-1-T_\mu(x_i,x_j);\ l=1,...,L_{ij}^{(u)})$,

with three values of α_{ij} : 0,3334; 0,4167; 0,5000 (i.e. approximately $\frac{4}{12},\frac{5}{12},\frac{6}{12}$) ($<i,j>\in R_m$);

• the number N of comparisons of each pair: 1, 3, 5, 7, 9.

The results of simulation comprise – for binary and multivalent comparisons:

• the fraction of errorless estimates (error (13) or (14) equal zero), which are singular solutions of the task (11) or (12);

• the fraction of errorless estimates, in the case of multiple solutions of the task (11) or (12);

• the average value of the error (13) and (14) in 100 simulations;

• the distributions of estimation errors (13), (14) in the case of multivalent comparisons, for both types of estimators (the frequencies of individual errors are presented in interval form – with except of zero value).

Results of comparisons have been obtained with the use of random number generator (independent comparisons). The total number of cases, analyzed in the survey, equals 90 for each type of estimator (3 distributions of comparisons errors, 5 values of N, 3 types of relation form, 2 types of comparisons). The number of simulations of each case equals 100 or 200; double number has been applied for the distributions of errors.

Some parameters used in the case of multivalent comparisons do not satisfy the assumption about the median of comparisons errors equal zero; it is so in the case: n=9 (linear order) and N=1 (singular comparison of each pair), $T_\mu(x_i,x_j)=m-1$ and probabilities: 0,3334, 0,4167. Examination of such cases is important from practical point of view, because probabilities of errorless (multivalent) comparisons may be lower than ½.

4 Results of simulation and their evaluation

The tables 1 – 6 comprise the results characterizing frequencies of errorless solutions and average errors for both estimators and comparisons types, obtained on the basis of 100 simulations. The average errors are presented for singular and multiple solutions of the tasks (11), (12).

Tables 7 – 15 comprise results characterizing distributions of estimation errors for multivalent comparisons, for both estimators, obtained on the basis of 200 simulations.

The following general conclusions can be drawn on the basis of the simulations obtained for binary comparisons:

- both estimators provide precise results for α assuming typical significance levels in statistical tests (at least 0,90) and three or more independent comparisons (value N) of each pair;
- increasing number of comparisons N indicates rapid improving of estimation precision – it reflects theoretical properties of the estimators, especially exponential convergence of the probabilities of errorless estimate to one (see Klukowski 2010);
- the median estimator requires more comparisons (at least plus 2) than the estimator based on sum of differences; it is efficient in the case: $\alpha \geq 0{,}95$, $N \geq 3$ or $\alpha \geq 0{,}85$, $N \geq 5$. Moreover, it produces more multiple estimates;
- average errors of the estimator based on sum of differences are significantly lower than the median estimator; low level of the errors indicate insignificant errors of estimates;
- the most precise estimates have been obtained for the relation with three subsets.

The general conclusions for multivalent comparisons are similar to the binary case. The differences concern values of probabilities of errorless comparisons (simulated values 0,3334, 0,4167, 0,5000), and the fact that the most precise results have been obtained for the relation form with nine subsets. It should be emphasized that the probabilities 0,3334, 0,4167 can provide errorless estimates for appropriate values of N.

Table 1. The efficiency of estimators based on binary comparisons, n=9 subsets

Number of comparisons N	The values	Probability of correct comparison: 0,85		Probability of correct comparison:0,90		Probability of correct comparison:0,95	
		Sum.	Median	Sum.	Median	Sum.	Median
1	% CR	20	20	29	29	49	49
	% of CRM	26	26	38	38	60	60
	AE	4,20	4,20	2,78	2,78	1,41	1,41
3	% CR	53	49	77	75	97	96
	% of CRM	56	55	78	80	97	97
	AE	0,88	1,13	0,38	0,45	0,03	0,03
5	% CR	82	82	92	92	99	99
	% of CRM	82	82	92	92	99	99
	AE	0,28	9,30	0,11	0,11	0,01	0,01
7	% CR	91	91	97	97	100	100
	% of CRM	91	91	97	97	100	100
	AE	0,10	0,10	0,03	0,03	0	0
9	% CR	95	95	100	100	100	100
	% of CRM	95	95	100	100	100	100
	AE	0,05	0,05	0	0	0	0

Computations by the author

Symbols: %CR – fraction of errorless singular estimates, % of CRM – fraction of errorless estimates taking into account multiple solutions, AE – average estimation error, taking into account multiple solutions

Table 2. The efficiency of estimators based on binary comparisons, n=6 subsets; number of elements: 1, 2, 1, 2, 1, 2

Number of comparisons N	The values	Probability of correct comparison:0,85		Probability of correct comparison:0,90		Probability of correct comparison:0,95	
		Sum.	Median	Sum.	Median	Sum.	Median
1	% CR	7	7	20	20	48	48
	% of CRM	20	20	33	33	58	58
	AE	5,10	5,10	3,23	3,23	1,39	1,39
3	% of CR	62	46	89	80	98	94
	% of CRM	79	73	96	93	98	95
	AE	0,57	1,11	0,11	0,35	0,02	0,12
5	% CR	88	68	99	92	99	98
	% of CRM	97	92	100	99	99	99
	AE	0,14	0,57	0,10	0,12	0,01	0,03
7	% CR	98	86	99	94	100	100
	% of CRM	99	95	99	98	100	100
	AE	0,02	0,02	0,01	0,09	0	0
9	% CR	100	100	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0	0	0	0	0	0

Computations by the author

Symbols: the same as in the Table 1

Table 3. The efficiency of estimators based on binary comparisons, n=3 subsets; number of elements: 2, 3, 4

Number of comparisons N	The values	Probability of correct comparison:0,85		Probability of correct comparison:0,90		Probability of correct comparison:0,95	
		Sum.	Median	Sum.	Median	Sum.	Median
1	% CR % of CRM AE	43 68 1,74	43 68 1,74	50 84 1,16	50 84 1,16	76 95 0,38	76 95 0,38
3	% CR % of CRM AE	94 99 0,09	84 98 0,38	94 97 0,06	93 97 0,08	99 99 0,01	99 99 0,01
5	% CR % of CRM AE	100 100 0	100 100 0	100 100 0	100 100 0	100 100 0	100 100 0
7	% CR % of CRM AE	100 100 0	100 100 0	100 100 0	100 100 0	100 100 0	100 100 0
9	% CR % of CRM AE	100 100 0	100 100 0	100 100 0	100 100 0	100 100 0	100 100 0

Computations by the author

Symbols: the same as in the Table 1

Table 4. The efficiency of estimators based on multivalent comparisons, n=9

Number of comparisons N	The values	Probability of correct comparison: 0,3334		Probability of correct comparison: 0,4167		Probability of correct comparison: 0,5000	
		Sum.	Median	Sum.	Median	Sum.	Median
1	% of CR % of CRM AE	17 32 43,68	17 32 43,68	31 51 27,70	31 51 27,70	60 78 12,50	60 78 12,50
3	% of CR % of CRM AE	78 85 6,09	58 74 12,35	91 95 1,11	79 93 3,39	97 100 0,26	92 98 1,19
5	% of CR % of CRM AE	95 98 0,72	71 93 4,0	99 100 0,08	97 100 0,20	100 100 0	100 100 0
7	% of CR % of CRM AE	98 99 0,12	92 99 0,55	100 100 0	100 100 0	100 100 0	100 100 0
9	% of CR % of CRM AE	100 100 0	100 100 0	100 100 0	100 100 0	100 100 0	100 100 0

Computations by the author

Symbols: the same as in the Table 1

Table 5. The efficiency of estimators based on multivalent comparisons, n=6 subsets, number of elements: 1, 2, 1, 2, 1, 2

Number of comparisons N	The values	Probability of correct comparison: 0,3334		Probability of correct comparison: 0,4167		Probability of correct comparison: 0,5000	
		Sum.	Median	Sum.	Median	Sum.	Median
1	% of CR	3	3	13	13	15	15
	% of CRM	15	15	39	39	57	57
	AE	38,10	38,10	22,13	22,13	17,00	17,00
3	% of CR	47	21	78	48	89	73
	% of CRM	74	64	89	87	99	96
	AE	6,59	11,53	2,14	5,69	0,56	1,98
5	% of CR	77	45	97	80	100	100
	% of CRM	86	85	98	98	100	100
	AE	1,95	5,52	0,20	1,06	0	0
7	% of CR	99	75	100	100	100	100
	% of CRM	100	96	100	100	100	100
	AE	0,04	1,52	0	0	0	0
9	% of CR	100	100	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0	0	0	0	0	0

Computations by the author

Symbols: the same as in the Table 1

Table 6. The efficiency of estimators based on multivalent comparisons, n=3 subsets; number of elements: 2, 3, 4

Number of comparisons N	The value	Probability of correct comparison: 0,3334		Probability of correct comparison: 0,4167		Probability of correct comparison: 0,5000	
		Sum.	Median	Sum.	Median	Sum.	Median
1	% of CR	4	4	12	12	24	24
	% of CRM	27	27	41	41	58	58
	AE	16,95	16,95	13,66	13,66	9,46	9,46
3	% of CR	57	28	80	54	93	84
	% of CRM	76	69	93	91	100	98
	AE	3,36	7,32	1,18	2,79	0,28	0,89
5	% of CR	90	59	92	87	100	100
	% of CRM	95	95	100	99	100	100
	AE	0,64	2,47	0,32	0,62	0	0
7	% of CR	93	67	100	100	100	100
	% of CRM	97	94	100	100	100	100
	AE	0,43	2,18	0	0	0	0
9	% of CR	100	100	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0	0	0	0	0	0

Computations by the author

Symbols: the same as in the Table 1

Table 7. Frequencies of estimation errors; n=9 subsets, α=0,3334

Value of error	N=1	N=3		N=5		N=7		N=9	
	Sum.	Sum.	Median	Sum.	Median	Sum.	Median	Sum.	Median
0	0,140	0,780	0,545	0,930	0,800	0,995	0,955	0,995	0,980
(0, 8]	0,035	0,075	0,175	0,035	0,100	0	0,020	0	0,010
(8, 16]	0,055	0,045	0,125	0,025	0,050	0	0,005	0,005	0,010
(16, 24]	0,040	0,040	0,050	0	0,030	0	0,015	0	0
(24, 32]	0,080	0,035	0,055	0	0,010	0,005	0,005	0	0
(32, 40]	0,120	0,005	0,045	0,005	0	0	0	0	0
(40, 48]	0,060	0,015	0,020	0	0,005	0	0	0	0
(48, 56]	0,105	0,005	0,020	0	0,005	0	0	0	0
(56, 64]	0,105	0	0,010	0,005	0	0	0	0	0
(64, 72]	0,080	0	0,005	0	0	0	0	0	0
>72	0,180	0	0	0	0	0	0	0	0

Computations by the author

Table 8. Frequencies of estimation errors; n=9 subsets, α=0,4167

Value of error	N=1	N=3		N=5		N=7		N=9	
	Sum.	Sum.	Median	Sum.	Median	Sum.	Median	Sum.	Median
0	0,350	0,890	0,770	0,990	0,965	1,0	0,985	1,0	1,0
(0.8]	0.090	0,065	0,075	0,005	0,015	0	0,010	0	0
(8.16]	0,085	0,010	0,050	0	0,015	0	0,005	0	0
(16,24]	0,100	0,015	0,040	0,005	0	0	0	0	0
(24,32]	0,065	0,015	0,030	0	0,005	0	0	0	0
(32,40]	0,035	0,005	0,025	0	0	0	0	0	0
(40,48]	0,085	0	0,010	0	0	0	0	0	0
(48.56]	0,050	0	0	0	0	0	0	0	0
(56,64]	0,045	0	0	0	0	0	0	0	0
(64,72]	0,035	0	0	0	0	0	0	0	0
>72	0,060	0	0	0	0	0	0	0	0

Computations by the author

Table 9. Frequencies of estimation errors; n=9 subsets, α=0,5000

Value of error	N=1	N=3		N=5		N=7		N=9	
	Sum.	Sum.	Median	Sum.	Median	Sum.	Median	Sum.	Median
0	0.600	0,985	0,920	1,0	0,995	1,0	1,0	1,0	1,0
(0,8]	0,095	0,010	0,040	0	0,005	0	0	0	0
(8,16]	0,080	0,005	0,015	0	0	0	0	0	0
(16,24]	0,025	0	0,010	0	0	0	0	0	0
(24,32]	0,050	0	0,010	0	0	0	0	0	0
(32,40]	0,040	0	0,005	0	0	0	0	0	0
(40,48]	0,025	0	0	0	0	0	0	0	0
(48,56]	0,035	0	0	0	0	0	0	0	0
(56,64]	0,015	0	0	0	0	0	0	0	0
(64,72]	0,015	0	0	0	0	0	0	0	0
>72	0,025	0	0	0	0	0	0	0	0

Computations by the author

Table 10. Frequencies of estimation errors; n=6 subsets, α=0,3334

Value of error	N=1	N=3		N=5		N=7		N=9	
	Sum.	Sum.	Median	Sum.	Median	Sum.	Median	Sum.	Median
0	0,025	0,420	0,155	0,775	0,485	0,915	0,635	0,990	0,825
(0,8]	0,045	0,220	0,240	0,185	0,329	0,075	0,275	0,010	0,165
(8,16]	0,100	0,165	0,245	0,015	0,125	0,005	0,070	0	0,005
(16,24]	0,190	0,110	0,195	0,010	0,060	0,005	0,020	0	0,005
(24,32]	0,195	0,040	0,055	0,015	0,005	0	0	0	0
(32,40]	0,165	0,025	0,075	0	0,01	0	0	0	0
(40,48]	0,090	0,015	0,020	0	0,005	0	0	0	0
(48,56]	0,100	0	0,010	0	0	0	0	0	0
(56,64]	0,025	0	0	0	0	0	0	0	0
(64,72]	0,030	0,005	0	0	0	0	0	0	0
>72	0,035	0	0	0	0	0	0	0	0

Computations by the author

Table 11. Frequencies of estimation errors; n=6 subsets, α=0,4167

Value of error	N=1	N=3		N=5		N=7		N=9	
	Sum.	Sum.	Median	Sum.	Median	Sum.	Median	Sum.	Median
0	0,115	0,790	0,560	0,970	0,815	0,995	0,945	1,0	0,970
(0,8]	0,090	0,155	0,260	0,025	0,145	0,005	0,055	0	0,030
(8,16]	0,170	0,020	0,120	0	0,030	0	0	0	0
(16,24]	0,190	0,025	0,004	0,005	0,005	0	0	0	0
(24,32]	0,150	0,010	0	0	0,005	0	0	0	0
(32,40]	0,100	0	0,015	0	0,005	0	0	0	0
(40,48]	0,090	0	0,005	0	0	0	0	0	0
(48,56]	0,025	0	0	0	0	0	0	0	0
(56,64]	0,035	0	0	0	0	0	0	0	0
(64,72]	0,020	0	0	0	0	0	0	0	0
>72	0,015	0	0	0	0	0	0	0	0

Computations by the author

Table 12. Frequencies of estimation errors; n=6 subsets, α=0,5000

Value of error	N=1	N=3		N=5		N=7		N=9	
	Sum.	Sum.	Median	Sum.	Median	Sum.	Median	Sum.	Median
0	0,255	0,880	0,775	1,000	0,950	1,000	1,000	1,000	1,000
(0,8]	0,180	0,100	0,155	0	0,050	0	0	0	0
(8,16]	0,220	0,015	0,045	0	0	0	0	0	0
(16,24]	0,115	0,005	0,015	0	0	0	0	0	0
(24,32]	0,060	0	0,010	0	0	0	0	0	0
(32,40]	0,075	0	0	0	0	0	0	0	0
(40,48]	0,050	0	0	0	0	0	0	0	0
(48,56]	0,020	0	0	0	0	0	0	0	0
(56,64]	0,015	0	0	0	0	0	0	0	0
(64,72]	0,010	0	0	0	0	0	0	0	0
>72	0	0	0	0	0	0	0	0	0

Computations by the author

Table 13. Frequencies of estimation errors; n=3 subsets, α=0,3334

Value of error	N=1	N=3		N=5		N=7		N=9	
	Sum.	Sum.	Median	Sum.	Median	Sum.	Median	Sum.	Median
0	0	0,530	0,280	0,840	0,540	0,955	0,720	0,980	0,840
(0,8]	0	0,345	0,385	0,125	0,345	0,045	0,275	0,020	0,16
(8,16]	0,015	0,105	0,235	0,035	0,095	0	0,005	0	0
(16,24]	0,120	0,015	0,090	0	0,020	0	0	0	0
(24,32]	0,125	0,005	0,010	0	0	0	0	0	0
(32,40]	0,145	0	0	0	0	0	0	0	0
(40,48]	0,230	0	0	0	0	0	0	0	0
(48,56]	0,225	0	0	0	0	0	0	0	0
(56,64]	0,105	0	0	0	0	0	0	0	0
(64,72]	0,035	0	0	0	0	0	0	0	0
>72	0	0	0	0	0	0	0	0	0

Computations by the author

Table 14. Frequencies of estimation errors; n=3 subsets, α=0,4167

Value of error	N=1	N=3		N=5		N=7		N=9	
	Sum.	Sum.	Median	Sum.	Median	Sum.	Median	Sum.	Median
0	0,145	0,785	0,530	0,970	0,820	0,990	0,920	1,000	1,000
(0,8]	0,255	0,190	0,360	0,030	0,165	0,010	0,080	0	0
(8,16]	0,340	0,020	0,100	0	0,010	0	0	0	0
(16,24]	0,170	0,005	0,010	0	0,005	0	0	0	0
(24,32]	0,045	0	0	0	0	0	0	0	0
(32,40]	0,020	0	0	0	0	0	0	0	0
(40,48]	0,015	0	0	0	0	0	0	0	0
(48,56]	0,005	0	0	0	0	0	0	0	0
(56,64]	0,005	0	0	0	0	0	0	0	0
(64,72]	0	0	0	0	0	0	0	0	0
>72	0	0	0	0	0	0	0	0	0

Computations by the author

Table 15. Frequencies of estimation errors; n=3 subsets, α=0,5000

Value of error	N=1	N=3		N=5		N=7		N=9	
	Sum.	Sum.	Median	Sum.	Median	Sum.	Median	Sum.	Median
0	0,310	0,915	0,795	0,995	0,940	1,000	0,975	1,000	1,000
(0,8]	0,325	0,085	0,190	0,005	0,060	0	0,025	0	0
(8,16]	0,215	0	0,015	0	0	0	0	0	0
(16,24]	0,130	0	0	0	0	0	0	0	0
(24,32]	0,015	0	0	0	0	0	0	0	0
(32,40]	0,005	0	0	0	0	0	0	0	0
(40,48]	0	0	0	0	0	0	0	0	0
(48,56]	0	0	0	0	0	0	0	0	0
(56,64]	0	0	0	0	0	0	0	0	0
(64,72]	0	0	0	0	0	0	0	0	0
>72	0	0	0	0	0	0	0	0	0

Computations by the author

The values presented in the tables 1 – 15 allow determining the number N guaranteeing necessary precision of estimates. E.g. in the case of: binary comparisons, relation form with six subsets and probability of correct comparison 0,90 - five comparisons (N=5) provide frequency of correct estimates greater than 90%; the precision of both estimators is not the same – the estimator based on sum of comparisons is more efficient. In the case of multivalent comparisons, similar precision is obtained for the probability of errorless comparisons equal 0,4167; however advantage of the estimator based on sum of differences is significant in the case of singular solutions of the task (11).

The analysis of simulations based on both types of comparisons indicates some essential conclusions about efficiency of the estimators proposed. The efficiency of the estimators based on multivalent comparisons exceeds, in general, efficiency of the binary estimators. Moreover, these estimators can be applied also in the case of multiple binary comparisons – using two-steps approach; the first step produces N differences of ranks on the basis of binary comparisons, the second – applies the estimators using differences of ranks. Such approach is applicable, if precision of binary estimates has to satisfy the requirements of multivalent estimators. Two-step estimator allows also combining binary and multivalent comparisons, e.g. results of statistical tests, experts, neural networks and other procedures.

The analysis of distributions of estimation errors, resulting from simulations, leads to following conclusions:

- the estimator based on sum of differences provides, in the case of multiple comparisons (N>1), better precision than the estimator based on medians. The better precision means higher frequency of errorless estimates and higher concentration of distributions of errors in neighborhood of zero. The difference in precision is insignificant only in the case of high frequency of errorless estimates generated by both estimators, i.e. greater than 95%;
- increasing number of comparisons N indicates rapid improving of precision of estimates. In the case of the estimator based on sum of differences and probability of errorless comparison α=0,5, three comparisons (N=3) guarantee the frequency of errorless estimate close to one. In the case of the median estimator, the number of comparisons has to be increased by two. Remaining values of α, i.e. 0,3334 and 0,4167 require the number of comparisons equal to – respectively: seven and five. The number of comparisons N from 7 to 9 guarantee frequency of errorless estimate close to 100%;
- the best precision of estimates has been obtained in the case of linear order, i.e. n=9; remaining cases, i.e. n=3 and n=6, have slightly lower – similar precision. In the case n=9 and

α=1/2, the acceptable precision of estimation (frequency of errorless comparison higher than 50%) is obtained for single comparison, i.e. N=1;

• frequencies of estimation errors are highly concentrated in the case of frequency of errorless estimate higher than 50%; the frequency of a value of error lower or equal than 16 is close to 100%. If the frequency of errorless estimate is higher than 75% the same property is valid for the error equal 8. Insignificant value of the error indicates slight difference between an estimate and relation form;

• in the case, when some distributions of comparisons errors do not satisfy the assumptions that zero is the median and mode of distributions of comparisons errors, the results of estimation become not acceptable. Such situation takes place in the case: N=1 and $\alpha \leq$0,4167 and $N \leq 3$ and α=0,3334.

In general, the survey confirms excellent efficiency of the estimator, in the form of sum of differences and based on multivalent comparisons. The experience gained in simulation experiments with n>9 (not presented in the paper) shows that above conclusions are valid for other relations forms.

5 Summary and conclusions

The simulation survey broadens significantly the range of theoretical properties of the estimators presented in Klukowski 2010. It confirms their good statistical properties, especially in the case of multiple comparisons of each pair. It should be emphasized the excellent efficiency of multivalent estimators (comparisons in the form of differences of ranks). It is clear that the results are also valid for remaining relation types, i.e. equivalence and tolerance.

Let us note that whole estimation process, i.e.: making pairwise comparisons (using statistical tests), solving of the discrete programming tasks, determining the properties of estimates (using simulation approach) and validation of estimates (see Klukowski 2011) can be computerized; thus the approach is close to data mining techniques.

References

[1] David H. A. (1988) The Method of Paired Comparisons, 2nd ed. Ch. Griffin, London.

[2] Klukowski L. (1994) Some probabilistic properties of the nearest adjoining order method and its extensions. *Annals of Operational Research*, **51**, 241 – 261.

[3] Klukowski L. (2007) Estimation of tolerance relation the basis of multiple pairwise comparisons with random errors. *Control and Cybernetics*, **36**, 443-466.

[4] Klukowski L. (2008a) Determination of tolerance relation – alternative approach to intuitionistic and fuzzy sets. *Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets an Related Topics*. Vol. II. K. Atanassov (et al), eds. Ac. Publ. House EXIT, Warsaw.

[5] Klukowski L. (2008b) Estimation of the preference relation the basis of multiple pairwise comparisons in the form of differences of ranks. *Control and Cybernetics*, **37**, 711-729.

[6] Klukowski L. (2008c) Tests for Relation Type - Equivalence or Tolerance – Based on Averaging of Multiple Binary Comparisons. In: *Development In Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics*. Foundations. Vol. I. K. Atanassov (et al), eds. Ac. Publ. House EXIT, Warsaw.

[7] Klukowski L. (2010) Estimation of the relations: preference, equivalence and tolerance on the basis of pairwise comparisons. Eighth Workshop on Intuitionistic Fuzzy Sets and generalized Nets. In: *Development In Fuzzy Sets, Intuitionistic Fuzzy Sets, Vol. I: Foundations*. IBS PAN, SRI PAS.

[8] Klukowski L. (2011) Tests for validation of estimates obtained on the basis of pairwise comparisons with random errors. In: *Recent Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics*. Vol. II: Applications, SRI PAS, IBS PAN.

[9] Slater P. (1961) Inconsistencies in a schedule of paired comparisons. *Biometrika*, **48**, 303 - 312msaleg L., Gros P., Berrani S-A. (2004) Robust Object Recognition in Images and the Related Database Problems. *Multimedia Tools and Applications*, 23, 1, Springer Netherlands, 221-235.