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Average case analysis of the set packing problem

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Abstract

The paper deals with the well known set packing problem and its special case when number of subsets is maximized. It is assumed that some of the problem coefficients are realizations of mutually independent random variables. Average case (i.e. asymptotical probabilistic) properties of selected problem characteristics are investigated for the variety of possible instances of the problem. The important results of the paper are:

- Behavior of the optimal solution values of the set packing problem is presented in the special asymptotic case, where mutual asymptotical relation between m (number of elements of the packed set) and n (number of sets provided) is playing essential role.
- Probability of reaching feasible solution is reasonably high (i.e. $\geqslant 2/e, 2/e \approx 0.736$); moreover it may be set arbitrary close to 1 (e.g. 0.999), however deterioration in the quality of approximation of the behavior of the optimal solution values may be substantial.
- Some relations between general case of the set packing problem and its maximization special case are investigated.

1 Introduction

Let us consider an m element set M and Φ a collection of n subsets M_i, $i = 1, \ldots, n$, of the set M, $\Phi = \{M_1, M_2, \ldots, M_n\}$. *Set packing problem* consists in finding set of disjoint subsets Ψ in Φ, $\Psi \subseteq \Phi$, where, $M_i, M_k \in \Psi$ if and only if $M_i \cap M_k = \emptyset$, for every i, k, $i \neq k$, $i, k \in \{1, \ldots, n\}$. Set packing problem is often formulated as the binary multiconstraint knapsack problem, see Nemhauser and Wolsey [7]:

$$\begin{aligned} & z_{OPT}(n) = \max \sum_{i=1}^{n} c_i \cdot x_i \\ & \text{subject to} \qquad \sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant 1 \\ & \text{where} \quad j = 1, \ldots, m, \quad x_i = 0 \text{ or } 1 \end{aligned} \tag{1}$$

It is assumed that:

$$c_i > 0,\ a_{ji} = 0 \text{ or } 1,\ i = 1,\dots,n,\ j = 1,\dots,m.$$

In fact a_{ji}, $i = 1,\dots,n$, $j = 1,\dots,m$, $j \in M$, are defining Φ, set of subsets of M, namely M_i, $i = 1,\dots,n$, $\Phi = \{M_1, M_2, \dots, M_n\}$, in the following way

$$a_{ji} = \begin{cases} 1 & \text{if } j \in M_i \\ 0 & \text{if } j \notin M_i \end{cases},$$

where c_i is the value expressing the preference assigned to the set M_i. Let us observe that definition of the sets M_i, $i = 1,\dots,n$, does not require them to be disjoint. Namely if there exists $j \in \{1,\dots,m\}$, $k \neq l$, $k,l \in \{1,\dots,n\}$, such that $a_{jk} = a_{jl} = 1$, then $j \in M$ belongs to both M_k and M_l, i.e. $M_k \cap M_l \neq \emptyset$. Choice of x_i, fulfilling the constraints imposed in (1) is defining the packing of the set M into disjoint subsets M_i, $M_i \in \Psi$,where $M_i \cap M_k = \emptyset\ i \neq k$, $i,k \in \{1,\dots,n\}$, for every M_i, $M_k \in \Psi$. Namely in (1)

$$\forall\ k,\ k \in \{1,\dots,n\},\ M_k \in \Psi,\ \text{if and only if } \exists\ j \in M_k : a_{jk} \cdot x_k = 1. \tag{2}$$

Each of the constraints $\sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant 1$, $j = 1,\dots,m$ is guaranteeing that each of the items j of the set M is assigned to at most one of the subsets M_i, $M_i \in \Psi$. Optimisation criteria in (1) is securing the choice of best possible packing according to preferences expressed by c_i, $i = 1,\dots,n$. If $c_i = c$, $i = 1,\dots,n$, c - constant (e.g. $c = 1$), then optimisation problem seeks for the maximum amount of subsets M_i to pack set M, known as *Maximum Set Packing Problem.* Maximum set packing problem maybe also formulated as the binary multiconstraint knapsack problem, similarly to (1), namely:

$$\begin{array}{ll} z_{OPT}(n) = \max \sum\limits_{i=1}^{n} x_i & \\ \text{subject to} \qquad \sum\limits_{i=1}^{n} a_{ji} \cdot x_i \leqslant 1 & \\ \text{where} \quad j = 1,\dots,m, \quad x_i = 0 \text{ or } 1 & \end{array} \tag{3}$$

Set packing problems arise in partitioning applications where there is strong demand that no elements of the set M are permitted to be included in more than one subset M_i. Set packing problem (1) is well known to be $\mathcal{NP}$ hard combinatorial optimisation problem, see Garey and Johnson [3]. Moreover Set Packing Problem is one of the 21 first Karp's $\mathcal{NP}$ complete problems, see [4]. There are also two closely related combinatorial problems, namely *set covering problem* and *set partitioning problem* (also known as *exact covering*),where in both of them one is looking for the subsets M_{kj}, $j = 1,\dots,r$, of the collection Φ of n subsets of M_i, $i = 1,\dots,n$, where demand $\bigcup_{j=1}^{r} M_{kj} = M$ holds, moreover in the set partitioning problem there is additional demand, namely that all M_{kj} are pairwise disjoint, i.e. $M_{k_j} \cap M_{k_l} = \emptyset$, for every k_j, k_l, $k_j \neq k_l$, $j,l \in \{1,\dots,r\}$. Both problems may be also formulated as special cases of the binary multiconstraint knapsack problem, see Nemhauser and Wolsey [7]. Maximum set packing problem is also known as *Maximum Hypergraph Matching.* As latter, under certain conditions, it is equivalent to well known *Maximum Clique problem*, see Ausiello, D'Atri and Protasi [1]. Another example of the application of the set packing problem in the graph theory is the so called *independent set*, i.e. set of graph vertices having no common edges.

Scheduling an airline flight crews to airplanes is good example of a practical application of the set packing problem. Each airplane must have a crew assigned to it, consisting of a pilot, copilot, and navigator. There is set of possible crew members, based on their training to operate relevant types of airplanes, as well as any personality conflicts. Considering all possible crews and airplanes combinations, each represented by a subset of items, our goal is to find such an assignment of crews to airplanes that each airplane and each crew member is in exactly one selected combination. From the mathematical point of view one is looking for a set packing, taking into account subset constraints. Simply in the considered time period the same crew members cannot be on two different airplanes and every airplane must have a crew, but not all of the crew members must be assigned. In the case of the set partitioning problem all of the crew members must be assigned and in the case of the set covering problem some crew members may be assigned to multiple airplanes.

As it was already mentioned set packing problem is often formulated as the binary multiconstraint knapsack problem, see (1) and (3). However the above formulations constitute rather special case of it, see Martello and Toth [5]. Its peculiarity consists in following facts:

- All the constraints left hand sides coefficients are equal either to 1 or to 0:

$$a_{ji} = 0 \text{ or } 1,\ i = 1, \ldots, n,\ j = 1, \ldots, m.$$

- All of the constraints right hand sides coefficients are equal to 1.
- Number of constraints m maybe arbitrarily big in comparison to n (number of decision variables).

In the general formulation of the binary multiconstraint knapsack problem it is only required that all of the knapsack problem coefficients, i.e. goal function, constraints left and right hand sides, are non-negative or, in order to avoid unclear interpretations, strictly positive. The latter especially applies to goal function and constraints right hand sides coefficients. It is usually also assumed that m (number of constraints) is not large with respect to number of decision variables n.

It does mean that results obtained for the general knapsack problem, e.g. in the case of Lagrange and dual estimations or asymptotic probabilistic analysis of the optimal solution value behavior, may not be valid in the case of set packing problem specific formulations provided in (1) or (3). In the present paper set packing problem (1) specific Lagrange and dual estimations are provided. Then for the random model of the (1) interesting results concerning the feasibility of the obtained solutions and asymtotical growth of the optimal solution values $z_{OPT}(n)$, when $n \to \infty$, are provided.

2 Definitions

The following definitions are necessary for the further presentation:

Definition 1 *We denote* $V_n \approx Y_n$, *where* $n \to \infty$, *if*

$$Y_n \cdot (1 - o(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o(1))$$

when V_n, Y_n are sequences of numbers, or

$$\lim_{n\to\infty} P\{Y_n \cdot (1-o(1)) \leqslant V_n \leqslant Y_n \cdot (1+o(1))\} = 1$$

when V_n is a sequence of random variables and Y_n is a sequence of numbers or random variables, where $\lim_{n\to\infty} o(1) = 0$ as it is usually presumed.

Definition 2 *We denote $V_n \preceq Y_n (V_n \succeq W_n)$ if*

$$V_n \leqslant (1+o(1)) \cdot Y_n \ (V_n \geqslant (1-o(1)) \cdot W_n)$$

when V_n, Y_n (W_n) are sequences of numbers, or

$$\lim_{n\to\infty} P\{V_n \leqslant (1+o(1)) \cdot Y_n\} = 1 \ (\lim_{n\to\infty} P\{V_n \geqslant (1-o(1)) \cdot W_n\} = 1)$$

when V_n is a sequence of random variables and Y_n (W_n) is a sequence of numbers or random variables, where $\lim_{n\to\infty} o(1) = 0$.

Definition 3 *We denote $V_n \approxeq Y_n$ if there exist constants $c'' \geqslant c' > 0$ such that*

$$c' \cdot Y_n \preceq V_n \preceq c'' \cdot Y_n$$

where Y_n, V_n are sequences of numbers or random variables.

The following random model of (1) will be considered in the paper:

- m, n, $0 < n \leqslant m!$, are arbitrary positive integers and moreover $n \to \infty$.
- c_i, a_{ji}, $i = 1, \ldots, n$, $j = 1, \ldots, m$, are realizations of mutually independent random variables and moreover c_i, are uniformly distributed over $(0, 1]$ and $P\{a_{ji} = 1\} = p$, where $0 < p \leq 1$.

Let us observe that asymptotical relations $0 < n \leqslant m!$ and $n \to \infty$ requires that also $m \to \infty$. As the matter of fact mutual asymtotical relation of the values of m and n may vary between 2 extreme cases $n/m \approx 0$ or $n \approx m!$ as $n \to \infty$

Under the assumptions made about c_i, a_{ji}, and taking into account (1) the following always hold

$$0 \leqslant z_{OPT}(n) \leqslant \sum_{i=1}^{n} c_i \leqslant n, \tag{4}$$

Moreover, from the strong law of large numbers it follows that

$$\sum_{i=1}^{n} c_i \approx E(c_1) \cdot n = n/2, \ \sum_{i=1}^{n} a_{ji} \approx p \cdot n, \ \sum_{j=1}^{m} a_{ji} \approx p \cdot m. \tag{5}$$

Therefore, it is justified to enhance formulas (4) and (5) in the following way:

$$0 \leqslant z_{OPT}(n) \preceq n/2, \ \sum_{i=1}^{n} a_{ji} \preceq 1, \text{ if } p < \frac{1}{n} \text{ or } \sum_{i=1}^{n} a_{ji} \succeq 1 \text{ when } p > \frac{1}{n}. \tag{6}$$

Formula (6) shows that random model of set packing problem (1) is complete in the sense that nearly all possible instances of the problem are considered.

The growth of $z_{OPT}(n)$ - value of the optimal solution of the problem (1) may be influenced by the problem coefficients, namely:

$$n,\ m,\ c_i,\ a_{ji},\ \text{where } i = 1, \ldots, n,\ j = 1, \ldots, m.$$

We have assumed that c_i, a_{ji} are realizations of the random variables and therefore their impact on the $z_{OPT}(n)$ growth is in this case indirect. Moreover, we have also assumed that m, n are arbitrary positive integers and $n \to \infty$.

The main aim of the present paper is to perform probabilistic analysis of the considered class of random set packing problems in the asymptotical case, i.e. when $n \to \infty$. For the considered random model probabilistic analysis has 2 strategic goals, namely:

- To examine existence of the feasible solutions.
- To investigate asymptotic behavior of $z_{OPT}(n)$.

  Existence of the feasible solution, provided by $x_1, \ldots, x_n$, means that $\sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant 1$ for all $j = 1, \ldots .m$. If any of the constraints is violated, i.e. $\exists\ j'$ such that $\sum_{i=1}^{n} a_{j'i} \cdot x_i \geqslant 1$, then solution, provided by $x_1, \ldots, x_n$, is not feasible.

3 Lagrange and dual estimations

When the general knapsack type problem, with one or many constraints, is considered then Lagrange function and the corresponding dual problems, see Averbakh [2], Meanti, Rinnooy Kan, Stougie and Vercellis [6], Szkatuła [8] and [9] are very useful tools to perform various kind of analyses of the original problem. In the specific case of the set packing problem (i.e. all of the constraints right hand sides coefficients equal to 1) Lagrange function of the problem (1) may be formulated as follows:

$$\begin{aligned} L_n(x) &= \sum_{i=1}^{n} c_i \cdot x_i + \sum_{j=1}^{m} \lambda_j \cdot \left(1 - \sum_{i=1}^{n} a_{ji} \cdot x_i\right) = \\ &= \sum_{j=1}^{m} \lambda_j + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{m} \lambda_j \cdot a_{ji}\right) \cdot x_i \end{aligned}$$

where $x = [x_1, \ldots, x_n]$ and $\Lambda = [\lambda_1, \ldots, \lambda_m]$ - vector of Lagrange multipliers. Moreover, let for every Λ, $\lambda_j \geq 0$, $j = 1, \ldots, m$:

$$\phi_n(\Lambda) = \max_{x \in \{0,1\}^n} L_n(x, \Lambda) = \max_{x \in \{0,1\}^n} \left\{ \sum_{j=1}^{m} \lambda_j + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{m} \lambda_j a_{ji}\right) x_i \right\}.$$

Taking the following notation:

$$x_i(\Lambda) = \begin{cases} 1 & \text{if } c_i - \sum\limits_{j=1}^{m} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

$$c_i(\Lambda) = \begin{cases} c_i & \text{if } c_i - \sum\limits_{j=1}^{m} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$a_{ji}(\Lambda) = \begin{cases} a_{ji} & \text{if } c_i - \sum\limits_{j=1}^{m} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

we have for every Λ, $\lambda_j \geq 0$, $j = 1, \ldots, m$:

$$\begin{aligned} \phi_n(\Lambda) &= \sum_{j=1}^{m} \lambda_j + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{m} \lambda_j \cdot a_{ji} \right) \cdot x_i(\Lambda) = \\ &= \sum_{j=1}^{m} \lambda_j + \sum_{i=1}^{n} \left(c_i(\Lambda) - \sum_{j=1}^{m} \lambda_j \cdot a_{ji}(\Lambda) \right) \end{aligned}$$

Obviously for $i = 1, \ldots, n$, $j = 1, \ldots, m$,

$$c_i(\Lambda) = c_i \cdot x_i(\Lambda), \quad a_{ji}(\Lambda) = a_{ji} \cdot x_i(\Lambda).$$

Dual problem to set packing problem (1) maybe formulated as follows:

$$\Phi_n^* = \min_{\Lambda \geq 0} \phi_n(\Lambda). \tag{8}$$

For every $\Lambda \geq 0$ the following holds:

$$z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + \sum_{j=1}^{m} \lambda_j (1 - s_j(\Lambda)). \tag{9}$$

Let us denote:

$$\begin{aligned} z_n(\Lambda) &= \sum_{i=1}^{n} c_i \cdot x_i(\Lambda) = \sum_{i=1}^{n} c_i(\Lambda), s_j(\Lambda) = \sum_{i=1}^{n} a_{ji} \cdot x_i(\Lambda) = \sum_{i=1}^{n} a_{ji}(\Lambda), \\ S_{nm}(\Lambda) &= \sum_{j=1}^{m} \lambda_j \cdot s_j(\Lambda), \ \tilde{\Lambda}(m) = \sum_{j=1}^{m} \lambda_j. \end{aligned}$$

By definition of $c_i(\Lambda)$ and $a_{ji}(\Lambda)$, see also (7), we have:

$$c_i(\Lambda) \geq \sum_{j=1}^{m} \lambda_j \cdot a_{ji}(\Lambda), \ i = 1, \ldots, n,$$

and therefore

$$z_n(\Lambda) \geq S_{nm}(\Lambda). \tag{10}$$

For certain Λ, $x_i(\Lambda)$ given by (7) may provide feasible solution of (1), i.e.:

$$s_j(\Lambda) \leq 1 \quad \text{for every} \quad j = 1, \ldots, m. \tag{11}$$

Then:

$$z_n(\Lambda) \leq z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + \tilde{\Lambda}(m) - S_{nm}(\Lambda). \tag{12}$$

If (11) holds, then the below inequality also holds:

$$\tilde{\Lambda}(m) - S_{nm}(\Lambda) \geq 0.$$

From (10) we get:

$$\frac{\phi_n(\Lambda)}{z_n(\Lambda)} = \frac{z_n(\Lambda)}{z_n(\Lambda)} + \frac{\tilde{\Lambda}(m) - S_{nm}(\Lambda)}{z_n(\Lambda)} \leq 1 + \frac{\tilde{\Lambda}(m) - S_{nm}(\Lambda)}{S_{nm}(\Lambda)}.$$

Therefore if (11) holds, then the following inequality also holds:

$$1 \leq \frac{z_{OPT}(n)}{z_n(\Lambda)} \leq \frac{\Phi_n^*}{z_n(\Lambda)} \leq \frac{\phi_n(\Lambda)}{z_n(\Lambda)} \leq \frac{\tilde{\Lambda}(m)}{S_{nm}(\Lambda)}. \tag{13}$$

Formula (13) shows, that if there exits such a set of Lagrange multipliers $\Lambda(n)$ which is fulfilling the formula (11) and if the formula below holds:

$$\lim_{n\to\infty} \frac{\tilde{\Lambda}(m)}{S_{nm}(\Lambda(n))} = 1 \tag{14}$$

then, due to (13), $\lim_{n\to\infty} \frac{z_{OPT}(n)}{z_n(\Lambda)} = 1$ and therefore $x_i(\Lambda(n))$, $i = 1, \ldots, n$, given by (7), is the asymptotically sub-optimal solution of the set packing problem (1). Moreover the value of $z_n(\Lambda(n))$ is an asymptotical approximation of the optimal solution value of the set packing problem i.e. $z_{OPT}(n)$.

In the case of Maximum set packing problem (3) $c_i \equiv 1$, $i = 1, \ldots, n$, and moreover c_i are no longer realizations of the random variables. Therefore in the case of Maximum set packing problem (3) in the above formulas c_i should be replaced with 1. As the consequence formulas where c_i was involved will look differently, e.g. in (7) $c_i - \sum_{j=1}^{m} \lambda_j \cdot a_{ji} > 0$ should be replaced by $1 - \sum_{j=1}^{m} \lambda_j \cdot a_{ji} > 0$. It does mean that:

$$\begin{aligned} c_i(\Lambda) = x_i(\Lambda) &= \begin{cases} 1 & \text{if } 1 - \sum\limits_{j=1}^{m} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \\ a_{ji}(\Lambda) &= \begin{cases} a_{ji} & \text{if } 1 - \sum\limits_{j=1}^{m} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} . \end{aligned} \tag{15}$$

In turn it means that $c_i(\Lambda) \equiv x_i(\Lambda)$, $i = 1, \ldots, n$, and therefore $z_n(\Lambda) = \sum_{i=1}^{n} x_i(\Lambda)$.

In either case, according to (2), $a_{ji}(\Lambda) = 1$ is guaranteeing that item j is assigned to set M_i. Obviously this also implies that $s_j(\Lambda) = 1$.

4 Probabilistic analysis

In the present section of the paper some probabilistic properties of the set packing problems (1) and (3) will be investigated. In the paper by Vercellis [10] there were some results of the probabilistic analysis of the set packing problems presented. In the present paper different approach is exploited. The random model of the specific knapsack problems (1) and (3) is significantly different from one considered in the case of the general knapsack problem in the earlier author papers, see Szkatuła [8] and [9]. Namely constraints left hand sides coefficients a_{ji}, $i = 1, \dots, n$, $j = 1, \dots, m$ have in present case discrete probability distribution while in the general case they have uniform (continuous) distribution. Moreover all of the constraints right hand sides coefficients are equal to 1 and m may be arbitrarily large in comparison to n. Therefore probabilistic analysis of the set packing problem (1) requires specific approach.

Let us first observe that due to the assumptions made the following holds, for $j = 1, \dots, m$:

$$\begin{aligned} P\{a_{ji} &= 1\} = p,\ P\{a_{ji} = 0\} = 1 - p,\ P\{a_{ji}(\Lambda) = 1\} = 1 - P\{a_{ji}(\Lambda) = 0\}, \\ P(c_i &< x) = \begin{cases} 0 & \text{when } x \leqslant 0 \\ x & \text{when } 0 < x \leqslant 1 \\ 1 & \text{when } x \geqslant 1 \end{cases}. \end{aligned} \tag{16}$$

Moreover for the random variable $\sum_{k=1,k\neq j}^{m} a_{ji}$, due to the binomial distribution, the following holds for every r - integer, $0 \leqslant r \leqslant m-1$:

$$P\left\{\sum_{k=1,k\neq j}^{m} a_{ki} = r\right\} = \binom{m-1}{r} \cdot p^r \cdot (1-p)^{m-r-1}. \tag{17}$$

Let us also assume that

$$\Lambda = \{\lambda, \cdots, \lambda\},\ \text{i.e. } \lambda_j = \lambda,\ \lambda \geqslant 0,\ j = 1, \cdots, m.$$

In the case of set packing problem (1) the following results hold.

Lemma 1 *If a_{ji} are realizations of mutually independent random variables where $P\{a_{ji} = 1\} = p$, $0 < p \leq 1$, then*

$$P\{a_{ji}(\Lambda) = 1\} = p - p\sum_{r=0}^{m-1} \binom{m-1}{r} \cdot p^r \cdot (1-p)^{m-r-1} \min\{1, \lambda(r+1)\}.$$

Moreover if $\lambda \leqslant 1/m$ then:

$$P\{a_{ji}(\Lambda) = 1\} = p \cdot (1 - \lambda \cdot (m \cdot p + 1 - p)).$$

Proof. From (7), (16) and (17) and taking into account that random variable $\sum_{k=1,k\neq j}^{m} a_{ji}$ may take any integer value r from the range $[0, m-1]$ with the probability given in (17) it follows that:

$$\begin{aligned} P\{a_{ji}(\Lambda)=0\} &= P\left\{a_{ji}=0 \cup a_{ji}=1 \cap c_i < \lambda\cdot\left(\sum_{k=1,k\neq j}^{m} a_{ki}+1\right)\right\} = \\ &= 1-p+p\cdot P\left\{c_i < \lambda\cdot\left(\sum_{k=1,k\neq j}^{m} a_{ki}+1\right)\right\} = \\ &= 1-p+p\sum_{r=0}^{m-1}\binom{m-1}{r}\cdot p^r\cdot(1-p)^{m-r-1}\min\{1,\lambda(r+1)\}. \end{aligned}$$

Due to the (16) the first formula of the Lemma is proven. Because

$$\binom{m-1}{r} = \frac{(m-1)!}{r!\cdot(m-1-r)!},$$

then when $\lambda \leqslant 1/m$ the following holds

$$P\{a_{ji}(\Lambda)=0\} = 1-p+\lambda\sum_{r=0}^{m-1}\frac{(m-1)!\cdot(r+1)}{r!\cdot(m-1-r)!}\cdot p^{r+1}\cdot(1-p)^{m-r-1} \qquad (18)$$

Let us observe that for every integers l, m, $l, > 1$, $m \geqslant 2$, and $0 \leqslant p \leqslant 1$ the following hold

$$\begin{aligned} \sum_{k=0}^{l}\binom{l}{k}\cdot p^k\cdot(1-p)^{l-k} &= (p+1-p)^l = 1 \\ r+1 &= m-(m-1-r). \end{aligned}$$

Using the above mentioned formulas (18) may be rewritten as:

$$\begin{aligned} P\{a_{ji}(\Lambda)=0\} &= 1-p+\lambda\cdot p\left(\sum_{r=0}^{m-1}\frac{(m-1)!\cdot m}{r!\cdot(m-1-r)!}\cdot p^r\cdot(1-p)^{m-1-r}-\right. \\ &\qquad \left.-\sum_{r=0}^{m-1}\frac{(m-1)!\cdot(m-1-r)}{r!\cdot(m-1-r)!}\cdot p^r\cdot(1-p)^{m-1-r}\right) = \\ &= 1-p+\lambda\cdot p\left(m\sum_{r=0}^{m-1}\binom{m-1}{r}\cdot p^r\cdot(1-p)^{m-1-r}-\right. \\ &\qquad \left.-p\cdot(m-1)\cdot(1-p)\sum_{r=0}^{m-2}\binom{m-2}{r}\cdot p^r\cdot(1-p)^{m-2-r}\right) = \\ &= 1-p+\lambda\cdot p\cdot(m-(m-1)\cdot(1-p)) = \\ &= 1-p+\lambda\cdot p\cdot(m\cdot p+1-p). \end{aligned}$$

Finally above formulas can be summarized as:

$$P\{a_{ji}(\Lambda)=0\} = 1-p+\lambda\cdot p\cdot(m\cdot p+1-p). \qquad (19)$$

Due to the formulas (16) and (19) we have

$$\begin{aligned} P\{a_{ji}(\Lambda)=1\} &= 1-P\{a_{ji}(\Lambda)=0\} = \\ &= p-\lambda\cdot p\cdot(m\cdot p+1-p) = p\cdot(1-\lambda\cdot(m\cdot p+1-p)). \end{aligned}$$

■

As the direct consequence of the above formulas we have

$$E(a_{ji}(\Lambda)) = 1 \cdot P\{a_{ji}(\Lambda) = 1\} + 0 \cdot P\{a_{ji}(\Lambda) = 0\} = P\{a_{ji}(\Lambda) = 1\}. \qquad (20)$$

Now instead of Λ we will consider $\Lambda(n)$. It does mean that for every value of integer n, we may consider different vector $\Lambda(n) = \{\lambda(n), \cdots, \lambda(n)\}$, $\lambda(n) \geqslant 0$. For every j, $j = 1, \cdots, m$, we have:

$$\begin{aligned} E(s_j(\Lambda(n))) &= \sum_{i=1}^{n} E(a_{ji}(\Lambda(n))) = n \cdot P\{a_{ji}(\Lambda(n)) = 1\} = \qquad (21) \\ &= n \cdot p(1 - \lambda(n) \cdot (m \cdot p + 1 - p)). \end{aligned}$$

The above equation (21) is providing the opportunity to determine $\lambda(n)$ solving $E(s_j(\Lambda(n)) = \alpha$, where $\alpha > 0$. When $\alpha = 1$ than $\lambda(n)$ is solving all of the constraints in the (1) as equations, in the sense of average (mean) values, $E(\sum_{i=1}^{n} a_{ji} \cdot x_i(\Lambda(n))) = 1$ for all $j = 1, \ldots .m$. Unfortunately there is no guarantee that solution obtained is feasible, i.e. $\sum_{i=1}^{n} a_{ji} \cdot x_i(\Lambda(n)) \leqslant 1$, for all $j = 1, \ldots .m$. Therefore one may try to consider smaller values of α, $0 < \alpha \leqslant 1$ in order to increase the chance to obtain the feasible solution of the set packing problem (1). Below those ideas are considered in formalized manner.

Lemma 2 *For every α, $\alpha > 0$ there exists m', $n' > 1$ such that for every $m \geqslant m'$ and $n \geqslant n'$, the following choice of $\lambda(n)$:*

$$\lambda(n) = \frac{1 - \alpha/(n \cdot p)}{m \cdot p + 1 - p} \text{ is solving the equations } E(s_j(\Lambda(n))) = \alpha.$$

Corollary 1 *If $E(s_j(\Lambda(n))) = \alpha$, then $P\{a_{ji}(\Lambda(n)) = 1\} = \alpha/n$.*

Proof. Proof of Lemma and Corollary follows immediately from formulas (20) and (21) and following fact that for all $m \geqslant m'$ and $n \geqslant n'$:

$$\lambda(n) \leqslant \frac{1}{m}.$$

■

Solution of the set packing problem (1) given by formula (7) is feasible (provides packing of the set M) if and only if the formula (11) holds.

Theorem 1 *For every α, $0 < \alpha \leqslant 1$ there exists m', n', m', $n' > 1$, such that for $\Lambda(n)$, providing $E(s_j(\Lambda(n))) = \alpha$, the following hold*

$$P\{s_j(\Lambda(n)) \leqslant 1\} = \left(1 - \frac{\alpha}{n}\right)^{n-1} \cdot (1 + \alpha - \frac{\alpha}{n})$$

Moreover for every fixed value of $\alpha, \alpha > 0$, we have

$$\lim_{n \to \infty} P\{s_j(\Lambda(n)) \leqslant 1\} = \frac{1 + \alpha}{e^{\alpha}}$$

Proof. As it was already mentioned solution of problem (1) given by formula (7) is feasible if and only if formula (11) holds i.e. $s_j(\Lambda(n)) = 0$ or $s_j(\Lambda(n)) = 1$. For every $\Lambda(n)$, random variable $s_j(\Lambda(n)) = \sum_{i=1}^{n} a_{ji}(\Lambda(n))$ may take any integer value r from the range $[0,n]$ with the probability given by the following formula:

$$P\left\{\sum_{i=1}^{n} a_{ji}(\Lambda(n)) = r\right\} = \binom{n}{r} \cdot \tilde{p}^{r} \cdot (1-\tilde{p})^{n-r}, \text{ where } \tilde{p} = P\{a_{ji}(\Lambda(n)) = 1\}.$$

From the above formula and Corollary 1 it follows that

$$\begin{aligned} P\{s_j(\Lambda(n)) \leqslant 1\} &= P\left\{\sum_{i=1}^{n} a_{ji}(\Lambda(n)) = 0 \cup \sum_{i=1}^{n} a_{ji}(\Lambda(n)) = 1\right\} = \qquad (22) \\ &= \left(1-\frac{\alpha}{n}\right)^{n} + \alpha\left(1-\frac{\alpha}{n}\right)^{n-1} = \left(1-\frac{\alpha}{n}\right)^{n-1} \cdot \left(1+\alpha-\frac{\alpha}{n}\right) \end{aligned}$$

The proof is finished by observing that $\lim_{n\to\infty}\left(1-\frac{\alpha}{n}\right)^{n-1} = e^{-\alpha}$ and $\lim_{n\to\infty}\frac{\alpha}{n} = 0$ ■

Corollary 2 *$P\{s_j(\Lambda(n)) \leqslant 1\} = 1$ if and only if $n = 1$. When $\alpha \to 0$ as $n \to \infty$ then*

$$\lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} = 1.$$

However if α, $\alpha > 0$, is a constant then:

$$\lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} < 1 \qquad (23)$$

Proof. Formula (23) follows immediately from the Theorem 1. ■

The above Theorem 1 and Corollary 2 to it have interesting interpretation, which may be observed on few examples presented below:

Example 1

$$\begin{aligned} \textit{When } \alpha = 0.01 \textit{ then } \lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} &= 0.999 \\ \textit{When } \alpha = 0.1 \textit{ then } \lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} &= 0.995 \\ \textit{When } \alpha = 0.5 \textit{ then } \lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} &= 0.9098 \\ \textit{When } \alpha = 1 \textit{ then } \lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} &= \frac{2}{e} \approx 0.736 \end{aligned}$$

Interpretation of the above examples is following. The closer the value of α is to 1, i.e. set packing problem (1) right-hand-side values the better approximation of the optimal solution values may be provided, however with less satisfactory value of the $\lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\}$. However, for any value α, $0 < \alpha \leqslant 1$, $\lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} \succeq 2/e$, where $2/e \approx 0.736$. Due approximations of the optimal solution values are provided in the next section.

In the case of maximum set packing problem (3) situation is significantly different. Namely according to (17), where $\gamma = \frac{1}{\lambda} - 1, \lambda = \frac{1}{\gamma+1}$:

$$\begin{aligned} P\{a_{ji}(\Lambda) = 1\} &= P\left\{a_{ji} = 1 \cap \lambda \cdot \left(\sum_{k=1,k\neq j}^{m} a_{ki} + 1\right) \leq 1\right\} = \qquad (24) \\ &= p \cdot P\left\{\lambda \cdot \left(\sum_{k=1,k\neq j}^{m} a_{ki} + 1\right) \leq 1\right\} = \\ &= p \cdot P\left\{\sum_{k=1,k\neq j}^{m} a_{ki} \leq \frac{1}{\lambda} - 1\right\} = \\ &= p \cdot \sum_{r=0}^{\lfloor\gamma\rfloor} P\left\{\sum_{k=1,k\neq j}^{m} a_{ki} = r\right\} = \\ &= p \cdot \sum_{r=0}^{\lfloor\gamma\rfloor} \binom{m-1}{r} \cdot p^r \cdot (1-p)^{m-r-1}. \end{aligned}$$

It is pretty obvious that only m values of γ, (and respectively λ) should be considered namely $\gamma = 0, 1, \ldots, m-1$, ($\lambda = \frac{1}{m}, \frac{1}{m-1}, \ldots, 1$) because

$$P\left\{\sum_{k=1,k\neq j}^{m} a_{ki} = r\right\} = 0 \text{ for } r < 0 \text{ and } r > m-1.$$

The above facts have very serious consequences for the probabilistic analysis of the maximum set packing problem (3). Namely using formula (24) with $\gamma = 0$ and $\gamma = m-1$ ($\lambda = \frac{1}{m}$ and $\lambda = 1$) and taking into account (20) it follows that

$$p \cdot (1-p)^{m-1} \leq E(a_{ji}(\Lambda)) = P\{a_{ji}(\Lambda) = 1\} \leq 1$$

The latter means that, when considering $\Lambda(n)$, $n \to \infty$, in order to solve

$$E(s_j(\Lambda(n))) = \alpha \text{ or } P\{a_{ji}(\Lambda(n)) = 1\} = \frac{\alpha}{n}, i = 1, \ldots, n,\ j = 1, \ldots, m \qquad (25)$$

the following condition should hold:

$$n \leq \frac{\alpha}{p \cdot (1-p)^{m-1}}.$$

As the matter of fact (25) is implying asymtotical relations between n, m, p and α. It may be difficult to obtain exact solution of (25) due to the finiteness of the set of values of Lagrange multipliers λ ($\lambda = \frac{1}{m}, \frac{1}{m-1}, \ldots, 1$) and the formula (24). Frequently there may exist only approximate solutions of (25).

5 Behavior of the optimal solution values

Main goal of this paper is to analyze the behavior of the optimal solution value of the set packing problem (1) in the asymtotical probabilistic case. Moreover it was author intention to use simple and easy to follow probabilistic apparatus.

In order to proceed with this analysis one may need to exploit the probabilistic properties of the random variables $c_i(\Lambda(n))$, $i = 1, \cdots, n$. The construction of the random variables $c_i(\Lambda(n))$ is defined by formulas (7) and (16) respectively. Distribution functions of the random variables $c_i(\Lambda(n))$, $i = 1, \cdots, n$ are given by the following formulas, where $0 < x \leq 1$:

$$\begin{aligned} P\{c_i(\Lambda(n)) < x\} &= P\{c_i < x \cup c_i \geq x \cap c_i \leq \Lambda(n) \cdot \sum_{j=1}^{m} a_{ji}\} = \qquad (26) \\ &= x + P\{x \leq c_i \leq \Lambda(n) \cdot \sum_{j=1}^{m} a_{ji}\}. \end{aligned}$$

Let us observe that $P\{x \leq c_i \leq \Lambda(n) \cdot \sum_{i=1}^{n} a_{ji}\}$ is by definition equal to zero if $c_i < x$ or $c_i > \Lambda(n) \cdot \sum_{i=1}^{n} a_{ji}$. Therefore (26) may be rewritten as

$$\begin{aligned} P\{c_i(\Lambda(n)) < x\} &= x + \sum_{r=1}^{m} P\{x \leq c_i \leq \Lambda(n) \cdot r \cap \sum_{j=1}^{m} a_{ji} = r\} = \qquad (27) \\ &= x + \sum_{r=1}^{m} (r\Lambda(n) - x)_{+} P\{\sum_{j=1}^{m} a_{ji} = r\}. \qquad (28) \end{aligned}$$

The above formula may enable us to calculate the mean value of the random variables $c_i(\Lambda(n))$, $i = 1, \cdots, n$. Namely:

$$\begin{aligned} E(c_i(\Lambda(n))) &= \int_0^1 x \cdot d(P\{c_i(\Lambda(n)) < x\}) = \qquad (29) \\ &= \frac{1}{2} + \int\limits_{0}^{\Lambda(n)\cdot m} x \cdot \left(\sum_{r=1}^{m} (r\Lambda(n) - x)'_{+} \cdot P\{\sum_{j=1}^{m} a_{ji} = r\} \right) = \\ &= \frac{1}{2} + \sum_{k=1}^{m} \int\limits_{\Lambda(n)\cdot(k-1)}^{\Lambda(n)\cdot k} x \left(\sum_{r=k}^{m} (r\Lambda(n) - x)'_{+} \cdot P\{\sum_{j=1}^{m} a_{ji} = r\} \right) dx = \\ &= \frac{1}{2} - \sum_{k=1}^{m} \int\limits_{\Lambda(n)\cdot(k-1)}^{\Lambda(n)\cdot k} x \cdot P\{\sum_{j=1}^{m} a_{ji} = r\} dx \end{aligned}$$

Let us observe that, similarly to the formula (17), the random variable $\sum_{k=1}^{m} a_{ji}$, due to its binomial distribution, has the following distribution function for every r - integer, $0 \leqslant r \leqslant m$:

$$P\left\{ \sum_{k=1}^{m} a_{ki} = r \right\} = \binom{m}{r} \cdot p^r \cdot (1-p)^{m-r} \text{ and moreover } \left(\sum_{k=1}^{r} (2k-1) \right) = r^2.$$

Therefore the formula (29) could be further simplified as follows:

$$\begin{aligned}
E(c_i(\Lambda(n))) &= \frac{1}{2} - \sum_{k=1}^{m} \left(\int\limits_{\Lambda(n)\cdot(k-1)}^{\Lambda(n)\cdot k} x dx \right) \cdot \left(\sum_{r=k}^{m} \binom{m}{r} \cdot p^r \cdot (1-p)^{m-r} \right) = \\
&= \frac{1}{2} - \frac{(\Lambda(n))^2}{2} \sum_{k=1}^{m} (2k-1) \cdot \left(\sum_{r=k}^{m} \binom{m}{r} \cdot p^r \cdot (1-p)^{m-r} \right) = \\
&= \frac{1}{2} - \frac{(\Lambda(n))^2}{2} \sum_{r=1}^{m} \left(\sum_{k=1}^{r} (2k-1) \right) \cdot \left(\binom{m}{r} \cdot p^r \cdot (1-p)^{m-r} \right) = \\
&= \frac{1}{2} - \frac{(\Lambda(n))^2}{2} \sum_{r=1}^{m} r^2 \cdot \left(\binom{m}{r} \cdot p^r \cdot (1-p)^{m-r} \right).
\end{aligned}$$

Let us observe that the following formula holds for $0 < p \leq 1$ and $m = 1, 2, \ldots$

$$\sum_{r=1}^{m} r^2 \cdot \left(\binom{m}{r} \cdot p^r \cdot (1-p)^{m-r} \right) = m \cdot p \cdot (1 + p \cdot (m-1))$$

From Lemma 2 (where $E(s_j(\Lambda(n))) = \alpha$, and $\lambda(n) = \frac{1-\alpha/(n\cdot p)}{m\cdot p+1-p}$) and due to the formula (9) we will therefore receive

$$\begin{aligned}
E(z_n(\Lambda)) &= \frac{n}{2} \left(1 - \left(\frac{1-\alpha/(n\cdot p)}{m\cdot p+1-p} \right)^2 \cdot m \cdot p \cdot (m \cdot p + 1 - p) \right) = \\
&= \frac{n}{2} \left(1 - \frac{m \cdot p \cdot (1 - \dfrac{\alpha}{n \cdot p})^2}{m \cdot p + 1 - p} \right) = \frac{n}{2} \left(1 - \frac{(1 - \dfrac{\alpha}{n \cdot p})^2}{1 + (1-p)/(m \cdot p)} \right).
\end{aligned}$$

If (11) holds then due to the formulas (12) and (13), where $\tilde{\Lambda}(m) = \sum_{j=1}^{m} \lambda_j(n) = m \cdot \lambda(n)$, $E(S_{nm}(\Lambda(n))) = \alpha \cdot m \cdot \lambda(n)$ and $\lambda(n) = \frac{1-\alpha/(n\cdot p)}{m\cdot p+1-p}$, one may receive much stronger results for $0 < \alpha \leqslant 1$, namely:

$$1 \leqslant E\left(\frac{z_{OPT}(n)}{z_n(\Lambda(n))} \right) \leqslant \frac{1}{\alpha}, \text{ where } E\left(\frac{\tilde{\Lambda}(m,n)}{S_{nm}(\Lambda(n))} \right) = \frac{1}{\alpha} \text{ and} \tag{30}$$

$$E(z_n(\Lambda(n))) = \frac{n}{2} \cdot \left(1 - \frac{(1-\alpha/(n\cdot p))^2}{1+(1-p)/(m\cdot p)} \right). \tag{31}$$

Formulas (30) and (31) may provide us with some estimations of the set packing problem (1) optimal solution values $z_{OPT}(n)$ growth, when $n \to \infty$. Corresponding to Example 1 estimations of the $E\left(\frac{z_{OPT}(n)}{z_n(\Lambda(n))}\right)$ for the different values of α are provided in the Example below, where appropriate value of $E(z_n(\Lambda(n)))$ is given in the formula (31):

Example 2

$$\begin{array}{lcl} \textit{When } \alpha & = & 0.01 \textit{ then } 1 \leqslant E\left(\dfrac{z_{OPT}(n)}{z_n(\Lambda(n))}\right) \leqslant 100 \textit{ with approx. probability } 0.999 \\ \textit{When } \alpha & = & 0.1 \textit{ then } 1 \leqslant E\left(\dfrac{z_{OPT}(n)}{z_n(\Lambda(n))}\right) \leqslant 10 \textit{ with approx. probability } 0.995 \\ \textit{When } \alpha & = & 0.5 \textit{ then } 1 \leqslant E\left(\dfrac{z_{OPT}(n)}{z_n(\Lambda(n))}\right) \leqslant 2 \textit{ with approx. probability } 0.9098 \\ \textit{When } \alpha & = & 1 \quad \textit{then } E\left(\dfrac{z_{OPT}(n)}{z_n(\Lambda(n))}\right) = 1 \textit{ with approx. probability } \dfrac{2}{e} \approx 0.736. \end{array}$$

The smaller is the value of α the higher is probability of providing feasible solution of the set packing problem (1) but quality of the approximation, provided by (30) and (31) is deteriorating. Obviously approximation is not "strict" in the sense that, as α increases,only the upper bound on the expected value of the approximation quality increases. However when α is very small, e.g. $\alpha = 0.01$ in the above example, then expected values of all constraints left hand sides in (1) are very small either, i.e. $E(s_j(\Lambda(n))) = \alpha$, $j = 1, \ldots, m$. This in turn may indicate that only trivial solution like $x_i(\Lambda(n)) = 0$, $i = 1, \ldots, n$, of the original problem may be provided. Anyhow moderate values of α, e.g. $\alpha = 0.5$ or $\alpha = 1$, in the example above are providing reasonable compromise between quality of the approximation and feasibility of the solution.

Since $n \leqslant m!$ and moreover $n \to \infty$ then obviously also $m \to \infty$. According to formula (31) asymptotic growth of the $E(z_n(\Lambda(n)))$ may be influenced by both n and m. Let us consider the following mutual asymptotic dependence of the both parameters:

$$n = \beta \cdot m^\gamma, \text{ where } \beta \text{ is constants}, 0 < \gamma \leqslant m, \beta > 0. \qquad (32)$$

If $0 < \gamma \leqslant m$ then condition $n \leqslant m!$ is always fulfilled asymptotically since, due to the Stirling's formula, for every constant $\beta > 0$ there exist constant $m' \geqslant 1$ such that for all $m \geqslant m'$ the inequality $n \leqslant m!$ holds. .

Under the above assumption the following Lemma holds

Lemma 3 *If asymptotical dependence (32) holds then:*

$$E(z_n(\Lambda(n))) \approx \frac{2 \cdot \alpha + \beta \cdot (1-p) \cdot m^{\gamma-1}}{2 \cdot p} \textit{ when } n \to \infty \qquad (33)$$

Proof. When (32) holds then (31) may be reformulated as follows:

$$E(z_n(\Lambda(n))) = \frac{2m \cdot \alpha \cdot \beta \cdot p + m^\gamma \cdot \beta^2 \cdot p \cdot (1-p) - \alpha^2 \cdot m^{-\gamma+1}}{2\beta \cdot p \cdot (m \cdot p + 1 - p)}$$

Taking into account previously made assumptions on α, β, γ and p proof of the formula (33) is straightforward. ■

Corollary 3 *Depending on the value of γ, $0 < \gamma \leqslant m$, the following cases of the asymtotical behavior of $E(z_n(\Lambda(n)))$ may be distinguished:*

$$\lim_{m \to \infty} E(z_n(\Lambda(n))) = \begin{cases} \dfrac{\alpha}{p} & \textit{when } 0 < \gamma < 1 \\ \dfrac{2\alpha + \beta \cdot (1-p)}{2p} & \textit{when } \gamma = 1 \\ \infty & \textit{when } \gamma > 1 \end{cases} \qquad (34)$$

Due to the formulas (13) and (30) $E(z_n(\Lambda(n)))$ is reasonable asymptotic approximation of the optimal solution of the set packing problem (1) i.e. $E\left(z_{OPT}(n)\right)$. The above Lemma and Corollary, especially formulas (33) and (34), provides interesting insight into asymptotical behavior of the value of $E(z_n(\Lambda(n)))$. Namely:

$$\text{When } n = o(m) \text{ then } \lim_{m\to\infty} E(z_n(\Lambda(n))) = \frac{\alpha}{p}.$$

It does mean that in this case values of β and γ are negligible so is the mutual asymptotic dependence of both n and m.

$$\text{When } n \approxeq m \text{ then } E(z_n(\Lambda(n))) \approx \frac{2\alpha + \beta\cdot(1-p)}{2p}.$$

In this case level of proximity of n and m is substantial and is expressed by value β.

$$\text{When } m = o(n) \text{ then } E(z_n(\Lambda(n))) \approx \frac{\beta\cdot(1-p)}{2\cdot p}\cdot m^{\gamma-1}$$

In the latter case dependence on α is negligible, β and p are defining constant multiplier.

In 2 first cases, where $\gamma \leqslant 1$, there is no asymptotical influence of the value of m (and therefore of n either) on the asymptotical value of $E(z_n(\Lambda(n)))$. However in the case when $\gamma > 1$, there is very strong dependence from both m and γ.

On the other hand parameters α, and p have substantial influence on the asymptotical behavior of $E(z_n(\Lambda(n)))$, when $\gamma \leqslant 1$. Namely the bigger is value of α, $\alpha > 0$, and/or smaller is value of p, $0 < p \leqslant 1$, the bigger is value of $E(z_n(\Lambda(n)))$. Consequence of the above statement is following

- The bigger is value of α the less probability of feasibility of the corresponding solution of the set packing problem (1) is, see Theorem 1.
- The smaller the value of p is the sparser the initial subsets M_i, $i = 1, \cdots, n$, of the original set M may be.

In the case of the maximum set packing problem (3) situation is different. Namely

$$\begin{aligned} P\{c_i(\Lambda)=1\} &= P\{x_i(\Lambda)=1\} = P\left\{\lambda\cdot\left(\sum_{k=1}^{m} a_{ki}+1\right)\le 1\right\} = \\ &= P\left\{\left(\sum_{k=1}^{m} a_{ki}+1\right)\le\frac{1}{\lambda}\right\} = \sum_{r=0}^{\lfloor 1/\lambda\rfloor} P\left\{\sum_{k=1,k\neq j}^{m} a_{ki} = r\right\} = \\ &= \sum_{r=0}^{\lfloor 1/\lambda\rfloor}\binom{m}{r}\cdot p^r\cdot(1-p)^{m-r} \text{ where } \lambda\in\left\{\frac{1}{m},\frac{1}{m-1},\ldots,1\right\} \end{aligned}$$

If there exist $\Lambda(n)$ and α solving (25), with sufficient level of accuracy, and assuring $s_j(\Lambda(n)) \leqslant 1$, $j = 1, \cdots, m$ then

$$E(z_n(\Lambda(n))) = n\cdot\sum_{r=0}^{\lfloor 1/\lambda(n)\rfloor}\binom{m}{r}\cdot p^r\cdot(1-p)^{m-r}$$

may serve as appropriate approximation of the value of $z_{OPT}(n)$ as it was in the case of the set packing problem (1) above.

6 Concluding remarks

In the present paper some results describing probabilistic properties of the set packing problem (1) and the maximum set packing problem (3) are summarized.

In the paper distribution functions of the various random variables representing important problems characteristics are presented. Moreover some results concerning the feasibility of the received solutions and estimations of the set packing problem (1) optimal solution values $z_{OPT}(n)$ growth, when $n \to \infty$ are provided.

Examples 1 and 2 shows that the higher is accuracy of approximation of the optimal solution value the lower is probability of the feasibility of corresponding solution. For example when $\alpha = 0.5$ the quality of approximation is pretty tolerable, with relatively high probability of the feasibility of the solution. Moreover when $\alpha = 1$ the quality of approximation is very good with reasonable probability of the feasibility of the solution, approximately equal to 0.736. Lemma 3 shows possible asymptotical behavior of the optimal solution values when there is certain mutual asymptotic dependence of the parameters n and m.

In the case of *Maximum Set Packing Problem* there are some problem specific peculiarities which have been preliminary investigated in the present paper.

Some of the important avenues for the future research is convergence of the approximate solutions to the optimal solution and possibility of investigating realistic approximations of their values.

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