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APPLICATION OF MATHEMATICAL PROGRAMMING METHODS FOR PUBLIC DEBT MANAGEMENT

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The paper presents examples of application of mathematical programming approach for optimization of public debt management in Poland. The results comprise: formulation and solving of appropriate mathematical programming problems on the basis of actual data. Basic tools applied for this purpose were: linear, convex and stochastic programming methods. The criterion functions (minimized) express servicing costs of a set of debt instruments; the cost of individual instrument is a product of its capital and profitability. The constraints of the problems comprise: budgetary requirements, measures of risk and other features of debt. The result of optimization determine a structure of debt, which minimizes servicing costs and satisfies constraints. The problems of debt management are of significant importance in Poland, because of high level of debt and its costs. Moreover, an administrative management generates excessive costs and gets worse budget situation.

Key words: mathematical programming application, nonlinear and stochastic progr amming, public debt management. MSC2010 codes: 90-02, 91G10

1. Introduction

Optimization approach plays an essential role in decisions appearing in the public debt management. Such the approach comprises formulation of appropriate mathematical programming problems (in numerical form), methods of their solving and - finally - application. Any optimization problem consist of a criterion function and constraints, determining a feasible set. Basic tools applied for such purposes in Poland were: linear, convex and stochastic programming.

The criterion functions of considered problems express servicing costs of debt instruments (bills and bonds); they subject to minimization. The cost of individual instrument is a product of its capital (nominal value minus discount) and profitability. The last component is determined by financial market, i.e. bids of investors at auctions of treasury securities. Thus, each criterion function expresses an average, annual servicing costs of optimized set of instruments. The constraints of the problems include: budgetary needs, measures of risk and other features of the debt issued. The results of optimization determine the structure of debt instruments, which minimize servicing costs and satisfy constraints.

The criterion functions of the problems corresponding to Polish conditions are typically: piecewise-linear (convex) or piecewise-nonlinear. The first kind of function corresponds only to treasury bills problem and one year investment horizon; such problems are easy to solve, especially in the case of linear or convex constraints. The nonlinear functions correspond to problems for treasury bonds, with investment horizon longer than one year. The consecutive pieces of such a function, corresponding to bids with the same price, have positive first derivative and negative second derivative. Second derivative is close to zero and converges rapidly to zero for increasing value of a bid (with the same price). Therefore, such kind of function can be approximated by a convex function, with a sufficient precision.

Typical constraints are linear, (concave) piecewise linear or convex (non-linear) functions; such features of the problems allow application of optimization algorithms for convex problems. Of course, original problems are discrete, because of fixed nominal value of bills and bonds; however approximation (rounding) of "continuous" optimal solutions guarantee sufficient precision.

The following groups of problems have been formulated, and solved, with the use of actual data, during the investigations in Poland:

- optimization of structure of instruments sold at individual auctions,
- optimization of transactions on existing debt,
- optimization of structure of instruments issued in some period of time.

Three examples of problems are presented concisely in the paper: optimization of structure of bonds sold at individual auction (the first group), optimization of premature bonds redemption associated with selling of new bonds (the second group) and optimization of three years strategy with stochastic constraints of budgetary needs (the third group).

The most important factor in optimization approach for debt management is accuracy of forecasts of interest rates of debt instruments. The horizon of the forecasts is often long 5 - 10 years or more. Accuracy of such forecasts can be poor – therefore multi-variants forecasts are also used. However, such kind of forecasts produces multi-variants optimal solutions. Determining of the unique solution is possible, but requires additional optimization tools. The two-person game with finite number of strategies with minimax solution (see e.g. Peters 2008) has been proposed for this purpose, by the author (see Klukowski 2003). The loss matrix, in the game, expresses increase of servicing costs resulting from incorrect variant of forecast. The solution of the game has usually random form (the probability distribution on the set of variants - strategies) and is obtained on the basis of linear programming problem. Such the approach guarantee minimal risk and reflects the features of the actual decision problem. It is also much more accurate than e.g. cost and risk techniques. Thus, the optimization approach can "overcame" uncertainty resulting from unknown probability distributions, without application of simulation techniques (Klukowski 2003).

The problems of debt management and debt strategy are of significant importance in Poland. It is so, because the debt level, in relation to GDP (the end of 2013 year), is close to 60%, deficit exceeds significantly 3% (during last several years) and servicing costs exceeds 2,5%. In such circumstances, optimization of debt management can provide considerable savings for the public finance; currently it is, about, 1% of total servicing costs (equal to approx. 12 billion of EURO). On the other hand "heuristic management" provides excessive costs and gets worse the budget situation.

The broad discussion of results of application of mathematical programming in debt management is presented in the book Klukowski 2003 (in Polish) and papers: Klukowski 2002, 2003, 2005, 2010 and Klukowski, Kuba 2001, 2004. Optimization approach to debt management has been initiated by: Cleassens S., J, Kreuser, L., Seigel L, R.J.-B. Wets (1998); simulation approach is presented in *Danish Government Borrowing and Debt* 2000.

The paper consists of five sections. Sections 2 – 4 present the formulation of the problems mentioned above and examples of their application. Last section summarizes the results and emphasizes attributes of optimization approach to debt management.

2. Optimization of structure of bonds sold at individual auction

Bonds for institutional investors are the basic tools for debt management. The set of these bonds comprises, in Poland, several instruments: two-years zero-coupon, five years fixed rate, ten years fixed rate, and others. They are sold at multi-price auctions, i.e. each investor pays his own price. The issuer announces the offered amount of each bond and determines minimal price on the bases of bids submitted. The price determines discount and profitability. The set of participants of auctions is limited to primary dealers, i.e. institutional investors, which have to satisfy some requirements. The typical amount offered at individual auction is currently about 3 billions of Polish zlotys (700 – 800 millions of EURO).

2.1. The problem of minimization of servicing costs

The problem of minimization of servicing costs of bonds sold at an auction consists of: the criterion function, expressing servicing costs, and constraints determining a feasible set. The case of two-years and five-years bonds is presented in this section. The criterion function is a sum of two components, corresponding to each bond they are products of the following factors: decision variable, i.e. number of units of a bond, an average price and profitability corresponding to the value of decision variable, assumed in the form of compound rate of return. The profitability of two-years bond is determined by discount, paid at redemption, the profitability of five-years bond – by coupons, paid each year and discount paid at redemption. The (average) price of a bond is the difference between the nominal value and the average discount $d_i(x_i)$ $(i = 1, 2)$, in the form:

$$d_i(x_i) = \begin{cases} v_{i,1}; & 0 < x_i \le \pi_{i,1}, \\ (\pi_{i,1} v_{i,1} + (x_i - \pi_{i,1}) v_{i,2}) / x_i; & \pi_{i,1} < x_i \le \sum_{j=1}^{2} \pi_{ij}, \\ (\pi_{i,1} v_{i,1} + \pi_{i,2} v_{i,2} + (x_i - \pi_{i,1} - \pi_{i,2}) v_{i,3}) / x_i; & \sum_{j=1}^{2} \pi_{ij} < x_i \le \sum_{j=1}^{3} \pi_{ij}, \\ \ldots \end{cases}$$

(1)

where:

v_{ij} - discount of i-th bond, in j-th bid (bids in ascending order: $v_{i,j+1} \ge v_{ij}$),

π_{ij} - number of units of i-th bond in j-th bid.

The compound rate of return (CRR) can be written in the form:

$$CRR = ((\sum_{t=1}^{n-1} S_t \prod_{\tau=i+1}^{n} (1 + r_\tau) + M) / I)^{1/n} - 1, \qquad (2)$$

where:

S_t - cash flow (interest or discount) at a time t,

r_τ - an interest rate in a year τ,

M – nominal value of one bond (1000 zł),

I – price of a bond (investment value),

n – horizon of investment (in years).

The compound rates of returns $\varphi_i(x_i)$ $(i = 1, 2)$ of the bonds considered, can be expressed for n equal 5, as follows:

$$\varphi_1(x_1) = ((N/(N - d_1(x_1))\prod_{r=3}^{5}(1 + r_{1,r}))^{1/5} - 1, \tag{3}$$

$$\varphi_2(x_2) = ((N * R\sum_{t=1}^{4}\prod_{r=t+1}^{5}(1 + r_{2,r}) + N(1 + R))/(N - d_2(x_2) + C_2))^{1/5} - 1, \tag{4}$$

where:

R - a coupon of five-years bond (percent of a nominal),

C_2 - accrued interests of five years bond paid at an auction $(C_1 = 0)$.

The price of five years bond used in (4), which include also accrued interests (dirty price), is adequate valuation of capital and find reflection in bids of investors.

The criterion function, expressing servicing costs corresponding to values of decision variables, assumes the form:

$$\min_{x_1, x_2}\{\sum_{i=1}^{2} x_i(M - d_i(x_i) + C_i)\varphi_i(x_i)\}, \tag{5}$$

where:

x_i $(i = 1, 2)$ - decision variables, i.e. number of units of i-th bond,

$d_i(x_i)$ $(i = 1, 2)$ - average discount of i-th bond,

$\varphi_i(x_i)$ - compound rate of return of i-th bond.

The constraints of the problem express budgetary needs and desired features of the debt, especially risk, resulting from variability of interest rates. Simple examples are:

$\sum_{i=1}^{2} x_i(N - d_i(x_i) + C_i) \geq B$, (budgetary needs) (6)

$a_i \leq x_i \leq b_i$ $(i = 1, 2)$, (minimum and maximum variables) (7)

$c \leq (2x_1 + 5x_2)/(x_1 + x_2) \leq d$, (average maturity of the bonds) (8)

$e \leq (\delta_1 x_1 + \delta_2 x_2)/(x_1 + x_2) \leq f$, (average duration of bonds, δ_i $(i = 1, 2)$ - duration of i-th bond). (9)

The function (5) expresses an average annual cost, corresponding to decision variables x_1, x_2, for assumed investment horizon n. It is a sum of non-linear piece-wise functions – any piece of an individual function relates to bids with a same price and is non-convex function, for

$n>1$. More precisely, its first derivative is positive, while the second negative; the second derivative approaches zero, as number of units of a bond, with the same price, $\pi_{ij} \to \infty$ (see (1)). Therefore, any component of the function (i.e. corresponding to each decision variable) can be approximated by a convex function, with a sufficient precision.

The constraint (6) is a piecewise linear concave function; the points determining pieces of the criterion function (5) and the constraint (6) are the same for each decision variable. The inequality (8) expressing an (weighted) average duration of bond portfolio is assumed in linear form, although duration of coupon bond is, in general, non-linear function of x_2 (in the case of bids with different prices). However, the range of variability is negligible and linear form is acceptable; duration of zero-coupon bond equals its life time.

The set of conditions (equalities/inequalities) determining feasibility set is in practice usually broader than (6) - (9), especially contains (nonlinear) constraints expressing risk. An example is the constraint based on covariance matrix (well-known Markovitz model), having quadratic convex form. The problem with convex approximation of criterion function and convex feasible set can be easily solved with the use of known algorithms (e.g. conjugate gradient, quasi-Newton). Of course, the original problem is discrete (number of units of bonds) and an optimal solution has to be rounded to natural number. Obviously, such a rounding does not influence the optimality of the solution for large values of the variables.

The numerical form of the problem requires values of parameters, especially forecasts of market interest rates (in (3) – (5)). They have to be determined in optimal way, because influence the solution in crucial way. The important role play also investment horizon n - it can be different than maturity of the bond with the longest life-time; the method of its determination is proposed in Klukowski 2003.

The optimization problem with two variables is the simplest one; the problems with higher number of variables have been also solved – the results are presented in Klukowski 2003.

2.2.Simple numerical example

The numerical example rests on actual data: results of an auction, budgetary constraints and forecasts of interest rates. The criterion function has been approximated with the use of polynomials (both components); the constraints assume the form:

$$\sum_{i=1}^{2} x_i (N - d_i(x_i) + C_i) \geq 1\,770\,457\,939\,,$$

$1\,100\,000 \le x_1 \le 1\,700\,000$,

$6\,000 \le x_2 \le 900\,000$,

$2{,}5 \le (1{,}78\,x_1 + 4{,}8\,x_2)/(x_1 + x_2)$,

$2{,}1 \le (1{,}78\,x_1 + 4{,}2\,x_2)/(x_1 + x_2)$.

The optimal solution $x_1^{\bullet}, x_2^{\bullet}$ of the above problem assumes the form (number of units): $x_1^{\bullet} = 1\,617\,064$, $x_2^{\bullet} = 600\,000$. It means that two-years bond provides lower costs than five years, for assumed forecasts of interest rates. The value of criterion function equals 226 110 349 zł and is lower than real decision – the relative difference equals 0,13%. The remaining properties of the solution are as follows: maturity 3,74, duration 3,53. Numerical form of such problems, especially criterion functions, depends on bids of investors; they are a kind of empirical functions.

3. Optimization of premature bonds redemption

Premature redemption of bonds issued in the past is a tool for debt management, which allows replacement of old bonds, having some non-desirable features, by new ones with appropriate properties; it enables also decreasing of servicing costs. Such replacement can be done with the use of two simultaneous auctions: redemption and sale. Total result of both auctions can be optimized – the decision variables express: redemption of old bonds and sale of new bonds, criterion function expresses the servicing costs of a mixture of old and new bonds (minimized), the feasible set determines necessary features of a "combined" portfolio. A result of the first auction determines the value and structure of redeemed bonds, while the second – the value and structure of new bonds providing capital for redemption purpose. Important feature of the approach is a "bidirectional" form of the criterion function: the first auction is aimed on maximization of profitability (discount), while the second - at minimization of profitability. The optimal solution can assume zero-form – nothing to redeem, nothing to sell. Thus, the optimal solution determines also the redemption time of old bonds: premature or original. The criterion function assumes, typically, non-linear form.

The idea of two simultaneous auctions and the form of optimization problem is proposition of the author. Currently, redemption is performed with constant - arbitrary price of redeemed bonds, while sale – at (ordinary) multi-price auction.

The example of the approach, presented in the point 3.2 is based on data from two actual (but separate) auctions (sale and redemption), performed in two consecutive days (not simultaneously). The financial market was stable at the time; therefore, the optimal solution is realistic.

3.1. Formulation of the problem of premature redemption

The optimization problem, reflecting above proposition, can be formulated as follows. To determine the optimal portfolio of bonds, comprising redeemed and issued bonds, under the following assumptions:

- the criterion function expresses total servicing costs of redeemed bonds and new bonds, in assumed time horizon,
- new bonds provide financial means for redemption purpose.

The constraints of the problem comprise: "budgetary" condition, values of decision variables and other features of the debt, after the auctions.

The criterion function can be expressed, in simplified form, as follows:

$$\min_{x,w}\{\sum_{i\in I_x} x_i(M - d_i(x_i))\varphi_i(x_i) + \sum_{j\in J_w}(W_j - w_j)P_j(W_j - w_j)\phi_j(W_j - w_j)\}, \quad (10)$$

where:

x_i $(i \in I_x)$ - decision variable - number of units of *i*-th bond (sale auction),

$d_i(x_i)$, $\varphi_i(x_i)$ - the same as in the function (5) (sale auction),

w_j $(j \in J_w)$ - decision variable - number of units of *j*-th bond (redemption),

W_j $(j \in J_w)$ - number of units of *j*-th bond at the market (constant),

$P_j(W_j - w_j)$ - average price of *j*-the non-redeemed bonds ($W_j - w_j$ units),

$\phi_j(W_j - w_j)$ - average profitability of *j*-th non-redeemed bond, for horizon *n*.

The average price of non-redeemed bonds is determined according to the formula: nominal value (M) minus an average discount of non-redeemed bonds (similarly as $(M - d_i(x_i))$). However, an average discount of redeemed bonds decreases, as w_j increases (in the range from zero to W_j), because prices at that auction are ordered in ascending way.

The profitability of non-redeemed bonds $\phi_j(W_j - w_j)$ consists of two profitabilities: from premature redemption to original redemption and from original redemption to the end of investment horizon n; the basis in both cases is the formula (2). The first profitability results from the prices $P_j(W_j - w_j)$, determined at premature redemption auction, the second – from forecasts of interest rates, with the horizon n, determined at the moment of premature redemption.

The components of the criterion function (10), corresponding to individual decision variables, can be approximated, with the appropriate precision, by convex (nonlinear) functions. Thus, their sum is also the convex function.

The constraints of the problem comprise:

- the (budgetary) constraint determining a capital necessary for redemption purpose:

$$\sum_{i \in I_x} x_i (M - d_i(x_i)) \geq \sum_{j \in J_w} w_j P_j^{(w)}(w_j), \qquad (11)$$

where: $P_j^{(w)}(w_j)$ - average price of redeemed bonds;

- the constraints on the values of decision variables: $x_i \ (i \in I_w), w_i \ (j \in J_w)$;
- other features of new debt.

The budgetary constraint (11) (the difference of left-hand side and right-hand side) is the piecewise linear concave function.

The optimal solution of the problem determines the results of both auctions with minimal servicing costs and desired features of a new debt. It is a “mixture” of old and new bonds; of course, some variables can assume values equal zero.

3.2. Example of application of premature redemption problem

An example of application of the problem of premature redemption is based on data from two actual (separate) auctions. Two bonds were redeemed: two-years w_1 and “special” assimilative three-years w_2; two bonds were sold: two-years x_1 and five-years x_2. The components of the criterion function (10), corresponding to the decision variables: $x_i \ (i = 1, 2)$ and $w_j \ (j = 1, 2)$, have been approximated with polynomials. The approximations corresponding to the variables $x_i \ (i = 1, 2)$ are convex increasing functions, the approximations corresponding to the variables $W_j - w_j$ $(j = 1, 2)$ - convex decreasing functions. The horizon of optimization was assumed equal five years, i.e. maturity of five-years (new) bonds.

Numerical form of the problem has been obtained on the basis of forecasts of: interest rates in investment horizon and results of future sale auctions (at original redemption time). Two optimal solutions, for two different feasible sets, are presented below; the first set is determined by the constraints on values of w_1, w_2 resulting from actual decision, the second is aimed at higher level of redemption. The first one corresponds to constraints:

$0 \le x_1 \le 1500000$,

$0 \le x_2 \le 3000000$,

$0 \le W_1 - w_1 \le 342197$,

$0 \le W_2 - w_2 \le 134734$.

The optimal solution of the above problem assumes the form (number of units): $x_1^* = 0$, x_2^* = 230 918, $w_1^* = 212\,945$, $w_2^* = 34\,568$; the value of the criterion function equals 34 637 898 zł, the value of capital constraint (inequality (11)) equals: 232 687 111 zł.

The second solution has been obtained for the set of constraints:

$0 \le W_1 - w_1 \le 212197$,

$0 \le W_2 - w_2 \le 24734$.

The optimal solution assumes the form: $x_1^* = 41\,232$, $x_2^* = 197\,1999$, $w_1^* = 130\,000$, w_2^* = 11 000; the value of the criterion function – equals 34 728 064 zł.

Both optimal solutions assume values w_j^* $(i = 1, 2)$ inside the feasible sets. The solutions are sensitive on values of the constraints, especially the second solution assumes non-zero value of the variable x_1^*. Such the properties of actual problem cannot be detected and analysed without optimization methods, especially in a short time - during auctions. Both solutions are different than actual decisions, which were made in two consecutive days. These decisions were aimed at minimization of prices of redeemed bonds; such prices correspond, simultaneously, to the highest profitability of new bonds. Thus, the "administrative approach" gives "benefits", resulting from low prices of redeemed bonds, in a period between premature and original redemption, i.e. usually 3 to 6 months. However, generates losses, resulting from high profitability in long period, i.e. several years. Therefore, the actual decisions, minimizing prices at premature redemption, are discordant with the main purpose of debt management.

4. Optimization of three years debt strategy with stochastic constraints

The optimization problems for debt management depend usually on parameters, which can assume random values. Examples are budgetary requirements, resulting from budgetary deficit - they cannot be determined precisely for future years; sometimes they require updating during a budgetary year. Thus, constant values of deficits can lead to non-acceptable solutions, outside of a feasible set. This drawback can be avoided by replacing the "deterministic" approach by stochastic one, with random levels of budgetary deficit. Optimization of stochastic problems is usually more complex, especially requires distributions of appropriate random variables and increases number of decision variables.

The optimization approach with random values (constraints) of budgetary requirements has been applied for the problem of three-years debt strategy. It determines optimal structure and level of debt instruments in three consecutive years. Random levels of budgetary deficit generate possibility of surplus or shortage (discrepancies between assumed level and actual value) and their costs. Such the costs are incorporated into a criterion function - together with servicing costs of the debt. The random form of the constraints generate additional decision variables and increase number of constraints. However, such the problem can be formulated and solved as the convex problem. The approach used in this section is based on the idea of the Dantzig - Madansky two stage problem (see Dantzig 1963, Grabowski 1980, Chap. 19).

4.1. Formulation of the problem of three-years strategy with stochastic constraints of budgetary requirements

The problem of optimization of three-years strategy can be stated as follows.

To determine three years schedule and structure of treasury bonds issue:

- aimed at minimizing of the criterion function comprising: servicing costs of the bonds and costs of shortage/surplus,
- assuming random values of budgetary constraints and advisable debt features.

The stochastic problem can be formulated as an extension of the deterministic problem (see Klukowski 2002), i.e. without costs of shortage/surplus. The deterministic problem (with budgetary constraints only) can be expressed as follows:

$$\min_{x} \{ \sum_{t=1}^{3} \sum_{i=1}^{\kappa} x_{it}(M - d_{it}(x_{it}))\varphi_{it}(x_{it}) \}, \tag{12}$$

$$\sum_{i=1}^{\kappa} x_{i,1}(M - d_{i,1}(x_{i,1})) = A_1, \tag{13}$$

$$\sum_{i=1}^{\kappa} x_{i,2}(M - d_{i,2}(x_{i,2})) = A_2, \tag{14}$$

$$\sum_{i=1}^{\kappa} x_{i,3}(M - d_{i,3}(x_{i,3})) - Mx_{1,1} = A_3, \tag{15}$$

$$x_{it}^{min} \leq x_{it} \leq x_{it}^{max} \ (i=1, \ldots, \kappa;\ t=1, 2, 3), \tag{16}$$

where:

x_{it} $(i = 1, \ldots, \kappa;\ t = 1, 2, 3)$ – sale of i-th bond in year t – decision variable;

x - vector of decision variables x_{it};

κ - number of bonds issued (equal 4);

$d_{it}(x_{it})$ – average discount of i-th bond corresponding to sale level x_{it};

$\varphi_{it}(x_{it})$ – compound rate of return of i-th bond, in t-th year, for sale level x_{it};

A_t – budgetary requirements in year t.

The constraint (15) includes additional term Mx_{11}, which reflects redemption of two-years bond, issued in the first year of the strategy; it increases budgetary requirements in the third year.

Stochastic form of budgetary requirements indicates replacement of the vector $\mathbf{A}' = [A_1, A_2, A_3]$ ($\mathbf{A}'$ - transposed vector) by a vector of random variables $\mathbf{\Lambda}' = [\Lambda_1, \Lambda_2, \Lambda_3]$. The distribution function of each variable Λ_t $(t = 1, 2, 3)$ is assumed in the form:

$$P(\Lambda_t = A_{tr}) = p_{tr} \qquad (r = 1, \ldots, s_t;\ s_t \geq 1), \qquad \sum_{r=1}^{s_t} p_{tr} = 1, \tag{17}$$

where:

A_{tr} $(t = 1, 2, 3; r = 1, \ldots, s_t)$ – a set of values of the random variable Λ_t, $s_t \geq 2$.

The optimization problem with random variables Λ_t, instead of constants A_t, allows a shortage or surplus in the constraints (14) – (16). More precisely, the case, when the value $\sum_{i=1}^{\kappa} x_{it}(M - d_{it}(x_{it}))$ is lower than actual budgetary requirement (realization of A_{tt}.), indicates shortage, opposite case – surplus. Both types of discrepancy can generate some costs; a shortage

– a higher rates of extra borrowing, surplus – deposits with lower rates than profitability of bonds. The costs of shortage γ_t and surplus η_t satisfy the inequalities:

$$\gamma_t, \eta_t \geq 0, \qquad \gamma_t + \eta_t > 0, \qquad t = 1, 2, 3. \tag{18}$$

The variables expressing shortage y_{tr} and surplus z_{tr}, included into a set of decision variables, are defined as follows $(t = 1, 2, 3; r = 1, ..., s_t)$:

$$y_{tr} = \max \{A_{tr} - \sum_{i=1}^{\kappa} x_{it} (M - d_{it}(x_{it})), \ 0\}, \tag{19}$$

$$z_{tr} = \max \{\sum_{i=1}^{\kappa} x_{it} (M - d_{it}(x_{it})) - A_{tr}, \ 0\}. \tag{20}$$

A cost resulting from the shortage y_{tr} is equal to $\lambda_t y_{tr}$, the cost resulting from the surplus equals $\eta_t z_{tr}$. Each of the values y_{tr} or z_{tr} can occur with the probability p_{tr}; thus, the expected cost of incorrect capital level equals $\sum_{t=1}^{3} \sum_{r=1}^{s_t} p_{tr}(\gamma_t y_{tr} + \eta_t z_{tr})$. This expression is incorporated into the criterion function (12).

The random levels of budgetary requirements imply modifications of feasible set of the problem (12) – (15); the differences: $y_{tr} - z_{tr}$ are added to left hand sides of the inequalities (12) – (14). The modifications implicate the problem:

$$\min_{\mathbf{x},\mathbf{y},\mathbf{z}} \ \sum_{t=1}^{3} \sum_{i=1}^{\kappa} x_{it} (M - d_{it}(x_{it})) \varphi_{it}(x_{it}) + \sum_{t=1}^{3} \sum_{r=1}^{s} p_r (\gamma_t y_{tr} + \eta_t z_{tr}), \tag{21}$$

$$\sum_{i=1}^{\kappa} x_{i,1}(M - d_{i,1}(x_{i,1})) + y_{1,r} - z_{1,r} = A_{1,r} \qquad (r = 1, ..., s_1), \tag{22}$$

$$\sum_{i=1}^{\kappa} x_{i,2}(M - d_{i,2}(x_{i,2})) + y_{2,r} - z_{2,r} = A_{2,r} \qquad (r = 1, ..., s_2), \tag{23}$$

$$\sum_{i=1}^{\kappa} x_{i,3}(M - d_{i,3}(x_{i,3})) + y_{3,r} - z_{3\,r} - M \cdot x_{1,1} = A_{3,r} \qquad (r = 1, ..., s_3), \tag{24}$$

$$x_{it}^{\min} \leq x_{it} \leq x_{it}^{\max} \qquad (i=1, ..., \kappa; t=1, 2, 3),$$

(25)

(y_{tr}, z_{tr} - defined in (19), (20); $\mathbf{y}, \mathbf{z}$ - vectors of the variables).

The stochastic problem generates additional decision variables and constraints. However, for moderate number of values of the variables Λ_t $(t = 1, 2, 3)$, i.e. several, it can be easily solved.

It is evident that the problem cannot be solved without optimization methods; any heuristic (administrative) procedure does not solve it even in rough way.

4.2. Example of application of stochastic problem

An example presented concisely in this point is based on actual data – from financial market and the budgetary projections.

The parameters and functions necessary to formulate numerical form of the problem (21) – (25) comprise:

a) the probability functions of the random variables Λ_t;

b) the rates (costs) γ_t, η_t;

c) the functions $d_{it}(x_{it})$ and $\varphi_{it}(x_{it})$;

d) intervals for decision variables x_{it} and other features of the debt.

The parameters (a), (b), (d) have been determined on the basis of budgetary projections and market forecasts (interest rates and shortage/surplus levels). The functions $d_{it}(x_{it})$, $\varphi_{it}(x_{it})$ and the intervals for the variables x_{it} have been assumed the same, as for the deterministic problem. The components of the criterion function, corresponding to individual bonds, have been approximated with the use of (convex) polynomials. Numerical form of whole problem is presented with details in Klukowski (2010). Some of the parameters of optimization problems are presented below in the Tables 1 - 3. The number of decision variables of the problem equals 30, number variables expressing sale of bonds – 12. These variables are aggregated in one year period; the actual number of auctions on treasury bonds generates about 100 variables. Solving of such problems is also possible with the use of usual computers.

The value of the criterion function, corresponding to the optimal solution, equals: 18 673 631 500; the optimal values of the decision variables are presented in Table 4 (sale of bonds) and Table 5 (shortage and surplus). Servicing costs of the debt assume the values (period 2003 – 2006, in Polish zlotys): 18 862 224 981; 22 116 427 533; 22 354 043 699;

22 247 884 741. The values of maturity and duration in optimal solution are presented in Table 6.

Table 1. Rates of shortage and surplus

	2002	2003	2004
Rate of shortage	0,1011	0,1004	0,0952
Rate of surplus	0,0101	0,0100	0,0095

Table 2. Variants of budgetary requirements in years 2002 – 2004 and their probability functions

Year	Variant I (r=1)	Variant II (r=2)	Variant III (r=3)
2002	61 719 000 000	63 719 000 000	59 719 000 000
2003	60 596 000 000	62 696 000 000	58 496 000 000
2004	56 554 000 000	58 854 000 000	54 254 000 000
Probab. function	0,5	0,3	0,2

Table 3. Constraints of the values of the decision variables

	x_{1t}	x_{2t}	x_{3t}	x_{4t}
$x_{it}^{\min}$ (t = 1,2, 3)	20000	30000	5000	1100
$x_{it}^{\max}$ (t = 1,2, 3)	35000	50000	12000	2000

Table 4. Optimal solution of the stochastic problem (sale of bonds)

Type of the bond	Absolute values in the year			Relative values (%) in the year		
	2002	2003	2004	2002	2003	2004
2-year bond (x_{1t})	20000	35000	35000	27,1	46,4	38,3
5-years bond (x_{2t})	45820	31328	43957	62,2	41,5	48,0
10-years (fixed) bond x_{3t})	6770	8030	10520	9,2	10,6	11,5
10-years (variable) bond (x_{4t})	1105	1139	2000	1,5	1,5	2,2

Table 5. Values of shortage y_{it} and surplus in the optimal solution z_{it}

Probability	2002		2003		2004	
	Shortage	surplus	shortage	surplus	shortage	surplus
0,5	0	0	0	2100	0	2300
0,3	2000	0	0	0	0	0
0,2	0	2000	0	4200	0	4600

Table 6. Values of average maturity and duration in optimal solution

Constraint	Year 2002	Year 2003	Year 2004
Average maturity	4,72	4,22	4,54
Duration	3,90	3,52	3,734

5. Summary and conclusions

The paper presents the examples of application of mathematical programming methods in the area of public debt management, in Poland. They are aimed at minimization of servicing costs of the debt under constraints on values of decision variables and debt features. The optimal solutions have been obtained on the basis of actual budgetary data and forecasts of market rates. The quality of optimal debt management exceeds in meaningful way traditional - administrative approach; especially some optimal decisions are reverse to actual decisions.

The main attributes of the optimization approach are as follows:

- it provides significant budgetary savings,
- increases transparency of decision process,
- reduces employment costs and hastens decision process.

The optimization approach does not eliminate the human, highly skilled work. The area of human contribution comprises:

- expert functions, i.e.: analysis, diagnosis and forecasts, which are the basis for the optimization problems,

- decision functions, i.e. evaluation of accuracy of solutions; inadequate solutions can be quickly modified or updated.

Empirical experience of the author, based on several optimization problems and several hundred of optimal solutions (Klukowski 2003), shows unquestionable practicability and present interest of the optimization approach in debt management. Let us notice that the level of budgetary deficit has been amended last year. The area of application can be significantly broadened, especially operational management can be combined with macroeconomic optimization, aimed on determining of a level of budgetary deficit, with the criterion function – maximization of economic growth, in assumed period. Such the approach indicates two-level optimal control model (see Klukowski 2005, Klukowski, Kuba 2004). Moreover, the broader set of current computer oriented tools can be applied, in particular the methods of computational intelligence. However, such the approach has been not applied in a routine work, in Poland.

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