Raport Badawczy

Research Report

RB/20/2008

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Warszawa 2008

ESTIMATION OF THE PREFERENCE RELATION ON THE BASIS OF MULTIPLE PAIRWISE COMPARISONS IN THE FORM OF DIFFERENCES OF RANKS

by

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The problem of estimation of the preference relation in a finite set on the basis of pairwise comparisons, in the form of difference of ranks with random errors, with the use of nearest adjoining order idea (NAO), is investigated in the paper. The results presented below are extension and correction of earlier works of the author; especially the case of multiple independent comparisons of each pair is examined. The comparisons of each pair are aggregated - two approaches are analysed: averaging of comparisons and median from comparisons. The estimated form of the relation is obtained in both cases on the basis of some discrete programming tasks. The properties of the estimators are obtained under weak assumptions about distributions of comparison errors, in particular the distributions may be unknown.

Keywords: multiple pairwise comparisons, nearest adjoining order method, difference of ranks

1. Introduction

The paper presents extensions of the method of ranking elements from a finite set on the basis of pairwise comparisons, in the form of difference of ranks with random errors, presented in Klukowski (2000). The results discussed in Klukowski (2000) relate to the case of one comparison of each pair and require some correction (see section 4). The extension examines the case of multiple comparisons; the comparisons of each pair are aggregated in two ways. The first way is simply averaging of (each pair) comparisons; the second approach is based on the median from the comparisons. In both cases the idea of nearest adjoining order (NAO) is applied (see Slater 1961, David 1988, section 2.2). The results obtained are based on weak assumptions about distributions of comparison errors, especially their distributions may be unknown. The properties of distributions of comparison errors assumed in the paper may be verified with the use of statistical tests (for unimodality, mode, median, symmetry). The basis for the properties of estimators are well known probabilistic inequalities: Hoeffding's inequality for sums of bounded independent random variables (Hoeffding 1963) and Tschebyshev inequality for expected value; some properties of the order statistics (David 1970) are also used. In the case of averaging approach the probability of the event that some random variable (defined in Section 3 below) corresponding to actual relation (the errorless estimation result) is lower than the variable corresponding to any other relation converges exponentially to one. Some asymptotic properties are obtained also for the median approach.

The empirical problems with such structure often appear in practice, e.g.: ranking organisms according to their age (in years), ranking writers according to artistic value of their works, etc.

Let us notice that the comparisons in the form of differences of ranks with the properties assumed can be obtained also on the basis of rankings (estimates) resulting from comparisons in the form of direction of preference (Klukowski 1994). It allows to construct two-stages estimators: the first step – to obtain estimates of the relation form with the use of comparisons indicating direction of preference and to determine differences of ranks for each estimate, the second step – to apply the algorithm based on differences of ranks (sections 4, 5 below). It seems that two-stages approach can be more efficient that the earlier approach presented in Klukowski (1994), section 5. Therefore, the examination of the estimator based on differences of ranks is needful.

The empirical results, based on actual data and initial simulation experiments, are promising – also for "inconvenient" form of distributions of comparison errors (asymmetric, with non-zero expected value).

The literature on ranking problems is quite extensive; for example the probabilistic approach is presented in David (1988), Marden (1995), learning approach - in Hastie et al. (2001) chapt. 14, Kamishima and Akaho (2006), fuzzy approach in Yager R., R. (2007).

The paper consists of six sections; main results: the problem formulation, definitions and notations, the form of estimators and their properties are presented in sections 2 – 5. Last section summarizes the results obtained.

2. Problem formulation

The formulation of the problem is an extension of the problem stated in Klukowski (2000) for the case of $N>1$ independent comparisons of each pair.

It is assumed, that in a finite set of elements $\mathbf{X}=\{x_1, ..., x_m\}$ $(m\geq3)$ there exists an unknown complete weak preference relation **R** of the form:

$$\mathbf{R} = \mathbf{I} \cup \mathbf{P}, \quad (1)$$

where:

I - the equivalence relation (reflexive, transitive, symmetric),

P - the strict preference relation (transitive, asymmetric).

The preference relation **R** generates from elements of the set **X** the family (sequence) of subsets $\chi_1^*, ..., \chi_n^*$ $(n\leq m)$, such that each element $x_i \in \chi_r^*$ is preferred to any element $x_j \in \chi_s^*$ $(r<s)$ and each subset χ_q^* $(1\leq q\leq n)$ comprises equivalent elements only.

The relation **R** can be characterised by the function $T: \mathbf{X} \times \mathbf{X} \rightarrow D_T$, $D_T = \{-(n-1), ..., 0, ..., n-1\}$, defined as follows:

$$T(x_i, x_j) = d_{ij} \quad \Leftrightarrow \quad x_i \in \chi_r^*, \quad x_j \in \chi_s^*, \quad d_{ij} = r - s. \quad (2)$$

The value of the function $T(x_i, x_j)$ expresses the difference of ranks of the elements x_i and x_j in the relation **R**. In the case $T(x_i, x_j)<0$, ($T(x_i, x_j)>0$) the element x_i precedes element x_j (element x_j precedes x_i), for d_{ij} positions. The value $T(x_i, x_j)=0$ means the equivalence of both elements (they belong to the same subset χ_q^*, $1\leq q\leq n$). It is obvious, that $T(x_i, x_j) = -T(x_j, x_i)$ for $T(\cdot)\neq 0$.

The relation form is to be determined (estimated) on the basis of pairwise comparisons of elements of the set **X** disturbed by random errors. Each pair $(x_i, x_j)\in \mathbf{X}$ is compared independently (in stochastic sense) N times; the result of *k-th* comparison ($k=1, \ldots, N$; $N>1$) is the value of the function:

$$g_k : \mathbf{X} \times \mathbf{X} \rightarrow D, \qquad D=\{-(m-1), ..., m-1\}; \tag{3}$$

the result $g_k(x_i, x_j)=c_{ijk}$ is an assessment of the difference of ranks in the pair (x_i, x_j), in k-th comparison. The set D can include values from the range: $-(m-1), ..., m-1$ because the number of subsets n is assumed unknown.

In the paper it is assumed, that each comparison $g_k(x_i, x_j)$ ($1\leq k\leq N$) can be disturbed by a random error; it means, that the difference $T(x_i, x_j)-g_k(x_i, x_j)$ may assume values different than zero - with some probabilities. The comparisons $g_k(x_i, x_j)$ and $g_l(x_r, x_s)$ are assumed independent, i.e.:

$$P((g_k(x_i, x_j)=c_{ijk})\cap(g_l(x_r, x_s)=c_{rsl}))=P(g_k(x_i, x_j)=c_{ijk})P(g_l(x_r, x_s)=c_{rsl}) \quad (k\neq l). \tag{4}$$

The probabilities, which characterize each distribution of comparison errors will be denoted with the symbols $\alpha_{ijk}(l)$, $\beta_{ijk}(l)$, $\gamma_{ijk}(l)$; the probabilities are defined as follows:

$$\alpha_{ijk}(l) = P(T(x_i, x_j) - g_k(x_i, x_j) = l;\ T(x_i, x_j) = 0) \qquad (-(m-1)\leq l\leq (m-1)), \tag{5}$$

$$\beta_{ijk}(l) = P(T(x_i, x_j) - g_k(x_i, x_j) = l;\ T(x_i, x_j) < 0) \qquad (-2(m-1)\leq l\leq 2(m-1)), \tag{6}$$

$$\gamma_{ijk}(l) = P(T(x_i, x_j) - g_k(x_i, x_j) = l;\ T(x_i, x_j) > 0) \qquad (-2(m-1) \le l \le 2(m-1)). \tag{7}$$

It is obvious, that the probabilities (5) - (7) have to fulfil the equalities:

$$\sum_{l=-(m-1)}^{(m-1)} \alpha_{ijk}(l) = 1, \qquad \sum_{l=-2(m-1)}^{2(m-1)} \beta_{ijk}(l) = 1, \qquad \sum_{l=-2(m-1)}^{2(m-1)} \gamma_{ijk}(l) = 1. \tag{8}$$

Moreover it is assumed that the following assumptions hold:

$$\sum_{l \le 0} P(T(x_i, x_j) - g_k(x_i, x_j) = l) > 1/2, \tag{9}$$

$$\sum_{l \ge 0} P(T(x_i, x_j) - g_k(x_i, x_j) = l) > 1/2, \tag{10}$$

$$P(T(x_i, x_j) - g_k(x_i, x_j) = l) \ge P(T(x_i, x_j) - g_k(x_i, x_j) = l+1); \qquad l \ge 0, \tag{11}$$

$$P(T(x_i, x_j) - g_k(x_i, x_j) = l) \ge P(T(x_i, x_j) - g_k(x_i, x_j) = l-1); \qquad l \le 0. \tag{12}$$

The conditions (9) – (12) guarantee, that: • zero is the median of each distribution, • each probability function is unimodal and • assumes maximum in zero. The expected value of any comparison error can differ from zero (especially, for $T(x_i, x_j)=\pm(n-1)$ the expected value of comparison is typically different than zero, because usually $P(T(x_i, x_j)-g_k(x_i, x_j)=0)\neq 1$.

The probabilities $\alpha_{ijk}(0)$, $\beta_{ijk}(0)$ and $\gamma_{ijk}(0)$ may be lower than ½; therefore the assumptions about errorless comparison are more general, than those in zero-one approach (see Klukowski 1994).

For simplification it is assumed, that distribution of any comparison error $T(x_i, x_j)$-$g_k(x_i, x_j)$ ($(x_i, x_j)\in \mathbf{X} \times \mathbf{X}$) is the same for each k, $1\le k\le N$ (as a result the comparisons of each pair $g_1(x_i, x_j)$, …, $g_N(x_i, x_j)$ are iid. random variables). Therefore, the index k will be omitted in symbols: $\alpha_{ijk}(l)$, $\beta_{ijk}(l)$, $\gamma_{ijk}(l)$. The relaxation of the assumption about identical distributions is not difficult.

The probabilities $\alpha_{ij}(l)$ ($-(m-1)\le l\le m-1$) determine the probability function of comparison errors for equivalent elements x_i and x_j (because $T(x_i, x_j)=0$). The probability $\alpha_{ij}(l)$ means, that a result of comparison assumes value l, when both elements are equivalent; especially $\alpha_{ij}(0)$ denotes the probability of errorless comparison (because $T(x_i, x_j)=g_k(x_i, x_j)=0$). In the case of known number of the relation subsets (index n) the interval of integers $[-(m-1), m-1]$ ("support" of comparisons $g_k(\cdot)$) ought to be replaced with the interval $[-(n-1), n-1]$. The interpretation of the probabilities $\beta_{ij}(l)$ and $\gamma_{ij}(l)$ ($-2(m-1)\le l\le 2(m-1)$) is similar, with the difference, that they both determine distributions of errors for non-equivalent elements.

The problem of estimation of the preference relation can be stated formally as follows. To determine the relation **R** (or, equivalently, the sequence of subsets $\chi_1^*, ..., \chi_n^*$) on the basis of the comparisons $g_k(x_i, x_j)$ ($k=1, \ldots, N$), made for each pair $(x_i, x_j)\in \mathbf{X}\times\mathbf{X}$.

Let us emphasize that, it is not assumed, that the probability functions of comparisons errors (probabilities $\alpha_{ij}(l)$, $\beta_{ij}(l)$, $\gamma_{ij}(l)$) and the number of subsets n are known.

3. Basic definition and notation

The following notation is introduced for further considerations:

- $t(x_i, x_j)$ - the function which determines the difference of ranks for each pair $(x_i, x_j)\in\mathbf{X}\times\mathbf{X}$ in any preference relation in the set **X**, i.e.:

$$t(x_i, x_j) = d_{ij} \Leftrightarrow x_i\in\chi_k,\quad x_i\in\chi_l;\quad d_{ij}=k-l. \tag{13}$$

- $I(\chi_1, ..., \chi_r)$, $P_1(\chi_1, ..., \chi_r)$, $P_2(\chi_1, ..., \chi_r)$ - the sets of pairs of indices $<i, j>$ generating a relation $(\chi_1, ..., \chi_r)$, i.e.:

$$I(\chi_1, ..., \chi_r)=\{<i,j> \mid t(x_i, x_j)=0; j>i\}, \tag{14}$$

$$P_1(\chi_1, ..., \chi_r)=\{<i,j> \mid t(x_i, x_j)<0; j>i\}, \tag{15}$$

$$P_2(\chi_1, \ldots, \chi_r)=\{<i,j> \mid t(x_i, x_j) > 0; j>i\}; \tag{16}$$

- $R_m = I(\chi_1, \ldots, \chi_r) \cup P_1(\chi_1, \ldots, \chi_r) \cup P_2(\chi_1, \ldots, \chi_r) = \{<i,j> \mid 1 \le i, j \le m; j>i\};$ (17)

- $M = m(m-1)/2 = \#(R_m)$ (18)

(the symbol $\#(\Xi)$ means number of elements of a set Ξ).

The properties of the estimators examined in the paper are based on random variables $U_{ij}^{(k)}(\chi_1, \ldots, \chi_r)$, $V_{ij}^{(k)}(\chi_1, \ldots, \chi_r)$, $Z_{ij}^{(k)}(\chi_1, \ldots, \chi_r)$, $W^{(k)}(\chi_1, \ldots, \chi_r)$ defined as follows:

$$U_{ij}^{(k)}(\chi_1, \ldots, \chi_r) = |g_k(x_i, x_j)|; \qquad t(x_i, x_j)=0, \tag{19}$$

$$V_{ij}^{(k)}(\chi_1, \ldots, \chi_r) = |t(x_i, x_j)-g_k(x_i, x_j)|; \qquad t(x_i, x_j)<0, \tag{20}$$

$$Z_{ij}^{(k)}(\chi_1, \ldots, \chi_r) = |t(x_i, x_j)-g_k(x_i, x_j)|; \qquad t(x_i, x_j)>0, \tag{21}$$

$$W^{(k)}(\cdot) = \sum_{<i,j>\in I(\cdot)} U_{ij}^{(k)}(\cdot) + \sum_{<i,j>\in P_1(\cdot)} V_{ij}^{(k)}(\cdot) + \sum_{<i,j>\in P_2(\cdot)} Z_{ij}^{(k)}(\cdot). \tag{22}$$

Random variables and other symbols corresponding to the relation **R** (errorless result of the estimation problem) will be marked with asterisks, i.e.: $U_{ij}^{(k)*}$, $V_{ij}^{(k)*}$, $Z_{ij}^{(k)*}$, I^*, P_1^*, P_2^*, $W^{(k)*}$, while variables and symbols corresponding to any other relation $\widetilde{X}_1, \ldots, \widetilde{X}_r$, different than errorless one, will be denoted: $\widetilde{U}_{ij}^{(k)}, \widetilde{V}_{ij}^{(k)}, \widetilde{Z}_{ij}^{(k)}, \widetilde{I}, \widetilde{P}_1, \widetilde{P}_2, \widetilde{W}^{(k)}$. For fixed k ($1 \le k \le N$) the difference $W^{(k)*} - \widetilde{W}^{(k)}$ can be written in the form[1]:

$$W^{(k)*} - \widetilde{W}^{(k)} = \sum_{I^* \cap (\widetilde{P}_1 - P_1^*)} (U_{ij}^{(k)*} - \widetilde{V}_{ij}^{(k)}) + \sum_{I^* \cap (\widetilde{P}_2 - P_2^*)} (U_{ij}^{(k)*} - \widetilde{Z}_{ij}^{(k)}) +$$

[1] The sum (23) comprises, in Klukowski 2000, six components only; it does not comprise the variables $Q_{ij}^{(k5)}$ $(<i,j>\in S_5)$ and $Q_{ij}^{(k8)}$ $(<i,j>\in S_8)$. Therefore, the evaluation (33) from Klukowski 2000 also requires correction (see formulas (46) and (63) in this paper).

$$+\sum_{P_1^*\cap(\tilde{I}-I^*)}(V_{ij}^{(k)*}-\tilde{U}_{ij}^{(k)})+\sum_{P_1^*\cap(\tilde{P}_2-P_2^*)}(V_{ij}^{(k)*}-\tilde{Z}_{ij}^{(k)})+\sum_{(P_1^*\cap\tilde{P}_1)\cap(T(\cdot)\neq\tilde{T}(\cdot))}(V_{ij}^{(k)*}-\tilde{V}_{ij}^{(k)})$$

$$+\sum_{P_2^*\cap(\tilde{I}-I^*)}(Z_{ij}^{(k)*}-\tilde{U}_{ij}^{(k)})+\sum_{P_2^*\cap(\tilde{P}_1-P_1^*)}(Z_{ij}^{(k)*}-\tilde{V}_{ij}^{(k)})+\sum_{(P_2^*\cap\tilde{P}_2)\cap(T(\cdot)\neq\tilde{T}(\cdot))}(Z_{ij}^{(k)*}-\tilde{Z}_{ij}^{(k)}) \quad (23)$$

or shortly in the form

$$W^{(k)*}-\tilde{W}^{(k)}=\sum_{\nu=1}^{8}\sum_{S_\nu}Q_{ij}^{(k\nu)},$$

where:

$$Q_{ij}^{(k,1)}=U_{ij}^{(k)*}-\tilde{V}_{ij}^{(k)}, \qquad S_1=\{<i,j>\,|\,<i,j>\in I^*\cap(\tilde{P}_1-P_1^*)\}, \quad (24)$$

$$Q_{ij}^{(k,2)}=U_{ij}^{(k)*}-\tilde{Z}_{ij}^{(k)}, \qquad S_2=\{<i,j>\,|\,<i,j>\in I^*\cap(\tilde{P}_2-P_2^*)\}, \quad (25)$$

$$Q_{ij}^{(k,3)}=V_{ij}^{(k)*}-\tilde{U}_{ij}^{(k)}, \qquad S_3=\{<i,j>\,|\,<i,j>\in P_1^*\cap(\tilde{I}-I^*)\}, \quad (26)$$

$$Q_{ij}^{(k,4)}=V_{ij}^{(k)*}-\tilde{Z}_{ij}^{(k)}, \qquad S_4=\{<i,j>\,|\,<i,j>\in P_1^*\cap(\tilde{P}_2-P_2^*)\}, \quad (27)$$

$$Q_{ij}^{(k,5)}=V_{ij}^{(k)*}-\tilde{V}_{ij}^{(k)}, \qquad S_5=\{<i,j>\,|\,<i,j>\in(P_1^*\cap\tilde{P}_1)\cap(T(x_i,x_j)\neq\tilde{T}(x_i,x_j))\}, \quad (28)$$

$$Q_{ij}^{(k,6)}=Z_{ij}^{(k)*}-\tilde{U}_{ij}^{(k)}, \qquad S_6=\{<i,j>\,|\,<i,j>\in P_2^*\cap(\tilde{I}-I^*)\}, \quad (29)$$

$$Q_{ij}^{(k,7)}=Z_{ij}^{(k)*}-\tilde{V}_{ij}^{(k)}, \qquad S_7=\{<i,j>\,|\,<i,j>\in P_2^*\cap(\tilde{P}_1-P_1^*)\}, \quad (30)$$

$$Q_{ij}^{(k,8)}=Z_{ij}^{(k)*}-\tilde{Z}_{ij}^{(k)}, \qquad S_8=\{<i,j>\,|\,<i,j>\in(P_2^*\cap\tilde{P}_2)\cap(T(x_i,x_j)\neq\tilde{T}(x_i,x_j))\}. \quad (31)$$

The following properties of the random variables $Q_{ij}^{(k,\nu)}$ are necessary for further considerations.

Lemma

The expected value of each random variable $Q_{ij}^{(k,\nu)}$ ($1 \le k \le N$; $<i,j> \in S_\nu$; $\nu=1, \ldots, 8$) satisfies the condition:

$$E(Q_{ij}^{(k,\nu)}) < 0. \tag{32}$$

Proof - see Appendix.

4. The case of averaged comparisons

The estimator presented in this section is based on averages from individual random variables $U_{ij}^{(k)}(\cdot)$, $V_{ij}^{(k)}(\cdot)$, $Z_{ij}^{(k)}(\cdot)$, i.e. the variables: $\overline{U}_{ij}(\cdot)$, $\overline{V}_{ij}(\cdot)$ and $\overline{Z}_{ij}(\cdot)$ defined in the following way:

$$\overline{U}_{ij}(\cdot) = \frac{1}{N} \sum_{k=1}^{N} U_{ij}^{(k)}(\cdot), \tag{33}$$

$$\overline{V}_{ij}(\cdot) = \frac{1}{N} \sum_{k=1}^{N} V_{ij}^{(k)}(\cdot), \tag{34}$$

$$\overline{Z}_{ij}(\cdot) = \frac{1}{N} \sum_{k=1}^{N} Z_{ij}^{(k)}(\cdot). \tag{35}$$

Similarly, the random variable $\overline{W}(\cdot)$ is defined, as follows:

$$\overline{W}(\cdot) = \sum_{<i,j> \in I(\cdot)} \overline{U}_{ij}(\cdot) + \sum_{<i,j> \in P_1(\cdot)} \overline{V}_{ij}(\cdot) + \sum_{<i,j> \in P_2(\cdot)} \overline{Z}_{ij}(\cdot). \tag{36}$$

The symbols corresponding to the relation $\chi_1^*, \ldots, \chi_n^*$ will be also denoted with asterisks, i.e. $\overline{U}_{ij}^*(\cdot)$, $\overline{V}_{ij}^*(\cdot)$, $\overline{Z}_{ij}^*(\cdot)$, $\overline{W}^*$, while the symbols corresponding to any other relation $\widetilde{\chi}_1, \ldots, \widetilde{\chi}_r$ will be denoted with tildas, i.e. $\widetilde{\overline{U}}_{ij}(\cdot)$, $\widetilde{\overline{V}}_{ij}(\cdot)$, $\widetilde{\overline{Z}}_{ij}(\cdot)$, $\widetilde{\overline{W}}$.

Let us notice, that the variables $\overline{U}_{ij}(\cdot)$, $\overline{V}_{ij}(\cdot)$ and $\overline{Z}_{ij}(\cdot)$ satisfy, under the assumption about identity of distribution functions $\alpha_{ijk}(l)$, $\beta_{ijk}(l)$, $\gamma_{ijk}(l)$ ($1 \le k \le N$), the conditions:

$$E(\overline{U}_{ij}(\cdot)) = E(U_{ij}^{(k)}(\cdot)), \tag{37}$$

$$E(\overline{V}_{ij}(\cdot)) = E(V_{ij}^{(k)}(\cdot)), \tag{38}$$

$$E(\overline{Z}_{ij}(\cdot)) = E(Z_{ij}^{(k)}(\cdot)), \tag{39}$$

$$Var(\overline{U}_{ij}(\cdot)) = \tfrac{1}{N} Var(U_{ij}^{(k)}(\cdot)), \tag{40}$$

$$Var(\overline{V}_{ij}(\cdot)) = \tfrac{1}{N} Var(V_{ij}^{(k)}(\cdot)), \tag{41}$$

$$Var(\overline{Z}_{ij}(\cdot)) = \tfrac{1}{N} Var(Z_{ij}^{(k)}(\cdot)). \tag{42}$$

The difference $\overline{W}^{*}(\cdot) - \tilde{\overline{W}}(\cdot)$ can be expressed in the form:

$$\begin{aligned}
\overline{W}^{*}(\cdot) - \tilde{\overline{W}}(\cdot) = & \sum_{I^{*}\cap(\tilde{P}_1 - P_1^{*})} (\overline{U}_{ij}^{*} - \tilde{\overline{V}}_{ij}) + \sum_{I^{*}\cap(\tilde{P}_2 - P_2^{*})} (\overline{U}_{ij}^{*} - \tilde{\overline{Z}}_{ij}) + \\
& + \sum_{P_1^{*}\cap(\tilde{I} - I^{*})} (\overline{V}_{ij}^{*} - \tilde{\overline{U}}_{ij}) + \sum_{P_1^{*}\cap(\tilde{P}_2 - P_2^{*})} (\overline{V}_{ij}^{*} - \tilde{\overline{Z}}_{ij}) + \sum_{(P_1^{*}\cap\tilde{P}_1)\cap(T^{*}(\cdot)\neq\tilde{T}(\cdot))} (\overline{V}_{ij}^{*} - \tilde{\overline{V}}_{ij}) \\
& + \sum_{P_2^{*}\cap(\tilde{I} - I^{*})} (\overline{Z}_{ij}^{*} - \tilde{\overline{U}}_{ij}) + \sum_{P_2^{*}\cap(\tilde{P}_1 - P_1^{*})} (\overline{Z}_{ij}^{*} - \tilde{\overline{V}}_{ij}) + \sum_{(P_2^{*}\cap\tilde{P}_2)\cap(T^{*}(\cdot)\neq\tilde{T}(\cdot))} (\overline{Z}_{ij}^{*} - \tilde{\overline{Z}}_{ij}) \\
& = \sum_{\nu=1}^{8} \sum_{S_\nu} \overline{Q}_{ij}^{(\nu)},
\end{aligned} \tag{43}$$

where:

$\overline{Q}_{ij}^{(1)} = \overline{U}_{ij}^{*} - \tilde{\overline{V}}_{ij}, \qquad <i, j> \in S_1,$

$\overline{Q}_{ij}^{(2)} = \overline{U}_{ij}^{*} - \tilde{\overline{Z}}_{ij}, \qquad <i, j> \in S_2,$

$\overline{Q}_{ij}^{(3)} = \overline{V}_{ij}^{*} - \tilde{\overline{U}}_{ij}, \qquad <i, j> \in S_3,$

$\overline{Q}_{ij}^{(4)} = \overline{V}_{ij}^{*} - \tilde{\overline{Z}}_{ij}, \qquad <i, j> \in S_4,$

$\overline{Q}_{ij}^{(5)} = \overline{V}_{ij}^{*} - \tilde{\overline{V}}_{ij}, \qquad <i, j> \in S_5,$

$\overline{Q}_{ij}^{(6)} = \overline{Z}_{ij}^{*} - \tilde{\overline{U}}_{ij}, \qquad <i, j> \in S_6,$

$$\overline{Q}^{(7)}_{ij} = \overline{Z}^{*}_{ij} - \tilde{\overline{V}}_{ij}, \qquad <i,j>\in S_7,$$

$$\overline{Q}^{(8)}_{ij} = \overline{Z}^{*}_{ij} - \tilde{\overline{Z}}_{ij}, \qquad <i,j>\in S_8,$$

(S_ν - the same as in (24) – (31)).

It results from the lemma presented in Section 3, that:

$$E(\overline{W}^{*}(\cdot) - \tilde{\overline{W}}(\cdot)) < 0. \tag{44}$$

Moreover, it can be determined the evaluation of the probability $P(\overline{W}^{*} < \tilde{\overline{W}})$; the Hoeffding's inequality (see Hoeffding 1963):

$$P(\sum_{k=1}^{N} Y_i - \sum_{k=1}^{N} E(Y_i) \geq Nt) \leq \exp\{-2Nt^2/(b-a)^2\}, \tag{45}$$

where:

Y_i (i=1, ..., N) – independent random variables satisfying the condition $P(a \leq Y_i \leq b) = 1$, ($-\infty<a<b<\infty$);

t – positive constant,

will be used as a basis for the evaluation.

Theorem 1.

The probability $P(\overline{W}^{*} < \tilde{\overline{W}})$ *satisfies the inequality*

$$P(\overline{W}^{*} < \tilde{\overline{W}}) \geq 1 - \exp\left\{ -2N \frac{(\sum_{\nu=1}^{8} \sum_{S_\nu} E(Q^{(1\nu)}_{ij}))^2}{(2\vartheta(m-1))^2} \right\}, \tag{46}$$

where:

ϑ *- the number of elements of the set* $\bigcup_{\nu=1}^{8} S_\nu$.

Proof.

The probability $P(\overline{W}^* < \widetilde{\overline{W}})$ can be expressed in the form:

$P(\overline{W}^* < \widetilde{\overline{W}}) = 1 - P(\overline{W}^* - \widetilde{\overline{W}} \geq 0)$ and

$$P(\overline{W}^* - \widetilde{\overline{W}} \geq 0) = P(\sum_{\nu=1}^{8} \sum_{S_\nu} \overline{Q}_{ij}^{(\nu)} \geq 0) =$$

$$= P(\sum_{\nu=1}^{8} \sum_{S_\nu} \frac{1}{N} \sum_{k=1}^{N} Q_{ij}^{(k\nu)} \geq 0) =$$

$$= P(\sum_{k=1}^{N} (\sum_{\nu=1}^{8} \sum_{S_\nu} Q_{ij}^{(k\nu)}) \geq 0), \qquad (47)$$

where:

$Q_{ij}^{(k1)} = U_{ij}^{(k)*} - \widetilde{V}_{ij}^{(k)}$,

....

$Q_{ij}^{(k8)} = Z_{ij}^{(k)*} - \widetilde{V}_{ij}^{(k)}$.

Last inequality in (47) can be transformed in the following way:

$$P(\sum_{k=1}^{N} (\sum_{\nu=1}^{8} \sum_{S_\nu} Q_{ij}^{(k\nu)}) \geq 0) = P(\sum_{k=1}^{N} (\sum_{\nu=1}^{8} \sum_{S_\nu} Q_{ij}^{(k\nu)}) - E(\sum_{k=1}^{N} (\sum_{\nu=1}^{8} \sum_{S_\nu} Q_{ij}^{(k\nu)})) \geq -E(\sum_{k=1}^{N} (\sum_{\nu=1}^{8} \sum_{S_\nu} Q_{ij}^{(k\nu)}))) =$$

$$= P(\sum_{k=1}^{N} (\sum_{\nu=1}^{8} \sum_{S_\nu} Q_{ij}^{(k\nu)}) - N \sum_{\nu=1}^{8} \sum_{S_\nu} E(Q_{ij}^{(1\nu)}) \geq -N \sum_{\nu=1}^{8} \sum_{S_\nu} E(Q_{ij}^{(1\nu)})). \qquad (48)$$

The equality (48) results from the assumption, that for any k ($1 \leq k \leq N$) the distributions of the random variables $U_{ij}^{(k)}(\cdot)$, $V_{ij}^{(k)}(\cdot)$ and $Z_{ij}^{(k)}(\cdot)$ are the same. Therefore, the expected values of the variables $Q_{ij}^{(k\nu)}$ ($1 \leq k \leq N$) are also the same.

The probability (48) can be evaluated on the basis of Hoeffding inequality (45) in the following way:

$$P\left(\sum_{k=1}^{N}\left(\sum_{\nu=1}^{8}\sum_{S_\nu}Q_{ij}^{(k\nu)}\right)-N\sum_{\nu=1}^{8}\sum_{S_\nu}E(Q_{ij}^{(1\nu)})\ge -N\sum_{\nu=1}^{8}\sum_{S_\nu}E(Q_{ij}^{(1\nu)})\right)\le \exp\left\{-2N\frac{\left(\sum_{\nu=1}^{8}\sum_{S_\nu}E(Q_{ij}^{(1\nu)})\right)^2}{(2\vartheta(m-1))^2}\right\}. \quad (49)$$

The evaluation (49) results from the following facts: that absolute value of each difference $|T(x_i, x_j)-g_k(x_i, x_j)|-|\tilde{T}(x_i, x_j)-g_k(x_i, x_j)|$ $((x_i, x_j)\in \mathbf{X}\times\mathbf{X})$ cannot exceed the value $2(m-1)$, the number of components of the sum $\sum_{\nu=1}^{8}\sum_{S_\nu}E(Q_{ij}^{(1\nu)})$ equals ϑ and each expected value $E(Q_{ij}^{(k\nu)})$ is negative (see lemma in the section 3). The evaluation is equivalent to the proved inequality (46).

□

The inequality (46) shows that $P(\overline{W}^* < \tilde{\overline{W}})$, i.e. the probability of the event that the value of the random variable $\overline{W}^*$ is lower than any other variable $\tilde{\overline{W}}$, converges exponentially to one, for $N\to\infty$. Moreover, each variance $Var(\overline{W}(\chi_1, ..., \chi_r))$ converges to zero, when $N\to\infty$. Therefore, any variable $\overline{W}(\chi_1, ..., \chi_r)$ converges in stochastic way to some constant $\overline{\varpi}(\chi_1, ..., \chi_r)$; the constant $\overline{\varpi}^*$, corresponding to the variable $\overline{W}^*$ (i.e. relation $\chi_1^*, ..., \chi_n^*$) assumes minimal value in the set $\{\overline{\varpi}(\chi_1, ..., \chi_r) \mid \chi_1, ..., \chi_r \in F_X$; F_X - the family of all preference relations in the set $\mathbf{X}\}$. This facts indicates the form of the estimator – to determine the relation $\hat{\chi}_1, ..., \hat{\chi}_{\hat{n}}$, which minimizes the value of the random variable $\overline{W}(\chi_1, ..., \chi_r)$ for given comparisons $g_k(x_i, x_j)$ ($k=1, ..., N$; $(x_i, x_j)\in\mathbf{X}\times\mathbf{X}$). Let us notice, that the value of the right-hand side of the inequality (46) depends on the form of the relation $\tilde{\chi}_1, ..., \tilde{\chi}_r$; an increase of "dissimilarity" between $\tilde{\chi}_1, ..., \tilde{\chi}_r$ and actual relation $\chi_1^*, ..., \chi_n^*$ increases the expected value $\sum_{\nu=1}^{8}\sum_{S_\nu}E(Q_{ij}^{(1\nu)})$ and - finally - decreases the probability

$P(\overline{W}^* \geq \tilde{\overline{W}})$. In other words - more dissimilar relation $\tilde{\chi}_1, ..., \tilde{\chi}_r$, in comparison to the relation $\chi_1^*, ..., \chi_n^*$, is less probable.

The optimization task for the case under examination assumes the form:

$$\min_{\chi_1^{(\iota)}, ..., \chi_{r(\iota)}^{(\iota)} \in F_X} \{ \sum_{<i,j> \in R_m} \sum_{k=1}^{N} |t^{(\iota)}(x_i, x_j) - g_k(x_i, x_j)| \}, \tag{50}$$

where:

F_X - the feasible set of the problem, i.e. the family of all preference relations in the set **X**,

$t^{(\iota)}(x_i, x_j)$ - the function describing the relation $\chi_1^{(\iota)}, ..., \chi_{r^{(\iota)}}^{(\iota)}$ from the feasible set F_X

(the factor $1/N$ is omitted, because it does not influence the form of the optimal solution).

The number of solutions of the problem (50) can be greater than one. In the case of multiple solutions, the inequality (46) relates to whole set of the solutions obtained. The unique form of the relation can be chosen randomly or with the use of an additional criterion, e.g. minimal value of the expression $\sum_{<i,j> \in I(\hat{\chi}_1, ..., \hat{\chi}_{\hat{n}})} \sum_{k=1}^{N} |\hat{t}(x_i, x_j) - g_k(x_i, x_j)|$ (the function $\hat{t}(\cdot)$ describes the estimate $\hat{\chi}_1, ..., \hat{\chi}_{\hat{n}}$).

The evaluation (46) is similar to those presented in Klukowski (1994) point 5.1, corresponding the case, when comparisons indicate the direction of preferences (not difference of ranks). The right-hand side of the probability (46) is better (assumes higher value), than the evaluation presented in Klukowski (1994) in the case, when:

$$(\sum_{\nu=1}^{8} \sum_{S_\nu} E(Q_{ij}^{(1\nu)}))^2 \Big/ (2\vartheta(m-1))^2 > (1/2-\delta)^2,$$

where δ denotes the maximal probability of error in comparisons expressing the direction of preference.

Numerical value of the right-hand side of the inequality (46) can be determined in the case of known distributions of comparison errors and the form of the relation $\chi_1^\bullet, ..., \chi_n^\bullet$. If not, they can be replaced with estimates or evaluations (see Klukowski 2007). The estimation requires sufficient number of comparisons N, at least several.

Let us notice that the evaluation (46) is, in general, significantly underestimated; its negative feature is dependence on number of elements m. Therefore, in the case of "reliable" estimation of the relation form (it is indicated by the minimal value of the function (50) close to zero), the value $m-1$ can be replaced with the estimate $\hat{n}$-1. The estimate can be usually "reliable" for moderate N, e.g. several, because of exponential form of the right-hand side of inequality (46).

Let us notice, that it is also possible to consider the estimation problem with the use quadratic function instead of absolute value, e.g.:

$$U_k^2(x_i, x_j) = (t(x_i, x_j) - g_k(x_i, x_j))^2; \quad t(x_i, x_j) = 0; \tag{51}$$

$$\overline{W}^2(x_i, x_j) = \frac{1}{N} \sum_{k=1}^{N} (\sum_{<i,j> \in I(\cdot)} U_k^2(x_i, x_j) + \sum_{<i,j> \in P_1(\cdot)} V_k^2(x_i, x_j) + \sum_{<i,j> \in P_2(\cdot)} Z_k^2(x_i, x_j)), \tag{52}$$

or with the use of the average $\overline{g}(x_i, x_j)$ instead of individual $g_k(x_i, x_j)$, e.g.:

$$\breve{U}(x_i, x_j) = |t(x_i, x_j) - \overline{g}(x_i, x_j)|; \quad t(x_i, x_j) = 0; \tag{53}$$

$$\breve{W}(x_i, x_j) = \sum_{<i,j> \in I(\cdot)} \breve{U}(x_i, x_j) + \sum_{<i,j> \in P_1(\cdot)} \breve{V}(x_i, x_j) + \sum_{<i,j> \in P_2(\cdot)} \breve{Z}(x_i, x_j), \tag{54}$$

where: $\overline{g}(x_i, x_j) = \frac{1}{N} \sum_{k=1}^{N} g_k(x_i, x_j)$, etc.

The minimization problems assume the following forms for these functions:

$$\min_{\chi_1^{(t)}, ..., \chi_{r(t)}^{(t)} \in F_X} \{ \sum_{<i,j> \in R_m} \sum_{k=1}^{N} (t^{(t)}(x_i, x_j) - g_k(x_i, x_j))^2 \}, \tag{55a}$$

$$\min_{\chi_1^{(t)},\ldots,\chi_{r(t)}^{(t)}\in F_X} \{ \sum_{<i,j>\in R_m} \left| t^{(t)}(x_i,x_j) - \bar{g}(x_i,x_j) \right| \}. \tag{55b}$$

The properties of such estimators need further investigation.

5. The median approach

In the case of the median approach it is assumed, that N is odd, i.e. $N=2\tau+1$ ($\tau=0, 1, \ldots$). The form of estimator is based on the random variables: $U_{me,N}(x_i,x_j)$, $V_{me,N}(x_i,x_j)$, $Z_{me,N}(x_i,x_j)$, $W_{me,N}(\chi_1,\ldots,\chi_r)$ defined in the following way:

$$U_{me,N}(x_i,x_j) = | t(x_i, x_j) - g_{me,N}(x_i, x_j) |; \qquad t(x_i, x_j) = 0, \tag{56}$$

$$V_{me,N}(x_i,x_j) = | t(x_i, x_j) - g_{me,N}(x_i, x_j) |; \qquad t(x_i, x_j) < 0, \tag{57}$$

$$Z_{me,N}(x_i,x_j) = | t(x_i, x_j) - g_{me,N}(x_i, x_j) |; \qquad t(x_i, x_j) > 0, \tag{58}$$

$$W_{me,N}(\chi_1,\ldots,\chi_r) = \sum_{I(\cdot)} U_{me,N}(x_i,x_j) + \sum_{P_1(\cdot)} V_{me,N}(x_i,x_j) + \sum_{P_2(\cdot)} Z_{me,N}(x_i,x_j), \tag{59}$$

where:

$g_{me,N}(x_i, x_j)$ – the median from comparisons $g_k(x_i, x_j)$ ($k=1, \ldots, N$), i.e. $g_{me,N}(x_i, x_j)= g_{((N+1)/2)}(x_i,x_j)$ and symbols $g_{(1)}(x_i, x_j), \ldots, g_{(N)}(x_i, x_j)$ denote the comparisons: $g_1(x_i, x_j), \ldots, g_N(x_i, x_j)$ ordered in non-decreasing manner.

The symbols corresponding to the actual relation $\chi_1^*, \ldots, \chi_n^*$ are marked with asterisks, i.e.: $U^*_{ij,me,N}$, $V^*_{ij,me,N}$, $Z^*_{ij,me,N}$, $W^*_{me,N}$, while corresponding to any other relation $\tilde{\chi}_1, \ldots, \tilde{\chi}_{\tilde{n}}$ - with tildas, i.e.: $\tilde{U}_{ij,me,N}$, $\tilde{V}_{ij,me,N}$, $\tilde{Z}_{ij,me,N}$, $\tilde{W}_{me,N}$.

Using such notations the difference $W^*_{me,N}$ - $\tilde{W}_{me,N}$ - the basis for the properties of the estimator based on medians - assumes the form:

$$W^{*}_{me,N} - \widetilde{W}_{me,N} = \sum_{I^{*}\cap(\widetilde{P}_1-P_1^{*})} (U^{*}_{ij,me,N} - \widetilde{V}_{ij,me,N}) + \sum_{I^{*}\cap(\widetilde{P}_2-P_2^{*})} (U^{*}_{ij,me,N} - \widetilde{Z}_{ij,me,N}) +$$

$$+ \sum_{P_1^{*}\cap(\widetilde{I}-I^{*})} (V^{*}_{ij,me,N} - \widetilde{U}_{ij,me,N}) + \sum_{P_1^{*}\cap(\widetilde{P}_2-P_2^{*})} (V^{*}_{ij,me,N} - \widetilde{Z}_{ij,me,N}) +$$

$$+ \sum_{(P_1^{*}\cap\widetilde{P}_1)\cap(T(\cdot)\neq\widetilde{T}(\cdot))} (V^{*}_{ij,me,N} - \widetilde{V}_{ij,me,N}) + \sum_{P_2^{*}\cap(\widetilde{I}-I^{*})} (Z^{*}_{ij,me,N} - \widetilde{U}_{ij,me,N}) +$$

$$+ \sum_{P_2^{*}\cap(\widetilde{P}_1-P_1^{*})} (Z^{*}_{ij,me,N} - \widetilde{V}_{ij,me,N}) + \sum_{(P_2^{*}\cap\widetilde{P}_2)\cap(T(\cdot)\neq\widetilde{T}(\cdot))} (Z^{*}_{ij,me,N} - \widetilde{Z}_{ij,me,N}), \tag{60}$$

equivalent to:

$$W^{*}_{me,N} - \widetilde{W}_{me,N} = \sum_{\nu=1}^{8} \sum_{S_\nu} Q^{(\nu)}_{ij,me,N}, \tag{61}$$

where:

$$Q^{(1)}_{ij,me,N} = U^{*}_{ij,me,N} - \widetilde{V}_{ij,me,N}, \quad \langle i,j\rangle \in S_1,$$

$$Q^{(2)}_{ij,me,N} = U^{*}_{ij,me,N} - \widetilde{Z}_{ij,me,N}, \quad \langle i,j\rangle \in S_2,$$

$$Q^{(3)}_{ij,me,N} = V^{*}_{ij,me,N} - \widetilde{U}_{ij,me,N}, \quad \langle i,j\rangle \in S_3,$$

$$Q^{(4)}_{ij,me,N} = V^{*}_{ij,me,N} - \widetilde{Z}_{ij,me,N}, \quad \langle i,j\rangle \in S_4,$$

$$Q^{(5)}_{ij,me,N} = V^{*}_{ij,me,N} - \widetilde{V}_{ij,me,N}, \quad \langle i,j\rangle \in S_5,$$

. . . .

$$Q^{(8)}_{ij,me,N} = Z^{*}_{ij,me,N} - \widetilde{Z}_{ij,me,N}, \quad \langle i,j\rangle \in S_8,$$

(S_ν - defined in (24) – (31)).

The properties of the difference $W^{*}_{me,N} - \widetilde{W}_{me,N}$ are determined in the following

Theorem 2.

The difference $W^{*}_{me,N} - \widetilde{W}_{me,N}$ *satisfies the inequalities:*

$$E(W^{*}_{me,N} - \widetilde{W}_{me,N}) < 0, \tag{62}$$

$$P(W^{*}_{me,N} < \widetilde{W}_{me,N}) \geq -\frac{E(\sum_{\nu=1}^{8}\sum_{S_\nu} Q^{(\nu)}_{ij,me,N})}{\lambda_1(m-1) + 2\lambda_2(m-1) + \lambda_3(m-2)}, \tag{63}$$

where:

$\lambda_1 = \#(S_1 \cup S_2 \cup S_3 \cup S_6)$; $\lambda_2 = \#(S_4 \cup S_7)$; $\lambda_3 = \#(S_5 \cup S_8)$

(symbol $\#(\Xi)$ *– means the number of elements of the set* Ξ*).*

Proof of the inequality (62).

The inequality (62) is true for $N=1$ (on the basis of the inequality (32) - see lemma in Section 3). For $N=2\tau+1$ ($\tau=1, \ldots,$) the proof is similar to the case examined in Klukowski (2007); therefore its draft is presented only below. The probability function $P(T(x_i, x_j) - g_{me,N}(x_i, x_j)) = l)$ ($N=2\tau+1$; $\tau=0, 1, \ldots,$) satisfies for each pair $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$ the inequalities:

$$P(T(x_i, x_j) - g_{me,N+2}(x_i, x_j) = 0) > P(T(x_i, x_j) - g_{me,N}(x_i, x_j) = 0), \tag{64a}$$

$$P(T(x_i, x_j) - g_{me,N+2}(x_i, x_j) = l) < P(T(x_i, x_j) - g_{me,N}(x_i, x_j) = l) \qquad (l \neq 0). \tag{64b}$$

The inequalities (64a) and (64b) result from the following facts. The probabilities $P(T(x_i, x_j) - g_{me,N}(x_i, x_j) = l)$ can be expressed in the form (see David (1970), section 2.4):

$$P(T(x_i, x_j) - g_{me,N}(x_i, x_j) = 0) =$$

$$= P(T(x_i, x_j) - g_{me,N}(x_i, x_j) \leq 0) - P(T(x_i, x_j) - g_{me,N}(x_i, x_j) \leq -1) =$$

$$=\frac{N!}{(((N-1)/2)!)^2}\int_{G(-1)}^{G(0)} t^{(N-1)/2}(1-t)^{(N-1)/2}dt, \tag{65a}$$

$$P(T(x_i,x_j)-g_{me,N}(x_i,x_j)=l)=$$

$$=P(T(x_i,x_j)-g_{me,N}(x_i,x_j)\le l)-P(T(x_i,x_j)-g_{me,N}(x_i,x_j)\le l-1)=$$

$$=\frac{N!}{(((N-1)/2)!)^2}\int_{G(l-1)}^{G(l)} t^{(N-1)/2}(1-t)^{(N-1)/2}dt, \tag{65b}$$

where:

$$G(l)=P(T(x_i,x_j)-g_k(x_i,x_j)\le l).$$

The expressions (65a) and (65b) are determined on the basis of beta distribution $B(p, q)$, with parameters $p=q=(N+1)/2$. The expected value and variance of the distribution assume the forms – respectively: ½ and $((N+1)/2)^2/(N+1)^2(N+2)=\frac{1}{4(N+2)}$. The variance of the distribution converges to zero for $N\to\infty$ and the integrand in integrals (65a), (65b) is symmetric around ½. These facts guarantee, that: the distributions of the random variables $T(x_i,x_j)-g_{me,N}(x_i,x_j)$ ($(x_i, x_j)\in\mathbf{X}\times\mathbf{X}$) are for each N unimodal, their probability functions assume maximum in zero (i.e. for $T(x_i,x_j)-g_{me,N}(x_i,x_j)=0$) and satisfy the inequalities (64a), (64b). Last two conditions are sufficient (see the assumptions (9) - (12) and lemma from section 3) for the inequality (62).

Proof of the inequality (63).

The inequality (63) is proved on the basis of Chebyshev inequality for expected value. For this purpose the left-hand side of the inequality is transformed to the form $P(W^*_{me,N}<\widetilde{W}_{me,N})=1-P(W^*_{me,N}-\widetilde{W}_{me,N}\ge 0)$ and each random variable $Q^{(\nu)}_{ij,me,N}$ is transformed to the form, which provides non-negative expected value:

$$Q'^{(\nu)}_{ij,me,N} = Q^{(\nu)}_{ij,me,N} + (m\text{-}1) \qquad (\nu\text{=}1, 2, 3, 6), \tag{66}$$

$$Q'^{(\nu)}_{ij,me,N} = Q^{(\nu)}_{ij,me,N} + 2(m\text{-}1) \qquad (\nu\text{=}5, 8), \tag{67}$$

$$Q'^{(\nu)}_{ij,me,N} = Q^{(\nu)}_{ij,me,N} + (m\text{-}2) \qquad (\nu\text{=}4, 7). \tag{68}$$

The probability $P(W^*_{me,N} - \widetilde{W}_{me,N} \geq 0)$ can be evaluated in the following way:

$$P(W^*_{me,N} - \widetilde{W}_{me,N} \geq 0) = P(\sum_{\nu=1}^{8} \sum_{S_\nu} Q^{(\nu)}_{ij,me,N} \geq 0) =$$

$$P(\sum_{\nu=1}^{8} \sum_{S_\nu} Q'^{(\nu)}_{ij,me,N} \geq \lambda_1(m-1) + 2\lambda_2(m-1) + \lambda_3(m-2)) \leq$$

$$\leq \frac{E(\sum_{\nu=1}^{8} \sum_{S_\nu} Q'^{(\nu)}_{ij,me,N})}{\lambda_1(m-1) + 2\lambda_2(m-1) + \lambda_3(m-2)} = 1 + \frac{E(\sum_{\nu=1}^{8} \sum_{S_\nu} Q^{(\nu)}_{ij,me,N})}{(\lambda_1 + 2\lambda_2)(m-1) + \lambda_3(m-2)}. \tag{69}$$

The inequality (69) is equivalent to proved inequality (63).

□

The right-hand side of the inequality (63) is included in the interval (0, 1). Its numerical value can be determined in the case of known distributions of comparison errors $P(T(\cdot) - g_{me,N}(\cdot) = \iota)$. In opposite case it can be estimated or approximated. An approximation procedure for this purpose, useful for moderate N (less than several), based on the formulas (65 a, b) and an assumption about symmetry of distribution tails, can be constructed in similar way, as in Klukowski (2007), for the tolerance relation. For N greater than several, unknown probability functions of comparison errors can be estimated.

The evaluation (63) based on the value m is usually underestimated (lower than actual probability) and – similarly as in the case of averaged comparisons – the values: m–1, 2(m –1) and m-2 can be replaced with the estimates based on $\hat{n}$. It seems also rational to use the estimates of the form: $\max_{\hat{T}(x_i,x_j)=0}\{|\tilde{T}(x_i,x_j)|\}$, $2\max_{\hat{T}(x_i,x_j)\cdot\tilde{T}(x_i,x_j)<0}\{|\hat{T}(x_i,x_j)-\tilde{T}(x_i,x_j)|\}$ and $\max_{\hat{T}(x_i,x_j)\cdot\tilde{T}(x_i,x_j)>0}\{|\hat{T}(x_i,x_j)-\tilde{T}(x_i,x_j)|\}$ (the expression $\hat{T}(x_i,x_j)\cdot\tilde{T}(x_i,x_j)$ means the product).

The minimisation task for the estimation of the preference relation is similar, as in the case N=1 (see (50)). It assumes the form:

$$\min_{\chi_1^{(\iota)},\ldots,\chi_{r(\iota)}^{(\iota)}\in F_X}\{\sum_{<i,j>\in R_m}|t^{(\iota)}(x_i,x_j)-g_{me,N}(x_i,x_j)|\}. \quad (70)$$

The number of solutions of the task (70) can exceed one.

It should be emphasized, that the evaluation (63) is based on rough probability inequality. However, it seems conceivable, that for some types of distributions of comparison errors, the efficiency of the median approach is similar to those corresponding to the averaging approach.

The right-hand side of the inequality (63) does not converge exponentially to one. However, the estimator, which guarantee such convergence can be constructed for medians (from the differences of ranks) on the basis of the approach presented in Klukowski 1994, point 5.2. The differences of ranks have to be transformed into comparisons indicating the direction of the preference, which satisfy the condition, that probability of errorless comparison is higher than ½. The idea of the transformation can be presented briefly in the following way. On the basis of the formulas (65a, b) it can be determined the minimal value (integer) κ, ($\kappa\leq N$), which guarantee, for each pair $(x_i, x_i)\in\mathbf{X}\times\mathbf{X}$, the condition:

$$P(T(\cdot)-g_{me,\kappa}(\cdot)=0) > ½, \quad (71)$$

where: $g_{me,\kappa}(\cdot)$ is the median in the subset of κ consecutive comparisons, i.e. $\{g_1(\cdot), \ldots, g_\kappa(\cdot)\}$ or $\{g_{\kappa+1}(\cdot), \ldots, g_{2\kappa}(\cdot)\}$, etc.

Let us define the random variables $U_{ij,\tau}(\chi_1, \ldots, \chi_r)$, $V_{ij,\tau}(\chi_1, \ldots, \chi_r)$, $Z_{ij,\tau}(\chi_1, \ldots, \chi_r)$ in the following way:

$$U_{ij,\tau}(\chi_1, \ldots, \chi_r) = \begin{cases} 0; & g_{me,\tau}(x_i, x_j) = t(x_i, x_j) \quad \text{for } \langle i, j\rangle \in I(\chi_1, \ldots, \chi_r); \\ 1; & g_{me,\tau}(x_i, x_j) \neq t(x_i, x_j) \quad \text{for } \langle i, j\rangle \in I(\chi_1, \ldots, \chi_r), \end{cases} \tag{72a}$$

$$V_{ij,\tau}(\chi_1, \ldots, \chi_r) = \begin{cases} 0; & g_{me,\tau}(x_i, x_j) = t(x_i, x_j) \quad \text{for } \langle i, j\rangle \in P_1(\chi_1, \ldots, \chi_r); \\ 1; & g_{me,\tau}(x_i, x_j) \neq t(x_i, x_j) \quad \text{for } \langle i, j\rangle \in P_1(\chi_1, \ldots, \chi_r), \end{cases} \tag{72b}$$

$$Z_{ij,\tau}(\chi_1, \ldots, \chi_r) = \begin{cases} 0; & g_{me,\tau}(x_i, x_j) = t(x_i, x_j) \quad \text{for } \langle i, j\rangle \in P_2(\chi_1, \ldots, \chi_r); \\ 1; & g_{me,\tau}(x_i, x_j) \neq t(x_i, x_j) \quad \text{for } \langle i, j\rangle \in P_2(\chi_1, \ldots, \chi_r), \end{cases} \tag{72c}$$

where:

$g_{me,\tau}(x_i, x_j)$ - the median in the set of comparisons $\{g_{(\tau-1)\cdot\kappa+1}(\cdot), \ldots, g_{\tau\kappa}(\cdot)\}$ ($\tau=1, \ldots, \vartheta$); the expression $a\cdot b$ - (index in $g_{a\cdot b}(\cdot)$) means the product of a and b,

ϑ - integer part of the quotient N/κ (odd number), i.e $\vartheta = \text{ent}(N/\kappa)$.

Now, the majority approach, introduced in Klukowski (1994) point 5.2 (equivalent to the median in the set comprising zero-one random variables), can be applied to the random variables $U_{ij,\tau}(\chi_1, \ldots, \chi_r)$, $V_{ij,\tau}(\chi_1, \ldots, \chi_r)$, $Z_{ij,\tau}(\chi_1, \ldots, \chi_r)$ ($\tau=1, \ldots, \vartheta$). As a result, one can obtain the variables $U_{ij,\vartheta}^{(me)}$, $V_{ij,\vartheta}^{(me)}$, $Z_{ij,\vartheta}^{(me)}$ defined as follows:

$$U_{ij,\vartheta}^{(me)}(\chi_1, \ldots, \chi_r) = \begin{cases} 0; & \sum_{\tau=1}^{\vartheta} U_{ij,\tau}(\chi_1, \ldots, \chi_r) < \vartheta/2 \quad \text{for } \langle i,j\rangle \in I(\chi_1, \ldots, \chi_r); \\ 1; & \sum_{\tau=1}^{\vartheta} U_{ij,\tau}(\chi_1, \ldots, \chi_r) > \vartheta/2 \quad \text{for } \langle i,j\rangle \in I(\chi_1, \ldots, \chi_r), \end{cases} \tag{73a}$$

$$V_{ij,\vartheta}^{(me)}(\chi_1, \ldots, \chi_r) = \begin{cases} 0; & \sum_{\tau=1}^{\vartheta} V_{ij,\tau}(\chi_1, \ldots, \chi_r) < \vartheta/2 \quad \text{for } \langle i,j\rangle \in P_1(\chi_1, \ldots, \chi_r); \\ 1; & \sum_{\tau=1}^{\vartheta} V_{ij,\tau}(\chi_1, \ldots, \chi_r) > \vartheta/2 \quad \text{for } \langle i,j\rangle \in P_1(\chi_1, \ldots, \chi_r), \end{cases} \tag{73b}$$

$$Z_{ij,\vartheta}^{(me)}(\chi_1, \ldots, \chi_r) = \begin{cases} 0; & \sum_{\tau=1}^{\vartheta} Z_{ij,\tau}(\chi_1, \ldots, \chi_r) < \vartheta/2 \quad \text{for } \langle i,j\rangle \in P_2(\chi_1, \ldots, \chi_r); \\ 1; & \sum_{\tau=1}^{\vartheta} Z_{ij,\tau}(\chi_1, \ldots, \chi_r) > \vartheta/2 \quad \text{for } \langle i,j\rangle \in P_2(\chi_1, \ldots, \chi_r). \end{cases} \tag{73c}$$

Let us apply the convention used in previous sections to the variables: $U_{ij,\vartheta}^{(me)}(\cdot)$, $V_{ij,\vartheta}^{(me)}(\cdot)$, $Z_{ij,\vartheta}^{(me)}(\cdot)$, i.e. the symbols corresponding to the actual relation $\chi_1^*, \ldots, \chi_n^*$ will be marked with asterisks: $U_{ij,\vartheta}^{(me)*}$, $V_{ij,\vartheta}^{(me)*}$, $Z_{ij,\vartheta}^{(me)*}$, while the symbols corresponding to any other relation $\widetilde{\chi}_1, \ldots, \widetilde{\chi}_r$ – with tildas: $\widetilde{U}_{ij,\vartheta}^{(me)}$, $\widetilde{V}_{ij,\vartheta}^{(me)}$, $\widetilde{Z}_{ij,\vartheta}^{(me)}$.

Finally let us define the random variables W_ϑ^* and $\widetilde{W}_\vartheta$:

$$W_\vartheta^* = \sum_{I^*} U_{ij,\vartheta}^{(me)*} + \sum_{P_1^*} V_{ij,\vartheta}^{(me)*} + \sum_{P_2^*} Z_{ij,\vartheta}^{(me)*}, \tag{74}$$

$$\widetilde{W}_\vartheta = \sum_{\widetilde{I}} \widetilde{U}_{ij,\vartheta}^{(me)} + \sum_{\widetilde{P}_1} \widetilde{V}_{ij,\vartheta}^{(me)} + \sum_{\widetilde{P}_2} \widetilde{Z}_{ij,\vartheta}^{(me)}. \tag{75}$$

On the basis of the results presented in Klukowski (1994), point 5.2, it is clear that:

$$P(W^*_\vartheta - \widetilde{W}_\vartheta < 0) > 1 - 2\lambda_\vartheta \,, \tag{76}$$

where:

$$\lambda_\vartheta = \exp\{-2\vartheta(1/2 - \delta^{(\kappa)}_{\max})^2\} \tag{77}$$

and

$$\delta^{(\kappa)}_{\max} = \max_{(x_i, x_j) \in \mathbf{X} \times \mathbf{X}} \{P(T(x_i, x_j) \neq g_{me,\kappa}(x_i, x_j))\}.$$

If $\kappa > 1$, then the convergence obtained as a result of the zero-one transformation is weaker, than those in Klukowski (1994), because $\vartheta < N$ in the equality (77) (in other words the exponent in the right-hand side of relationship (76) "decreases with the step κ"). The case $\kappa = 1$ is not excluded, in general, but it is satisfied only in the case $P(T(\cdot) - g_k(\cdot) = 0) > 1/2$ for each $(x_i, x_l) \in \mathbf{X} \times \mathbf{X}$.

It seems viable to prove, that efficiency of the median approach in the case of difference of ranks is not worse than those based on the transformations (72a) – (73c); the problem needs further investigations.

7. Summary

The paper presents two approaches to estimation of the preference relation on the basis of multiple pairwise comparisons in the form of difference of ranks. The results are extensions and completion of the case $N=1$ (one comparison of each pair) considered in Klukowski (2000); the extension is based on the ideas similar to those developed in Klukowski (1994) (for the case of comparisons indicating the direction of preference). The algorithms presented in the paper are based on weak assumptions about distributions of comparison errors. The properties of the averaging approach, especially exponential convergence of the probability $P(\overline{W}^* < \widetilde{\overline{W}})$ to one for $N \rightarrow \infty$, are meaningful. On the other hand, the optimisation problem corresponding to the median approach is easy to solve. The question about efficiency of the median approach, in comparison to averaging approach needs further investigations. It seems reasonable to investigate the properties of the estimators, difficult to analytic examination, with the use of simulation approach.

Appendix

The proof of the lemma (section 3)

Lemma

The expected value of each random variable $Q_{ij}^{(k,\nu)}$ ($1 \le k \le N$; $<i,j> \in S_\nu$; $\nu=1, \ldots, 8$), defined in (24) – (31), satisfy the condition:

$$E(Q_{ij}^{(k,\nu)}) < 0. \tag{A1}$$

Proof.

The proof of the inequality (A1) is elementary (but cumbersome).

Let us consider the cases $\nu=1, 3, 5, 7$. In the case $\nu=1$ the random variable $Q_{ij}^{(k,1)}$ assumes the form:

$$Q_{ij}^{(k,1)} = U_{ij}^{(k)*} - \widetilde{V}_{ij}^{(k)} = | g_k(x_i, x_j) | - | \widetilde{T}(x_i, x_j) - g_k(x_i, x_j) | \qquad (<i,j> \in I^* \cap (\widetilde{P}_1 - P_1^*)). \tag{A2}$$

The facts $T(\cdot)=0$ and $\widetilde{T}(\cdot)<0$ indicate three possible situations:

(i) $g_k(\cdot) \le \widetilde{T}(\cdot)$;

(ii) $\widetilde{T}(\cdot) < g_k(\cdot) < T(\cdot)$;

(iii) $g_k(\cdot) \ge T(\cdot)$.

For the values $g_k(\cdot) \le \widetilde{T}(\cdot)$ (the case (i)) the difference $U_{ij}^{(k)*}$-$\widetilde{V}_{ij}^{(k)}$ equals: -$\widetilde{T}(\cdot)$>0 with the probability: $\sum_{l \le \widetilde{T}(\cdot)} P(g_k(\cdot) = l) < ½$. In the case (iii) the difference (A2) is equal to: $\widetilde{T}(\cdot)$<0 with the probability $\sum_{l \ge T(\cdot)} P(g_k(\cdot) = l) > ½$. The inequality (ii) indicates, that the difference (A2) equals $\widetilde{T}(\cdot)$-2$g_k(\cdot)$. The expression $\widetilde{T}(\cdot)$-2$g_k(\cdot)$ ($\widetilde{T}(\cdot)$<$g_k(\cdot)$<$T(\cdot)$) satisfy the condition:

$$\widetilde{T}(\cdot) < \widetilde{T}(\cdot) - 2g_k(\cdot) < -\widetilde{T}(\cdot)$$

and assume values from the set {$\widetilde{T}(\cdot)$+2, ..., -$\widetilde{T}(\cdot)$-2} with probabilities $P(\widetilde{T}(\cdot) - 2g_k = \iota) = P(g_k = (\widetilde{T}(\cdot) - \iota)/2)$. The expression $\widetilde{T}(\cdot)$-2$g_k(\cdot)$ ($\widetilde{T}(\cdot)$<$g_k(\cdot)$<$T(\cdot)$) assumes values placed symmetrically around zero; their probabilities satisfy the conditions:

$$P(\widetilde{T}(\cdot) - 2g_k(\cdot) = -\iota) \ge P(\widetilde{T}(\cdot) - 2g_k(\cdot) = \iota) \qquad (\iota > 0);$$

last inequality results from the fact, that in the case $\widetilde{T}(\cdot)$-2$g_k(\cdot)$=-ι the difference $T(\cdot)$-$g_k(\cdot)$ is closer to zero than in the case $\widetilde{T}(\cdot)$-2$g_k(\cdot)$=ι (in other words the value $g_k(\cdot)$ is closer to $T(\cdot)$). Assembling the facts concerning the case under consideration ($T(\cdot)$=0 and $\widetilde{T}(\cdot)$<0), i.e.:

$$P(Q_{ij}^{(k,1)} = \widetilde{T}(\cdot)) = \sum_{l \ge T(\cdot)} P(g_k(\cdot) = l) > ½,$$

$$P(Q_{ij}^{(k,1)} = -\widetilde{T}(\cdot)) = \sum_{l \le \widetilde{T}_1(\cdot)} P(g_k(\cdot) = l) < ½,$$

$$P(\widetilde{T}(\cdot) - 2g_k(\cdot) = -\iota) \ge P(\widetilde{T}(\cdot) - 2g_k(\cdot) = \iota) \qquad (\iota > 0),$$

one can obtain:

$$E(Q_{ij}^{(k,1)}) < 0. \qquad \text{(A3)}$$

The random variable $Q_{ij}^{(k,3)}$ assumes the form:

$$Q_{ij}^{(k,3)} = V_{ij}^{(k)*} - \widetilde{U}_{ij}^{(k)} = | T(x_i, x_j) - g_k(x_i, x_j) | - | g_k(x_i, x_j) | \qquad (<i,j> \in P_1^* \cap (\widetilde{I} - I^*)). \qquad (A4)$$

The facts $T(\cdot)<0$ and $\widetilde{T}(\cdot)=0$ indicates three possible situations:

(iv) $g_k(\cdot) \le T(\cdot)$;

(v) $T(\cdot)< g_k(\cdot) < \widetilde{T}(\cdot)$;

(vi) $g_k(\cdot) \ge \widetilde{T}(\cdot)$.

For the values $g_k(\cdot) \le T(\cdot)$ (the case (iv)) the difference $V_{ij}^{(k)*} - \widetilde{U}_{ij}^{(k)}$ equals: $T(\cdot)<0$ with the probability: $\sum_{l \le T(\cdot)} P(g_k(\cdot)=l) = \sum_{l \ge 0} P(T(\cdot)-g_k(\cdot)=l) > ½$. In the case (vi) the difference (A4) is equal to: $-T(\cdot)>0$ with the probability $\sum_{l \ge \widetilde{T}(\cdot)} P(g_k(\cdot)=l) < ½$. The inequality (v) indicates, that the difference (A4) equals $2g_k(\cdot) - T(\cdot)$. The expression $2g_k(\cdot) - T(\cdot)$ $(T(\cdot)< g_k(\cdot) < \widetilde{T}(\cdot))$ satisfy the condition:

$$T(\cdot)<2g_k(\cdot)-T(\cdot)<-T(\cdot)$$

and assume values from the set $\{T(\cdot)+2, \ldots, -T(\cdot)-2\}$ with probabilities $P(2g_k(\cdot)-T(\cdot)=\iota) = P(g_k(\cdot)=(T(\cdot)+\iota)/2)$. The values of the expression $2g_k(\cdot)-T(\cdot)$ $(T(\cdot)< g_k(\cdot) < \widetilde{T}(\cdot))$ are placed symmetrically around zero; their probabilities satisfy the conditions:

$$P(2g_k(\cdot)-T(\cdot)=-\iota) \ge P(2g_k(\cdot)-T(\cdot)=\iota) \qquad (\iota>0);$$

last inequality results from the fact that in the case $2g_k(\cdot)-T(\cdot)=-\iota$, the difference $T(\cdot)-g_k(\cdot)$ is closer to zero, than in the case $2g_k(\cdot)-T(\cdot)=\iota$. Assembling the facts concerning the case under consideration ($T(\cdot)=0$ and $\widetilde{T}(\cdot)<0$), i.e.:

$$P(Q_{ij}^{(k,3)}=T(\cdot))=\sum_{l\leq T(\cdot)} P(g_k(\cdot)=l) > ½,$$

$$P(Q_{ij}^{(k,3)}=-T(\cdot))=\sum_{l\geq \tilde{T}_1(\cdot)} P(g_k(\cdot)=l) < ½,$$

$$P(2g_k(\cdot)-T(\cdot)=-\iota) \geq P(2g_k(\cdot)-T(\cdot)=\iota),$$

one can obtain:

$$E(Q_{ij}^{(k,3)})<0. \tag{A5}$$

The random variable $Q_{ij}^{(k,5)}$ assumes the form:

$$Q_{ij}^{(k,5)} = V_{ij}^{(k)*} - \tilde{V}_{ij}^{(k)} =$$

$$= |T(x_i, x_j) - g_k(x_i, x_j)| - |\tilde{T}(x_i, x_j) - g_k(x_i, x_j)| \quad (<i,j>\in(P_1^{*}\cap\tilde{P}_1)\cap(T(x_i, x_j)\neq\tilde{T}(x_i, x_j))). \tag{A6}$$

The facts $T(\cdot)<0$ and $\tilde{T}(\cdot)<0$ ($T(\cdot)\neq\tilde{T}(\cdot)$) indicates two systems of conditions. The first one corresponds to the inequality $T(\cdot)<\tilde{T}(\cdot)<0$ and one of the conditions:

(vii') $g_k(\cdot)\leq T(\cdot)$;

(viii') $T(\cdot)<g_k(\cdot)<\tilde{T}(\cdot)$;

(ix') $g_k(\cdot)\geq\tilde{T}(\cdot)$.

In the case (vii') the difference $V_{ij}^{(k)*} - \tilde{V}_{ij}^{(k)}$ equals: $T(\cdot)-\tilde{T}(\cdot)<0$ with the probability: $\sum_{l\leq T(\cdot)} P(g_k(\cdot)=l) > ½$. In the case (ix') the difference (A6) equals: $-T(\cdot)+\tilde{T}(\cdot)>0$ with the probability $\sum_{l\geq \tilde{T}(\cdot)} P(g_k(\cdot)=l) < ½$. In the case (viii') the expression (A6) equals: $2g_k(\cdot)-T(\cdot)-\tilde{T}(\cdot)$ ($T(\cdot)<g_k(\cdot)<\tilde{T}(\cdot)$) and satisfy the condition:

$$T(\cdot)-\tilde{T}(\cdot)<2g_k(\cdot)-T(\cdot)-\tilde{T}(\cdot)<-T(\cdot)+\tilde{T}(\cdot).$$

Moreover it assumes the values from the set $\{T(\cdot)-\tilde{T}(\cdot)+2, \ldots, -T(\cdot)+\tilde{T}(\cdot)-2\}$ with probabilities $P(2g_k(\cdot)-T(\cdot)-\tilde{T}(\cdot)=\iota)=P(g_k(\cdot)=(T(\cdot)+\tilde{T}(\cdot)+\iota)/2)$. The expression $2g_k(\cdot)-T(\cdot)-\tilde{T}(\cdot)$ ($T(\cdot)<g_k(\cdot)<\tilde{T}(\cdot)$) assumes values placed symmetrically around zero; their probabilities satisfy the conditions:

$$P(2g_k(\cdot)-T(\cdot)-\tilde{T}(\cdot)=-\iota)\geq P(2g_k(\cdot)-T(\cdot)-\tilde{T}(\cdot)=\iota) \qquad (\iota>0);$$

last inequality results from the fact, that in the case $2g_k(\cdot)-T(\cdot)-\tilde{T}(\cdot)=-\iota$, the difference $T(\cdot)-g_k(\cdot)$ is closer to zero, than in the case $2g_k(\cdot)-T(\cdot)-\tilde{T}(\cdot)=\iota$. Assembling the facts concerning the case: $T(\cdot)=0$ and $\tilde{T}(\cdot)<0$, i.e.:

$$P(Q_{ij}^{(k,5)}=T(\cdot)-\tilde{T}(\cdot))=\sum_{l\leq T(\cdot)}P(g_k(\cdot)=l)>½,$$

$$P(Q_{ij}^{(k,5)}=-T(\cdot)+\tilde{T}(\cdot))=\sum_{l\geq \tilde{T}_1(\cdot)}P(g_k(\cdot)=l)<½,$$

$$P(2g_k(\cdot)-T(\cdot)-\tilde{T}(\cdot)=-\iota)\geq P(2g_k(\cdot)-T(\cdot)-\tilde{T}(\cdot)=\iota),$$

one can obtain in the case $T(\cdot)<\tilde{T}(\cdot)<0$ the inequality:

$$E(Q_{ij}^{(k,5)})<0. \qquad \text{(A7)}$$

The second case, i.e. $\tilde{T}(\cdot)<T(\cdot)<0$ is similar to the previous one; the inequality $\tilde{T}(\cdot)<T(\cdot)$ may occur together with one of the conditions:

(vii") $g_k(\cdot)\leq\tilde{T}(\cdot)$;

(viii'') $\tilde{T}(\cdot)<g_k(\cdot)<T(\cdot)$;

(ix'') $g_k(\cdot)\geq T(\cdot)$.

In the case (vii'') the difference $V_{ij}^{(k)*}$ - $\widetilde{V}_{ij}^{(k)}$ equals: $T(\cdot)$-$\widetilde{T}(\cdot)$>0 with the probability: $\sum_{l\leq\widetilde{T}(\cdot)} P(g_k(\cdot)=l)$<½. In the case (ix'') the difference (A6) equals: $-T(\cdot)+\widetilde{T}(\cdot)$<0 with the probability $\sum_{l\geq T(\cdot)} P(g_k(\cdot)=l)$> ½. In the case (viii') the expression (A6) equals: $2g_k(\cdot)$ -$T(\cdot)$-$\widetilde{T}(\cdot)$ ($T(\cdot)<g_k(\cdot)<\widetilde{T}(\cdot)$) and satisfy the condition:

$$-T(\cdot)+\widetilde{T}(\cdot)<2g_k(\cdot)-T(\cdot)-\widetilde{T}(\cdot)<T(\cdot)-\widetilde{T}(\cdot).$$

Moreover, it assumes the values from the set $\{-T(\cdot)+\widetilde{T}(\cdot)+2, \ldots, T(\cdot)-\widetilde{T}(\cdot)-2\}$ with probabilities $P(2g_k(\cdot)-T(\cdot)-\widetilde{T}(\cdot)=\iota)=P(g_k(\cdot)=(T(\cdot)+\widetilde{T}(\cdot)+\iota)/2)$. The expression $2g_k(\cdot)$ - $T(\cdot)$-$\widetilde{T}(\cdot)$ ($T(\cdot)<g_k(\cdot)<\widetilde{T}(\cdot)$) assumes the values placed symmetrically around zero; their probabilities satisfy the conditions:

$$P(2g_k(\cdot)-T(\cdot)-\widetilde{T}(\cdot)=-\iota)\geq P(2g_k(\cdot)-T(\cdot)-\widetilde{T}(\cdot)=\iota) \qquad (\iota>0);$$

last inequality results from the fact, that in the case $2g_k(\cdot)-T(\cdot)-\widetilde{T}(\cdot)=-\iota$, the difference $T(\cdot)$-$g_k(\cdot)$ is smaller (closer to zero), than in the case $2g_k(\cdot)-T(\cdot)-\widetilde{T}(\cdot)=\iota$. Assembling the facts concerning the case under consideration ($T(\cdot)$=0 and $\widetilde{T}(\cdot)$<0), i.e.:

$$P(Q_{ij}^{(k,5)}=-T(\cdot)+\widetilde{T}(\cdot))=\sum_{l\geq T(\cdot)} P(g_k(\cdot)=l)>½,$$

$$P(Q_{ij}^{(k,5)}=T(\cdot)-\widetilde{T}(\cdot))=\sum_{l\leq\widetilde{T}_1(\cdot)} P(g_k(\cdot)=l)<½,$$

$$P(2g_k(\cdot)-T(\cdot)-\widetilde{T}(\cdot)=-\iota)\geq P(2g_k(\cdot)-T(\cdot)-\widetilde{T}(\cdot)=\iota),$$

one can obtain in the case $\widetilde{T}(\cdot)<T(\cdot)<0$ the inequality:

$$E(Q_{ij}^{(k,5)})<0. \qquad \text{(A8)}$$

The random variable $Q_{ij}^{(k,7)}$ assumes the form:

$$Q_{ij}^{(k,7)} = Z_{ij}^{(k)*} - \widetilde{V}_{ij}^{(k)} = | T(x_i, x_j) - g_k(x_i, x_j) | - | \widetilde{T}(x_i, x_j) - g_k(x_i, x_j) | \quad (<i,j> \in P_2^{*} \cap (\widetilde{P}_1 - P_1^{*})). \text{ (A9)}$$

The facts $T(\cdot)>0$ and $\widetilde{T}(\cdot)<0$ indicates three possible situations:

(x) $g_k(\cdot) \geq T(\cdot)$;

(xi) $\widetilde{T}(\cdot) < g_k(\cdot) < T(\cdot)$;

(xii) $g_k(\cdot) \leq \widetilde{T}(\cdot)$.

For the values $g_k(\cdot) \geq T(\cdot)$ (the case (x)) the difference $Z_{ij}^{(k)*} - \widetilde{V}_{ij}^{(k)}$ equals: $-T(\cdot)+\widetilde{T}(\cdot)<0$ with the probability: $\sum_{l \geq T(\cdot)} P(g_k(\cdot)=l) > ½$. In the case (xii) the difference (A9) is equal to: $T(\cdot)-\widetilde{T}(\cdot)>0$ with the probability $\sum_{l \leq \widetilde{T}(\cdot)} P(g_k(\cdot)=l) < ½$. The inequality (xi) indicates, that the difference (A9) equals $T(\cdot)+\widetilde{T}(\cdot)-2g_k(\cdot)$. The expression $T(\cdot)+\widetilde{T}(\cdot)-2g_k(\cdot)$ $(T(\cdot)<g_k(\cdot)<\widetilde{T}(\cdot))$ satisfy the condition:

$$-T(\cdot)+\widetilde{T}(\cdot) < T(\cdot)+\widetilde{T}(\cdot)-2g_k(\cdot) < T(\cdot)-\widetilde{T}(\cdot)$$

and assume values from the set $\{-T(\cdot)+\widetilde{T}(\cdot)+2, \ldots, T(\cdot)-\widetilde{T}(\cdot)-2\}$ with probabilities $P(T(\cdot)+\widetilde{T}(\cdot)-2g_k(\cdot)=\iota) = P(g_k(\cdot)=(T(\cdot)+\widetilde{T}(\cdot)-\iota)/2)$. The expression $T(\cdot)+\widetilde{T}(\cdot)-2g_k(\cdot)$ assumes values placed symmetrically around zero; their probabilities satisfy the conditions:

$$P(T(\cdot)+\widetilde{T}(\cdot)-2g_k(\cdot)=-\iota) \geq P(T(\cdot)+\widetilde{T}(\cdot)-2g_k(\cdot)=\iota) \qquad (\iota>0).$$

Last inequality results from the fact, that in the case $T(\cdot)+\widetilde{T}(\cdot)-2g_k(\cdot)=-\iota$, the difference $T(\cdot)-g_k(\cdot)$ is closer to zero, than in the case $T(\cdot)+\widetilde{T}(\cdot)-2g_k(\cdot)=\iota$. Assembling the facts concerning the case under consideration ($T(\cdot)>0$ and $\widetilde{T}(\cdot)<0$), i.e.:

$$P(Q_{ij}^{(k,7)} = -T(\cdot) + \tilde{T}(\cdot)) = \sum_{l \geq T(\cdot)} P(g_k(\cdot) = l) > ½,$$

$$P(Q_{ij}^{(k,7)} = T(\cdot) - \tilde{T}(\cdot)) = \sum_{l \leq \tilde{T}_1(\cdot)} P(g_k(\cdot) = l) < ½,$$

$$P(2g_k(\cdot) - T(\cdot) = -\iota) \geq P(2g_k(\cdot) - T(\cdot) = \iota),$$

one can obtain: $E(Q_{ij}^{(k,7)})<0$.

The proofs of the inequalities $E(Q_{ij}^{(k,\nu)})<0$ for ν=2, 4, 6, 8 are similar; negative expectations $E(Q_{ij}^{(k,\nu)})$ for ν=1, …, 8 are sufficient for the inequality (32).

□

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