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Modelowanie matematyczne, symulacja komputerowa i identyfikacja dynamiki przepływu cieczy w wannie szklarskiej

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W Raporcie przedstawiono dwa artykuły dotyczące modelowania matematycznego i komputerowej identyfikacji dynamiki przepływu masy szklanej w piecu wannowym do produkcji szkła okiennego. Artykuły te zostały zaprezentowane w postaci wykładów w Szkole Letniej nt. Zaawansowanych Problemów Mechaniki (Summer School on Advanced Problems in Mechanics – APM'2001), która była zorganizowana w lipcu br. w Repinie koło Petersburga (21-30.07.2001). Organizatorami Szkoły Letniej były: Instytut Problemów Inżynierii Mechanicznej Rosyjskiej Akademii Nauk z Petersburga (Institute for Problems in Mechanical Engineering of Russian Academy of Sciences) oraz niemieckie Towarzystwo Matematyki i mechaniki Stosowanej (Gesellschaft fuer Angewandte Mathematik und Mechanik – GAMM). Artykuły ukażą się w materiałach Szkoły Letniej w 2002 r. (Proceedings of the XXIX Summer School on Advanced Problems in Mechanics, St. Petersburg (Repino), IPME RAS 2002).

Spis treści

On the solution of a nonlinear Navier-Stokes problem using the finite difference method

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Summary. In this paper the solvability analysis and numerical calculations of a quasi-linear, two-dimensional problem of a viscous liquid flow in a rectangle are discussed. The flow is described by the Navier-Stokes, energy and continuity equations. The analysis occurs with help of the ε-approximation and some finite differences are used to get the numerical solution of the problem. The theorems of the existence, uniqueness and convergence of the solution are prooved. The calculations are made for some real data from a glass tank furnace. In the paper an attemp to reconcile the theoretical investigation with practical applications is made.

Key words: Navier-Stokes equations, solvabilty analysis, mathematical modelling, computer simulations, dynamic systems, glas mass currents.

1. Introduction

The problem under consideration is

$$\begin{cases} L(u) \equiv -\Delta u + A\sum_{i=1}^{2} u_i \dfrac{\partial u}{\partial x_i} + B\,\mathrm{grad}p + Fu = f \\ \widetilde{\mathrm{div}}\,u = 0 & x \in \Omega \\ u = \psi & x \in \Gamma \end{cases} \tag{1.1}$$

where the domain Ω is a bounded rectangle in R^2 with boundary Γ, i.e. $\overline{\Omega} = \Omega \cup \Gamma$, $\Omega = \{x: x = (x_1, x_2),\ 0 \le x_i \le m_i,\ i = 1,2\}$ and $\Gamma = \bigcup_{i=1}^{4} \Gamma_i$ (see Fig. 1.1). Let us denote in (1.1) $u = (u_1, u_2, u_3) = (v_1, v_2, T)$, $f = -(0,1,0)\rho g \beta T_0 / \mu$ and $\widetilde{\mathrm{div}}\,u = \sum_{i=1}^{2} \dfrac{\partial u_i}{\partial x_i}$. ψ is a vector function of the boundary conditions,

$$A = \frac{\rho}{\mu}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c_v\mu/\lambda \end{bmatrix}, \quad B = \frac{1}{\mu}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = -\frac{\rho g \beta}{\mu}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

v_1, v_2 are velocity components, T is temperature, p is pressure and ρ, g, β, T_o, μ, c_v, λ are density, acceleration of gravity, coefficient of expansion, reference temperature, viscosity, specific heat and thermal conductivity, respectively.

System (1.1) arises from the steady-state Navier-Stokes problem upon complementing it with the energy equation. The joint equations describe the two-dimensional viscous liquid flow in which both forced and free convections (and not only the forced one as in the case of the pure Navier-Stokes equation) are taken into consideration. They create a nonlinear boundary value problem with non-homogenous boundary conditions that fits well for modelling e.g. the molten glass currents in a glass tank furnace.

The analysis of system (1.1) consists in approximating the original equations by a new boundary value problem with a small parameter $\varepsilon>0$ and then in constructing finite difference approximation for the equations obtained. The existence and uniqueness of the solutions of both differential and difference problems are established and the appropriate convergences of the solutions are shown. Subsequently some results of computer simulation for a simplified version of a glass tank mathematical model are presented. The dimensions of the model and the values of the coefficients which appear in the equations refer to real technological data from a conventional glass tank furnace.

2. The ε-approximation

Let us define the Hilbert space H^1 of vector functions $u=(u_1,u_2,u_3)$ equipped with the scalar product and the norm

$$\begin{cases} (u,v)_{H^1} = \sum_{i=1}^{3}\int_{\Omega}\left(u_i v_i + \sum_{j=1}^{2} D_j u_i D_j v_i \right) dx \\ \|u\|_{H^1} = (u,u)_{H^1}^{1/2} \end{cases} \tag{2.1}$$

respectively, and the subspace $H_o^1 \subset H^1$ that complements with the H^1-norm the set C_o^∞ of the vector functions the carriers of which are closed and contained in Ω. After introducing into the set C_o^∞ the following scalar product

$$(u,v)_U = \sum_{i=1}^{3}\int_{\Omega}\sum_{j=1}^{2} D_j u_i D_j v_i dx$$

and complementing it with the norm

$$\|u\|_U = (u,u)_U^{1/2}$$

we get the Hilbert space $\overset{o}{U}$ whose norm is equivalent to the H_o^1-norm for bounded Ω [7].

According to Temam [15] we can assume the existence in $\overline{\Omega}$ of an auxiliary vector function $\overline{w}\in H^1$ (implicitly defined) which satisfies the conditions: $\operatorname{d\tilde{i}v}\overline{w}=0$ on Ω and $\overline{w}=\psi$ on Γ. Upon substituting $u_w = u-\overline{w}$ we get from (1.1) the following homogenous problem

$$
\begin{cases}
L(u_w) \equiv -\Delta u_w + A\sum_{i=1}^{2} u_{wi}\dfrac{\partial u_w}{\partial x_i} + B\mathrm{grad}p + C\mathrm{grad}u_w + Gu_w + f_w \\
\mathrm{d\tilde{i}v}u_w = 0 & x \in \Omega \\
u_w = 0 & x \in \Gamma
\end{cases}
\tag{2.2}
$$

where $C\mathrm{grad}u_w = A\sum_{i=1}^{2} \overline{w}_i \dfrac{\partial u_w}{\partial x_i}$, $Gu_w = Fu_w + A\sum_{i=1}^{2} u_{wi}\dfrac{\partial \overline{w}}{\partial x_i}$, $f_w = f - L_w(\overline{w})$.

$L_w(\overline{w}) = -\Delta\overline{w} + A\sum_{i=1}^{2} \overline{w}_i \dfrac{\partial \overline{w}}{\partial x_i} + F\overline{w}$.

By resolving the non-homogenous problem (1.1) to the homogenous one (2.2) we can simplify radically the subsequent investigations.

By applying the ε-approximation with small parameter $\varepsilon > 0$, Temam [1966], to (2) we arrive at the following transformed problem

$$
\begin{cases}
\mathrm{L}_\varepsilon(u_\varepsilon) \equiv -\Delta u_\varepsilon + P_\varepsilon(u_\varepsilon) - \dfrac{1}{\varepsilon}B\mathrm{grad}(\mathrm{d\tilde{i}v}u_\varepsilon) + C\mathrm{grad}u_\varepsilon + Gu_\varepsilon = f_\varepsilon & x \in \Omega \\
u_\varepsilon = 0 & x \in \Gamma
\end{cases}
\tag{2.3}
$$

where

$$
P_\varepsilon(u_\varepsilon) = \frac{1}{2}A\sum_{i=1}^{2}(u_{\varepsilon i}\mathrm{D}_i u_\varepsilon + \mathrm{D}_i(u_{\varepsilon i}u_\varepsilon))
\tag{2.4}
$$

The ε-approximation consists in replacing the condition $\mathrm{d\tilde{i}v}u_w = 0$ with $\mathrm{d\tilde{i}v}u_\varepsilon = -\varepsilon p_\varepsilon$ and in addition the expression $\dfrac{1}{2}Au_\varepsilon \mathrm{d\tilde{i}v}u_\varepsilon$ to $L(u_w)$, where $f_\varepsilon \equiv f_w$ and u_ε, p_ε are some approximations of the functions u_w, p.

The benefits of the ε-approximation are: reduction of the number of variables in the equations (removal of the p-function), reduction of the number of equations (removal of the separate continuity condition), elimination of the nonlinear components in the relations during the farther investigation concerning the existence and uniqueness of the problem solution.

3. Solvability of the differential boundary problem

Definition 3.1. *Function $u_w \in \overset{0}{\mathrm{U}}$ is a solution of the problem (2.2) if for any $v \in \overset{0}{\mathrm{U}}$ the integral identity*

$$(u_w, v)_U + \frac{1}{2}(\sum_{i=1}^{2} Au_{wi} D_i u_w, v) - \frac{1}{2}\sum_{i=1}^{2}(Au_w u_{wi}, D_i v) + (C \mathrm{grad} u_w, v) + (Gu_w, v) = (f_w, v) \tag{3.1}$$

holds where $\widetilde{\mathrm{div}}\, u_w = \widetilde{\mathrm{div}}\, v = 0$ *and* $(.,.)$ *is the scalar product in the space* $L^2(\Omega)$.

Definition 3.2. *Function* $u_\varepsilon \in \overset{0}{U}$ *is a solution of the problem (2.3) if for any* $v \in \overset{0}{U}$ *the integral identity*

$$(u_\varepsilon, v)_U + (P_\varepsilon(u_\varepsilon), v) + \frac{1}{\varepsilon\mu}(\widetilde{\mathrm{div}}\, u_\varepsilon, \widetilde{\mathrm{div}}\, v) + (C \mathrm{grad} u_\varepsilon, v) + (Gu_\varepsilon, v) = (f_\varepsilon, v) \tag{3.2}$$

holds.

For farther consideration we will need the following Cauchy inequality

$$ab \le \frac{\eta}{p} a^p + \frac{1}{q\eta} b^q; \qquad a,b \in R, \qquad \eta, p, q > 0, \qquad \frac{1}{p} + \frac{1}{q} = 1 \tag{3.3}$$

and the Hölder inequalities:

$$\left|\int_\Omega \prod_{k=1}^{n} u_k dx\right| \le \prod_{k=1}^{n}\left(\int_\Omega |u_k|^{p_k} dx\right)^{1/p_k}; \qquad u_k \in L^{p_k}(\Omega) \tag{3.4}$$

$$\left|\sum_{i=1}^{m}\sum_{k=1}^{n} u_{ik}\right| \le \prod_{k=1}^{n}\left(\sum_{i=1}^{m} |u_{ik}|^{p_k}\right)^{1/p_k}; \qquad u_{ik} \in R \tag{3.5}$$

and also the following lemmas:

Lemma 3.1. *For any function* $u \in H_0^1(\Omega)$, $\Omega \subset R^2$, *the inequality*

$$\|u\|_4^4 \le 2\|u\|^2 \|u\|_U^2$$

holds where $\|.\|, \|.\|_4$ *mean the norms in* $L^2(\Omega)$ *and* $L^4(\Omega)$, *respectively* [7].

Lemma 3.2. *For any function* $u \in H_0^1(\Omega)$ *with the domain* Ω *bounded the inequality*

$$\|u\|^2 \le \frac{1}{\nu_1}\|u\|_U^2$$

holds where ν_1 *is the smallest eigenvalue of the Laplace operator* $-\Delta$ *in* Ω *for the zero boundary conditions* [7].

Lemma 3.3. *A sequence of functions* $\{u^n\}$ *that converges weakly in* $H_o^1(\Omega)$ *for* $\Omega \subset R^2$ *is converging strongly in* $L^2(\Omega)$ *and* $L^4(\Omega)$ [7].

Lemma 3.4. *A system of the nonlinear equations*

$$H_i(c) \equiv H_i(c_1,...,c_k) = h_i, \qquad i=1,2,...,k, \qquad c \in R^k,$$

has at least one solution if the inequality $\sum_{i=1}^{k} H_i(c)c_i \ge a_o|c|^p - K_o$ *holds for* $a_o > 0$, $|c|^p = c_1^p + ... + c_k^p$, $p > 1$ *and* $K_o \ge 0$ [16].

Theorem 3.1. *For any* $f_\varepsilon \in L^2$ *and for any* $\varepsilon > 0$

(a) the problem (2.3) has at least one solution $u_\varepsilon \in \overset{o}{U}$ *if the inequality*

$$\chi_0 = \frac{1}{2} - \chi_1 - \chi_2 \|\overline{w}\|_U > 0 \tag{3.6}$$

holds, where $\chi_1 = \frac{|F|}{\nu_1}$, $\chi_2 = \frac{2^{3/4}|A|}{\nu_1^{1/2}}$, $|F|$ *and* $|A|$ *are some norms of the matrices F and A, respectively;*

(b) for any solution $u_\varepsilon \in \overset{o}{U}$ *the following estimation*

$$\chi_0 \|u_\varepsilon\|_U^2 + \frac{1}{\varepsilon\mu} \|\operatorname{d\tilde{i}v} u_\varepsilon\|^2 \le \frac{1}{2} \|f_\varepsilon\|_{U'}^2 \tag{3.7}$$

is true, where $\|f_\varepsilon\|_{U'} = \sup_v \frac{(f_\varepsilon, v)}{\|v\|_U}$ *for any function* $v \in \overset{o}{U}$;

(c) the series of the solutions of (2.3) determined approximately by the Galerkin method converges to any u_ε *according to the* $\overset{o}{U}$*-norm.*

We precede the proof of the theorem by some lemmas.

Lemma 3.5. *For any function* $u \in \overset{o}{U}$ *the equality*

$$(P_\varepsilon(u), u) = 0$$

holds.

This equality we get after multiplying the expression $P_\varepsilon(u)$ *(see (2.4)) by u in* L^2 *and using subsequently the Green formulas.*

Lemma 3.6. *For any function* $u \in \overset{o}{U}$ *the equality*

$$(C\mathrm{grad}u, u) = 0$$

holds.

This equality we get after multiplying the expression Cgradu (see (2.2)) by u in L^2 *and using subsequently the Green formulas and the assumption* $\mathrm{d\tilde{i}v}\overline{w} = 0$.

Lemma 3.7. *For any functions* $u, v, w \in \overset{o}{U}$ *the inequality*

$$\left|\left(\sum_{i=1}^{2} Au_i D_i v, w\right)\right| \le \chi_3 \|u\|_4 \|v\|_U \|w\|_4$$

holds where $\chi_3 = 2^{1/4}|A|$.

This inequality we get after using successively the Hölder inequalities (3.4) and (3.5).

Lemma 3.8. *For any function* $u \in \overset{o}{U}$ *the inequality*

$$|(Gu, u)| \le (\chi_1 + \chi_2\|\overline{w}\|_U)\|u\|_U^2$$

holds.

This inequality we get after estimating the expression Gu (see (2.2)) from above and using the estimates from lemmas 3.7, 3.2 and 3.1 successively, i.e.

$$|(Gu, u)| \le |(Fu, u)| + \left|\left(A\sum_{i=1}^{2} u_i D_i \overline{w}, u\right)\right| \le |F|\|u\|^2 + \chi_3\|u\|_4\|\overline{w}\|_U\|u\|_4 \le \chi_1\|u\|_U^2 + \chi_2\|\overline{w}\|_U\|u\|_U^2 .$$

Proof. According to the Galerkin method we can write the approximate solution of (2.3) in the form $u_\varepsilon^k = \sum_{i=1}^{k} c_{ki} v_i$ where $c_{ki} \in R$, k=1,2,... and $\{v_i\}_{i=1}^{\infty}$ is a system of vector functions that is complete in $\overset{o}{U}$. For u_ε^k and the set of v_i we get from (2.3) an equivalent system of nonlinear equations

$$(L_\varepsilon u_\varepsilon^k, v_i) \equiv -(Du_\varepsilon^k, v_i) + (P_\varepsilon(u_\varepsilon^k), v_i) - \frac{1}{\varepsilon}(B\mathrm{graddi\tilde{v}}u_\varepsilon^k, v_i) + \\ +(C\mathrm{grad}u_\varepsilon^k, v_i) + (Gu_\varepsilon^k, v_i) = (f_\varepsilon, v_i) \tag{3.8}$$

After multiplying each equation of (3.8) by an appropriate c_{ki} and summing all equations the following relation

$$\left\|u_\varepsilon^k\right\|_U^2+(P_\varepsilon(u_\varepsilon^k),u_\varepsilon^k)+\frac{1}{\varepsilon\mu}\left\|\mathrm{d}\tilde{\mathrm{i}}\mathrm{v}u_\varepsilon^k\right\|^2+(C\mathrm{grad}u_\varepsilon^k,u_\varepsilon^k)+(Gu_\varepsilon^k,u_\varepsilon^k)=(f_\varepsilon,u_\varepsilon^k) \tag{3.9}$$

results. Using the lemmas 3.5, 3.6 and 3.8 we transform (3.9) into the inequality

$$\left\|u_\varepsilon^k\right\|_U^2(1-\chi_4)+\frac{1}{\varepsilon\mu}\left\|\mathrm{d}\tilde{\mathrm{i}}\mathrm{v}u_\varepsilon^k\right\|^2\le(f_\varepsilon,u_\varepsilon^k) \tag{3.10}$$

where $\chi_4=\chi_1+\chi_2\|\overline{w}\|_U$. The solvability of (3.8) for $1-\chi_4>0$ results from (3.10) and the lemma 3.4. Using the Young inequality (3.3) with $\eta=1$ to estimate $(f_\varepsilon,u_\varepsilon^k)$ from above we get from (3.10) the relation

$$\chi_0\left\|u_\varepsilon^k\right\|_U^2+\frac{1}{\varepsilon\mu}\left\|\mathrm{d}\tilde{\mathrm{i}}\mathrm{v}u_\varepsilon^k\right\|^2\le\frac{1}{2}\|f_\varepsilon\|_U^2. \tag{3.11}$$

which means that the sequence $\left\{u_\varepsilon^k\right\}_k$ is uniform bounded in $\overset{o}{\mathrm{U}}$ for $\chi_0>0$. Consequently a function $u_\varepsilon\in\overset{o}{\mathrm{U}}$ and a subsequence $\left\{u_\varepsilon^{kn}\right\}$ exist and u_ε^{kn} converges weakly to u_ε in $\overset{o}{\mathrm{U}}$ and it converges strongly to u_ε in L^2 and L^4 according to the lemma 3.3.

To show that u_ε is a solution of the problem (2.3) we will find the limits of the components of (3.8) for $kn\to\infty$ and for i fixed. To find the limit for $(\mathrm{P}_\varepsilon(\mathrm{u}_\varepsilon^k),\mathrm{v}_i)$ the following transformation

$$(P_\varepsilon(u_\varepsilon^k),v_i)=\frac{1}{2}(A\sum_{j=1}^{2}(u_{\varepsilon j}^k-u_{\varepsilon j})\mathrm{D}_ju_\varepsilon^k,v_i)+\frac{1}{2}(A\sum_{j=1}^{2}u_{\varepsilon j}\mathrm{D}_ju_\varepsilon^k,v_i)-\frac{1}{2}\sum_{j=1}^{2}(Au_{\varepsilon j}^k(u_\varepsilon^k-u_\varepsilon),\mathrm{D}_jv_i)-$$
$$-\frac{1}{2}\sum_{j=1}^{2}(Au_{\varepsilon j}^ku_\varepsilon,\mathrm{D}_jv_i) \tag{3.12}$$

and subsequently the lemma 3.7 and the fact of the strong convergence of u_ε^{kn} to u_ε in L^4 should be used. As a result we get

$$\lim_{kn\to\infty}(P_\varepsilon(u_\varepsilon^{kn}),v_i)=(P_\varepsilon(u_\varepsilon),v_i) \tag{3.13}$$

Finding the limits for other components of (3.8) we take into consideration the weak convergence of u_ε^{kn} to u_ε in $\overset{o}{\mathrm{U}}$ and its strong convergence to u_ε in L^4 and also the completeness of $\{v_i\}_{i=1}^{\infty}$ in $\overset{o}{\mathrm{U}}$. Consequently we get from (3.8) the identity (3.2) that shows the truth of thesis (a).

We get the estimation (3.7) from (3.2) after inserting $v\equiv u_\varepsilon$ and then following the same way as by getting the relation (3.11). This shows the truth of thesis (b).

To show the strong convergence of u_ε^{kn} to u_ε in $\overset{\circ}{U}$ we transform (3.2) into the form

$$(L_\varepsilon u_\varepsilon - L_\varepsilon u_\varepsilon^k, u_\varepsilon - u_\varepsilon^k) = (u_\varepsilon - u_\varepsilon^k, u_\varepsilon - u_\varepsilon^k)_U + (P_\varepsilon(u_\varepsilon), u_\varepsilon - u_\varepsilon^k) - (P_\varepsilon(u_\varepsilon^k), u_\varepsilon - u_\varepsilon^k) +$$
$$+\frac{1}{\varepsilon\mu}(\mathrm{d\tilde{i}v}(u_\varepsilon - u_\varepsilon^k), \mathrm{d\tilde{i}v}(u_\varepsilon - u_\varepsilon^k)) + (C\mathrm{grad}(u_\varepsilon - u_\varepsilon^k), u_\varepsilon - u_\varepsilon^k) + (G(u_\varepsilon - u_\varepsilon^k), u_\varepsilon - u_\varepsilon^k) \tag{3.14}$$

from which we get

$$\|u_\varepsilon - u_\varepsilon^k\|_U^2 + \frac{1}{\varepsilon\mu}\|\mathrm{d\tilde{i}v}(u_\varepsilon - u_\varepsilon^k)\|^2 = (L_\varepsilon u_\varepsilon, u_\varepsilon - u_\varepsilon^k) - (C\mathrm{grad}(u_\varepsilon - u_\varepsilon^k), u_\varepsilon - u_\varepsilon^k) -$$
$$-(G(u_\varepsilon - u_\varepsilon^k), u_\varepsilon - u_\varepsilon^k) - (f_\varepsilon, u_\varepsilon - u_\varepsilon^k) - (P_\varepsilon(u_\varepsilon), u_\varepsilon - u_\varepsilon^k) + (P_\varepsilon(u_\varepsilon^k), u_\varepsilon) - (P_\varepsilon(u_\varepsilon^k), u_\varepsilon^k) \equiv I_k$$

We can find now the limit of I_k for $kn \to \infty$ using for it successively the lemmas 3.5, 3.6, 3.8, the relation 3.12 and the property of the strong convergence of u_ε^{kn} to u_ε in L^2. Then we get $\lim\limits_{kn\to\infty} |I_k| = 0$. It shows the truth of thesis (c) and ends the proof of theorem 3.1 •

Theorem 3.2. *If the estimation*

$$1 - \chi_4 - \chi_5 \|f_\varepsilon\|_{U'} \geq \delta_1 > 0 \tag{3.15}$$

holds where $\chi_5 = \dfrac{\chi_2}{(2\chi_0)^{1/2}}$ *then the approximated solution of the problem (2.3) is unique.*

We precede the proof of the theorem by some lemmas.

Lemma 3.9. *For any functions u, v the inequality*

$$|(P_\varepsilon(u) - P_\varepsilon(v), u - v)| \leq \chi_2 \|u\|_U \|u - v\|_U^2$$

holds.

We get this inequality using successively the lemmas 3.7, 3.1 and 3.2 to estimate from above the following expression

$$(P_\varepsilon(u) - P_\varepsilon(v), u - v) = \frac{1}{2}(A\sum_{i=1}^{2}(u_i - v_i)\mathrm{D}_i u, u - v) + \frac{1}{2}(A\sum_{i=1}^{2} v_i \mathrm{D}_i(u - v) -$$
$$-\frac{1}{2}\sum_{i=1}^{2}(A(u_i - v_i)u, \mathrm{D}_i(u - v)) - \frac{1}{2}\sum_{i=1}^{2}(Av_i(u - v), \mathrm{D}_i(u - v))$$

that results from (2.4) like (3.12) does.

Lemma 3.10. *For any solutions* u_ε^l, u_ε^k *of the problem (2.3) the inequality*

$$(L_\varepsilon u_\varepsilon^l - L_\varepsilon u_\varepsilon^k, u_\varepsilon^l - u_\varepsilon^k) \geq (1 - \chi_4 - \chi_5 \|f_\varepsilon\|_{U^*}) \|u_\varepsilon^l - u_\varepsilon^k\|_U^2$$

holds.

After inserting $u_\varepsilon \equiv u_\varepsilon^l$ into (3.13) and using successively the lemmas 3.9, 3.6 and 3.8 we get the relation

$$(L_\varepsilon u_\varepsilon^l - L_\varepsilon u_\varepsilon^k, u_\varepsilon^l - u_\varepsilon^k) \geq \|u_\varepsilon^l - u_\varepsilon^k\|_U^2 + \frac{1}{\varepsilon\mu}\|\tilde{\mathrm{div}}(u_\varepsilon^l - u_\varepsilon^k)\|^2 - \chi_2 \|u_\varepsilon^l\|_U \|u_\varepsilon^l - u_\varepsilon^k\|_U^2 - \chi_4 \|u_\varepsilon^l - u_\varepsilon^k\|_U^2$$

from which the lemma's inequality results after considering the estimation (3.7).

Proof. Let us assume that two different solutions u_ε^1, u_ε^2 of the problem (2.3) exist and we will show that they are equal in $\overset{\circ}{U}$. For the solutions and for any function $v \in \overset{\circ}{U}$ the following relation

$$(L_\varepsilon u_\varepsilon^1 - L_\varepsilon u_\varepsilon^2, v) \equiv (u_\varepsilon^1 - u_\varepsilon^2, v)_U + (P_\varepsilon(u_\varepsilon^1) - P_\varepsilon(u_\varepsilon^2), v) + \frac{1}{\varepsilon\mu}(\tilde{\mathrm{div}}(u_\varepsilon^1 - u_\varepsilon^2), v) +$$
$$+ (C\mathrm{grad}(u_\varepsilon^1 - u_\varepsilon^2), v) + (G(u_\varepsilon^1 - u_\varepsilon^2), v) = 0 \equiv I_1$$

results from (3.2). After inserting $v = u_\varepsilon^1 - u_\varepsilon^2$ and using the lemma 3.10 we get

$$I_1 \geq (1 - \chi_4 - \chi_5 \|f_\varepsilon\|_{U^*}) \|u_\varepsilon^1 - u_\varepsilon^2\|_U^2 = I_2$$

from where with the help of (3.14) the inequality

$$0 \geq I_2 \geq \delta_1 \|u_\varepsilon^1 - u_\varepsilon^2\|_U^2 \geq 0$$

holds. It means that $u_\varepsilon^1 = u_\varepsilon^2$ in $\overset{\circ}{U}$ and this ends the proof •

Theorem 3.3. *If $\varepsilon \to 0$ then from the sequence $\{u_\varepsilon\}$ of the solutions of the problems family (2.3) one can choose a subsequence that converges strongly in $\overset{\circ}{U}$ to the solution u_w of the problem (2.2).*

Proof. The proof consists of two steps. We show the weak convergence of the solutions $\{u_\varepsilon\}$ in $\overset{\circ}{U}$ to u_w in the first step and the strong convergence of $\{u_\varepsilon\}$ in $\overset{\circ}{U}$ to u_w in the second one.

We can see from (3.7) that the sequence $\{u_\varepsilon\}$ is uniformly bounded in $\overset{\circ}{U}$ i.e. such a function $u_w \in \overset{\circ}{U}$ and a subsequence $\{u_{\varepsilon n}\}$ exist that $\{u_{\varepsilon n}\}$ converges weakly in $\overset{\circ}{U}$ to u_w

for $\varepsilon_n \to 0$. Then $\{u_{\varepsilon n}\}$ converges strongly to u_w in L^2 and L^4 according to the lemma 3.3 and also $\{\tilde{\mathrm{div}}\, u_{\varepsilon n}\}$ converges strongly to $\tilde{\mathrm{div}}\, u_w$ in L^2. Using (3.7) we get the existence of a subsequence $\{\tilde{\mathrm{div}}\, u_{\varepsilon nm}\}$ that converges strongly in L^2 to 0 for $\varepsilon_{nm} \to 0$. From it $\tilde{\mathrm{div}}\, u_w=0$ holds. To show that u_w is a solution of (2.2.) we write down the identity (3.2) for $u_{\varepsilon nm} \in \ddot{U}$ and for any $v \in \ddot{U}$ for which $\tilde{\mathrm{div}}\, v = 0$ holds. Then we have the relation

$$(L_\varepsilon u_{\varepsilon nm}, v) \equiv (u_{\varepsilon nm}, v)_U + (P_\varepsilon(u_{\varepsilon nm}), v) + \frac{1}{\varepsilon\mu}(\tilde{\mathrm{div}}\, u_{\varepsilon nm}, \tilde{\mathrm{div}}\, v) + (C\mathrm{grad} u_{\varepsilon nm}, v) +$$
$$+(Gu_{\varepsilon nm}, v) = (f_\varepsilon, v)$$

Finding for it the limit for $\varepsilon_{nm} \to 0$ under consideration of (3.14), (2.3) and of the condition $\tilde{\mathrm{div}}\, v = 0$ we get the identity (3.1). This ends the first step of the proof.

Now we form from (2.2) and (2.3) the relation

$$(Lu_w - L_\varepsilon u_{\varepsilon nm}, u_w - u_{\varepsilon nm}) \equiv (u_w - u_{\varepsilon nm}, u_w - u_{\varepsilon nm})_U + (P(u_w), u_w - u_{\varepsilon nm}) - (P_\varepsilon(u_{\varepsilon nm}), u - u_{\varepsilon nm}) +$$
$$+(B\mathrm{grad}(p - p_{\varepsilon nm}), u_w - u_{\varepsilon nm}) + (C(u_w - u_{\varepsilon nm}), u_w - u_{\varepsilon nm}) + (G(u_w - u_{\varepsilon nm}), u_w - u_{\varepsilon nm}) = 0 \tag{3.16}$$

where $P(u_w) = A\sum_{i=1}^{2} u_{wi} D_i u_w$, $p_{\varepsilon nm} = -\frac{1}{\varepsilon_{nm}}\tilde{\mathrm{div}}\, u_{\varepsilon nm}$. If we find now the limit for (3.16) for $\varepsilon_{nm} \to 0$ taking for it under consideration (3.13), the lemmas 3.5, 3.6, 3.8, the condition $\tilde{\mathrm{div}}\, u_w = 0$ and the estimation (3.7) successively, then we get $\lim_{\varepsilon_{nm} \to 0} \|u_w - u_{\varepsilon nm}\|_U^2 = 0$. That ends the proof of the theorem •

4. The difference approximation

To solve approximately the problem (2.3) the finite difference method will be used. In the rectangle $\overline{\Omega}$ we determine the following grid

$$\overline{\Omega}_h \equiv \{x\colon x = (ih_1, jh_2),\ i = 0,1,\ldots,N_1,\ j = 0,1,\ldots,N_2,\ N_k h_k = m_k,\ k = 1,2\}.$$

Let us introduce the Hilbert spaces L_h^2 and U_h of grid vector-functions $u_h^T = (u_{h1}, u_{h2}, u_{h3})$ determined on $\overline{\Omega}_h$. The scalar product and the norm for L_h^2 are as follows

$$\begin{cases} (u_h, v_h)_h = h_1 h_2 \sum_{i=1}^{3} \sum_{x \in \Omega_h} u_{hi}(x) v_{hi}(x), \qquad x = (x_1, x_2) \\ \|u_h\|_h = (u_h, u_h)_h^{1/2} \end{cases} \tag{4.1}$$

and for U_h are as follows

$$\begin{cases}(u_h,v_h)_{U_h}=(B_h u_h,v_h)_h\\ \|u_h\|_{U_h}=(B_h u_h,u_h)_h^{1/2}\end{cases} \tag{4.2}$$

where $B_h\equiv-\Delta_h$ is the Laplace difference operator.

Let us introduce also the space $\overset{o}{L}{}^2_h\subset L_h^2$ equipped with the norm (4.1) and defined for the functions u_h that are determined on Ω_h and are zero on the grid bound Γ_h, $\overline{\Omega}_h=\Omega_h\cup\Gamma_h$. We introduce either the space $\overset{o}{U}_h\subset U_h$ equipped with the norm (4.2) and defined for the operator B_h that is determined for the vector functions being zero on Γ_h.

We take the following denotation

$$u_h(x)\equiv u_h(ih_1,jh_2)=u_h^{ij},\quad \partial_1 u_h\equiv(u_h^{i+1,j}-u_h^{ij})/h_1,\quad \overline{\partial}_1 u_h\equiv(u_h^{ij}-u_h^{i-1,j})/h_1,$$
$$\tilde{\partial}_1 u_h\equiv(\partial_1 u_h+\overline{\partial}_1 u_h)/2,\quad \overline{\partial}_1\partial_1 u_h\equiv(u_h^{i+1,j}-2u_h^{ij}+u_h^{i-1,j})/h_1^2$$

whereby the finite defferences concerning the variable x_2 are to write down analogically. We have as well

$$\partial_x u_h=\{\partial_1 u_h,\partial_2 u_h\},\quad \overline{\partial}_x u_h=\{\overline{\partial}_1 u_h,\overline{\partial}_2 u_h\},\quad -\Delta_h u_h\equiv-\overline{\partial}_x\partial_x u_h=\sum_{i=1}^{2}\overline{\partial}_i\partial_i u_h .$$

We can approximate now the differential problem (2.3) with the following difference problem

$$\begin{cases}L_{\varepsilon h}u_h\equiv-\Delta_h u_h+P_{\varepsilon h}(u_h)-\dfrac{1}{\varepsilon}B\mathrm{grad}_h\mathrm{d\tilde{i}v}_h u_h+C\mathrm{grad}_h u_h+Gu_h=f_h, & x\in\Omega_h\\ u_h=0, \quad x\in\Gamma_h\end{cases} \tag{4.3}$$

where

$$\begin{cases}P_{\varepsilon h}(u_h)=\dfrac{1}{2}A\sum_{i=1}^{2}(u_{hi}\tilde{\partial}_i u_h+\tilde{\partial}_i(u_{hi}u_h))\\ C\mathrm{grad}_h u_h=A\sum_{i=1}^{2}\overline{w}_{hi}\tilde{\partial}_i u_h, \quad Gu_h=Fu_h+A\sum_{i=1}^{2}u_{hi}\tilde{\partial}_i\overline{w}_h\\ \mathrm{grad}_h\mathrm{d\tilde{i}v}_h u_h=(D_1,D_2), \quad D_i=\dfrac{1}{2}(\partial_i\mathrm{d\bar{\tilde{i}}v}_h u_h+\overline{\partial}_i\mathrm{d\tilde{i}v}_h u_h), \quad i=1,2\\ \mathrm{d\tilde{i}v}_h u_h=\sum_{i=1}^{2}\partial_i u_{hi}, \quad \mathrm{d\bar{\tilde{i}}v}_h u_h=\sum_{i=1}^{2}\overline{\partial}_i u_{hi}\end{cases} \tag{4.4}$$

The functions $f_h^T=(f_{h1},f_{h2},f_{h3})\in L_h^2$ and $\overline{w}_h^T=(\overline{w}_{h1},\overline{w}_{h2},\overline{w}_{h3})\in U_h$ are some approximations of the functions $f_\varepsilon\in L^2$ and $\overline{w}\in H^1$.

5. Solvability of the difference boundary problem

We will show now the existence and uniqueness of the solution of the problem (4.3) and the convergence of this solution to the solution of (2.3). For the farther consideration we will need the following lemmas:

Lemma 5.1. *For any function* $u_h \in \overset{o}{U}_h$ *the inequality*

$$\|u_h\|_{h4}^4 \le 2\|u_h\|_h^2 \|u_h\|_{U_h}^2$$

holds where $\|\cdot\|_{h4}$ *means the norm in* L_h^4 *that is the grid approximation of* L^4 [1].

Lemma 5.2. *For any function* $u_h \in \overset{o}{U}_h$ *the inequality*

$$\|u_h\|_h^2 \le \frac{c_0}{2}\|u_h\|_{U_h}^2, \qquad c_0 = m_1 m_2$$

holds [1].

One can see that these lemmas are difference equivalents to the lemmas 3.1 and 3.2.

Theorem 5.1. *For any function* $f_h \in L_h^2$ *and for any* $h = (h_1, h_2) > 0$

(a) the problem (4.3) has at least one solution $u_h \in \overset{o}{U}_h$ *if the inequality*

$$\chi_{00} = \frac{1}{2} - \chi_{11} - \chi_{22}\|\overline{w}_h\|_{U_h} > 0 \tag{5.1}$$

holds where $\chi_{11} = \dfrac{|F|c_0}{2}$ and $\chi_{22} = 2^{1/4}|A|c_0^{1/2}$;

(b) for any solution $u_h \in \overset{o}{U}_h$ *the following estimation*

$$\chi_{00}\|u_h\|_{U_h}^2 + \frac{1}{2\varepsilon\mu}\left(\|\mathrm{d}\tilde{\mathrm{i}}\mathrm{v}_h u_h\|_h^2 + \|\mathrm{d}\bar{\mathrm{i}}\mathrm{v}_h u_h\|_h^2\right) \le \frac{1}{2}\|f_h\|_{U_h^*}^2 \tag{5.2}$$

is true where $\|f_h\|_{U_h^*}^2 = \sup\limits_{v_h} \dfrac{(f_h, v_h)_h}{\|v_h\|_{U_h}}$ *for any function* $v_h \in \overset{o}{U}_h$.

We precede the proof of the theorem by 4 lemmas which are the difference equivalents to the lemmas 3.5, 3.6, 3.7, 3.8, respectively, and which are to prove similarly as the last ones.

Lemma 5.3. *For any function* $u_h \in \overset{o}{U}_h$ *the equality*

$$(P_{\varepsilon h}(u_h), u_h) = 0$$

holds.

Lemma 5.4. *For any function* $u_h \in \overset{o}{U}_h$ *the equality*

$$(C\mathrm{grad}_h u_h, u_h)_h = 0$$

holds.

To get this equality the conditions $\mathrm{d}\tilde{\mathrm{i}}\mathrm{v}_h \overline{w}_h = \mathrm{d}\overline{\tilde{\mathrm{i}}}\mathrm{v}_h \overline{w}_h = 0$ *are to be used.*

Lemma 5.5. *For any functions* $u_h,\ v_h,\ w_h \in \overset{o}{U}_h$ *the inequality*

$$\left|\left(\sum_{i=1}^{2} Au_{hi}\tilde{\partial}_i v_h, w_h\right)_h\right| \le \chi_{33}\|u_h\|_{h4}\|v_h\|_{U_h}\|w_h\|_{h4}$$

holds where $\chi_{33} = 2^{1/4}|A| = \chi_3$.

Lemma 5.6. *For any functions* $u_h,\ v_h \in \overset{o}{U}_h$ *the inequality*

$$\left|(Gu_h, v_h)_h\right| \le (\chi_{11} + \chi_{22}\|\overline{w}_h\|_{U_h})\|u_h\|_{U_h}\|v_h\|_{U_h}$$

holds.

To get this inequality the lemmas 5.5, 5.1 and 5.2 must be used.

Proof. The problem (4.3) is a system of N=$(N_1-1)(N_2-1)$ nonlinear algebraical equations determined in $\overset{o}{U}_h$. A solution of this system one can formulate as a grid function $u_h = \sum_{i=1}^{N} c_i v_h^i$ where $c_i \in R$ and $\left\{v_h^i\right\}_{i=1}^{N}$ is a base in $\overset{o}{U}_h$. Then we can write down the vector equation (4.3) as the equivalent system of scalar equations

$$(L_{\varepsilon h}u_h, v_h^i)_h = (f_h, v_h^i), \qquad i=1,\dots,N \tag{5.3}$$

After multiplying each equation of (5.3) by corresponding c_i and summing all equations we get the relation

$$(L_{\varepsilon h}u_h, u_h)_h \equiv \|u_h\|_{U_h}^2 + (P_{\varepsilon h}(u_h), u_h)_h + \frac{1}{2\varepsilon\mu}(\|\mathrm{d}\tilde{\mathrm{i}}\mathrm{v}_h u_h\|_h^2 + \|\mathrm{d}\overline{\tilde{\mathrm{i}}}\mathrm{v}_h u_h\|_h^2) +$$
$$+(C\mathrm{grad}_h u_h, u_h)_h + (Gu_h, u_h)_h = (f_h, u_h)_h$$

from which with the help of the lemmas 5.3, 5.4 and 5.6 the inequality

$$\|u_h\|_{U_h}^2(1-\chi_{44})+\frac{1}{2\varepsilon\mu}\left(\left\|\mathrm{d\tilde{i}v}_h u_h\right\|_h^2+\left\|\mathrm{d\bar{\bar{i}}v}_h u_h\right\|_h^2\right)\le(f_h,u_h)_h \tag{5.4}$$

results where $\chi_{44}=\chi_{11}+\chi_{22}\|\overline{w}_h\|_U$. One can see from (5.4) and the lemma 3.4 that the equation system (5.3) and also the problem (4.3) has at least one solution for $1-\chi_{44}>0$ and for $h=(h_1,h_2)$ fixed. This proves the thesis (a).

If we will use the inequality (3.3) with $\eta=1$ to estimate (5.4) we will get the estimation (5.2). It shows the truth of the thesis (b) and ends the proof of the whole theorem •

Theorem 5.2. *If the estimation*

$$1-\chi_{44}-\chi_{55}\|f_h\|_{U_h^*}\ge\delta_{11}>0 \tag{5.5}$$

holds where $\chi_{55}=\dfrac{\chi_{22}}{(2\chi_{00})^{1/2}}$ *then the solution of the problem (4.3) is unique.*

We precede the proof of the theorem by 2 lemmas which are the difference equivalents to the lemmas 3.9 and 3.10 and which are also simirlarly to prove.

Lemma 5.7. *For any functions* $u_h,\ v_h\in\overset{o}{\mathrm{U}}_h$ *the inequality*

$$(P_{\varepsilon h}(u_h)-P_{\varepsilon h}(v_h),u_h-v_h)_h\le\chi_{22}\|u_h\|_{U_h}\|u_h-v_h\|_{U_h}^2$$

holds.

We get this inequality after transforming (4.4) similarly as it was done in the lemma 3.9 and using successively the lemmas 5.5, 5.1 and 5.2 to estimate the transformation.

Lemma 5.8. *For any functions* $u_h,\ v_h\in\overset{o}{\mathrm{U}}_h$ *the inequality*

$$(L_{\varepsilon h}u_h-L_{\varepsilon h}v_h,u_h-v_h)_h\ge(1-\chi_{44}-\chi_{55}\|f_h\|_{U_h^*})\|u_h-v_h\|_{U_h}^2$$

holds.

From (4.3) we will get the relation

$$\begin{aligned}(L_{\varepsilon h}u_h-L_{\varepsilon h}v_h,u_h-v_h)_h&=-(\Delta_h(u_h-v_h),u_h-v_h)_h+(P_{\varepsilon h}(u_h)-P_{\varepsilon h}(v_h),u_h-v_h)_h+\\&+\frac{1}{2\varepsilon\mu}\Big((\mathrm{d\tilde{i}v}_h(u_h-v_h),\mathrm{d\tilde{i}v}_h(u_h-v_h))_h+(\mathrm{d\bar{\bar{i}}v}_h(u_h-v_h),\mathrm{d\bar{\bar{i}}v}_h(u_h-v_h))_h\Big)+\\&+(C\mathrm{grad}_h(u_h-v_h),u_h-v_h)_h+(G(u_h-v_h),u_h-v_h)_h\end{aligned} \tag{5.6}$$

from which the lemma's inequality results after using successively the lemmas 5.7, 5.4 and 5.6 and considering the estimation (4.6).

Proof. Let us assume that for a fixed $h=(h_1,h_2)$ two different solutions u_h^1, u_h^2 of the problem (4.3) exist. For any function $v_h \in \overset{o}{U}_h$ we get from (4.3) the expression $(L_{\varepsilon h}u_h^1 - L_{\varepsilon h}u_h^2, v_h)_h \equiv 0 = I$ from which (5.6) results after inserting $v_h \equiv u_h^1 - u_h^2$. Using the lemma 5.8 and considering (5.5) we will get from (5.6) the estimation $I \geq \delta_{11}\left\|u_h^1 - u_h^2\right\|_{U_h}^2 \geq 0$ which shows that $u_h^1 \equiv u_h^2$ in $\overset{o}{U}_h$. This ends the proof •

Theorem 5.3. *If*

(a) the function $u_h \in \overset{o}{U}_h$ is a solution of the problem (4.3) and

(b) the functions $\tilde{u}_h$, $\partial_i\tilde{u}_h$ are the segmental constant extensions in Ω of the functions u_h, $\partial_i u_h$, respectively,

then from the set $\{h\}$ of different discretization steps of Ω one can choose such a sequence $\{h_n\}_n$ converging to zero that $\tilde{u}_{hn}$ and $\partial_i\tilde{u}_{hn}$ are converging strongly in $L^2(\Omega)$ to u_ε and $D_i u_\varepsilon$, respectively, where u_ε is the solution of the problem (2.3).

Proof. We obtain from (4.6) that the sequence $\{u_h\}$ of the solutions of (4.3) is uniformly bounded in $\overset{o}{U}_h$ and also in $\overset{o}{L}{}_h^2(\Omega_h)$ for $\chi_{00} > 0$. The result is that the sequences of the step functions $\{\tilde{u}_h\}$, $\{\partial_i\tilde{u}_h\}$ which are induced by the grid functions u_h, $\partial_i u_h$, respectively, are uniform bounded in $L^2(\Omega)$ [5]. It means that such a function $u_\varepsilon \in H_0^1$ and a subsequence $\{\tilde{u}_{hn}\} \in \{\tilde{u}_h\}$ exist that $\tilde{u}_{hn}$ converges strongly to u_ε and $\partial_i\tilde{u}_{hn}$ converges weakly to $D_i u_\varepsilon$ in L^2 for $h_n \to 0$.

For showing the strong convergence of $\partial_i\tilde{u}_{hn}$ to $D_i u_\varepsilon$ in L^2 we form the grid function $u_{\varepsilon h}$ that is a grid cutting of the limit function u_ε. Then from the sequence of the step functions $\{\tilde{u}_{\varepsilon h}\}$ which are induced by $u_{\varepsilon h}$ one can choose such a subsequence $\{\tilde{u}_{\varepsilon hm}\}$ that $\tilde{u}_{\varepsilon hm}$ converges strongly to u_ε and $\partial_i\tilde{u}_{\varepsilon hm}$ converges weakly to $D_i u_\varepsilon$ in L^2 for $h_m \to 0$. It means that $(\tilde{u}_{hnm} - \tilde{u}_{\varepsilon hnm})$ converges strongly to 0 and $(\partial_i\tilde{u}_{hnm} - \partial_i\tilde{u}_{\varepsilon hnm})$ converges weakly to 0 in L^2 and also that $(u_{hnm} - u_{\varepsilon hnm})$ converges strongly to 0 and $(\partial_i u_{hnm} - \partial_i u_{\varepsilon hnm})$ converges weakly to 0 in L_h^2 [17]. We can write down the expression

$$(L_{\varepsilon h}u_{hnm} - L_{\varepsilon h}u_{\varepsilon hnm}, u_{hnm} - u_{\varepsilon hnm})_h = (f_h, u_{hnm} - u_{\varepsilon hnm})_h - (L_{\varepsilon h}u_{\varepsilon hnm}, u_{hnm} - u_{\varepsilon hnm})_h$$

from which using (4.3) for $u_h \equiv u_{hnm}$ and using (5.6) for $u_h \equiv u_{hnm}$ and $v_h \equiv u_{\varepsilon hnm}$ we get the relation

$$\sum_{i=1}^{2}\left\|\partial_i(u_{hnm}-u_{\varepsilon hnm})\right\|_h^2+\frac{1}{2\varepsilon\mu}\left(\left\|\mathrm{d\tilde{i}v}_h(u_{hnm}-u_{\varepsilon hnm})\right\|_h^2+\left\|\mathrm{d\tilde{\tilde{i}}v}_h(u_{hnm}-u_{\varepsilon hnm})\right\|_h^2\right)=$$

$$=(f_h,u_{hnm}-u_{\varepsilon hnm})_h-\sum_{i=1}^{2}(\partial_i u_{\varepsilon hnm},\partial_i(u_{hnm}-u_{\varepsilon hnm}))_h- \tag{5.7}$$

$$-\frac{1}{2\varepsilon\mu}\left((\mathrm{d\tilde{i}v}_h u_{\varepsilon hnm},\mathrm{d\tilde{i}v}_h(u_{hnm}-u_{\varepsilon hnm}))_h+(\mathrm{d\tilde{\tilde{i}}v}_h u_{\varepsilon hnm},\mathrm{d\tilde{\tilde{i}}v}_h(u_{hnm}-u_{\varepsilon hnm}))_h\right)-$$

$$-(P_{\varepsilon h}(u_{hnm}),u_{hnm}-u_{\varepsilon hnm})_h-(C\mathrm{grad}_h u_{hnm},u_{hnm}-u_{\varepsilon hnm})_h-(Gu_{hnm},u_{hnm}-u_{\varepsilon hnm})_h$$

Finding the limit for (5.7) for $h_{nm}\to 0$ we get that the right side of the relation converges 0. It means that $\partial_i(u_{hnm}-u_{\varepsilon hnm})$ converges strongly to 0 in L_h^2 and subsequently $(\partial_i\tilde{u}_{hnm}-\partial_i\tilde{u}_{\varepsilon hnm})$ converges strongly to 0 in L^2 and finally $\partial_i\tilde{u}_{hnm}$ converges strongly to $\mathrm{D}_i u_\varepsilon$ in L^2.

In the second step we show that $u_\varepsilon\in H_0^1$ is the solution of the problem (2.3). Using (4.3) we can write down the expression

$$(L_{\varepsilon h}\tilde{u}_{\varepsilon h},\tilde{v}_h)\equiv-(\sum_{i=1}^{2}\bar{\partial}_i\partial_i\tilde{u}_{\varepsilon h},\tilde{v}_h)+(P_{\varepsilon h}(u_{\varepsilon h}),\tilde{v}_h)-\frac{1}{\varepsilon}(B\mathrm{grad}_h d\tilde{i}v_h\tilde{u}_{\varepsilon h},\tilde{v}_h)+ \tag{5.8}$$

$$+(C\mathrm{grad}_h\tilde{u}_{\varepsilon h},\tilde{v}_h)+(G\tilde{u}_{\varepsilon h},\tilde{v}_h)-(\tilde{f}_h,\tilde{v}_h)$$

where the functions $\tilde{u}_{\varepsilon h}$, $\tilde{v}_h\in L^2$ mean segmentally constant extensions of the grid functions $u_{\varepsilon h}$, $v_h\in L_h^2$ which are some grid cuttings of u_ε and of any function $v\in H_0^1$, respectively. Finding for (5.8) the limit for $h\to 0$ we get the integral identity (3.2). This ends the proof •

6. Iterative algorithm

The difference approximation of the differential problem (2.3) leads to the system (4.3) of nonlinear algebraical equations which should be solved approximately with iterative methods. One can use here the following algorithm [3]

$$B_h u_h^{n+1}=B_h u_h^n-\gamma(L_{\varepsilon h}u_h^n-f_h),\qquad n=0,1,... \tag{6.1}$$

where $B_h:\overset{\circ}{U}_h\to\overset{\circ}{U}_h$ is a positiv, linear and self conjugated operator, i.e. $B_h=B_h^*>0$. We can take $B_h\equiv-\Delta_h$.

For farther consideration we will need the following lemmas [4]:

Lemma 6.1. *For any functions* u_h, $v_h\in\overset{\circ}{U}_h$ *the inequality*

$$(L_{\varepsilon h}u_h-L_{\varepsilon h}v_h,u_h-v_h)_h\geq\delta_2\left\|u_h-v_h\right\|_{U_h}^2 \tag{6.2}$$

holds where $\delta_2 = 1 - \chi_{44} - \chi_{33}\|u_h\|_{U_h}$.

We will get this inequality from the relation (5.6) after using successively the lemmas 5.7, 5.4 and 5.6.

Lemma 6.2. *For any functions u_h, $v_h \in \overset{o}{U}_h$ and for $B_h \equiv -\Delta_h$ the inequality*

$$(B_h^{-1}(L_{\varepsilon h}u_h - L_{\varepsilon h}v_h), L_{\varepsilon h}u_h - L_{\varepsilon h}v_h)_h \le \delta_3\|u_h - v_h\|_{U_h}^2 \tag{6.3}$$

holds where

$$\delta_3 = (1 + \chi_{22}(2\|u_h\|_{U_h} + \|u_h - v_h\|_{U_h}) + \frac{2}{\varepsilon\mu} + \chi_{11} + \chi_{22}(c_0^{-1/4}\|\overline{w}_h\|_{h4} + \|\partial_x \overline{w}_h\|_h))^2.$$

For proving the lemma we introduce the auxiliary function

$$s_h(x) \equiv \begin{cases} B_h^{-1}(L_{\varepsilon h}u_h - L_{\varepsilon h}v_h), & x \in \Omega_h \\ 0, & x \in \Gamma_h \end{cases}$$

and with the help of it and of (4.3) we can formulate the relation

$$\begin{aligned}
&(B_h^{-1}(L_{\varepsilon h}u_h - L_{\varepsilon h}v_h), L_{\varepsilon h}u_h - L_{\varepsilon h}v_h)_h \equiv (L_{\varepsilon h}u_h - L_{\varepsilon h}v_h, s_h)_h = \\
&= (B_h(u_h - v_h), s_h)_h + (P_{\varepsilon h}(u_h) - P_{\varepsilon h}(v_h), s_h)_h + \\
&+ \frac{1}{2\varepsilon\mu}\Big((\widetilde{\mathrm{di}}\mathrm{v}_h(u_h - v_h), \widetilde{\mathrm{di}}\mathrm{v}_h s_h)_h + (\overline{\overline{\mathrm{di}}}\mathrm{v}_h(u_h - v_h), \overline{\overline{\mathrm{di}}}\mathrm{v}_h s_h)_h\Big) + \\
&+ (C\mathrm{grad}_h(u_h - v_h), s_h)_h + (G(u_h - v_h), s_h)_h
\end{aligned} \tag{6.4}$$

We will get (6.3 after estimating the components of the right side of (6.4) using successively the Hölder inequality (3.5) and the lemmas 5.1, 5.2, 5.5 and 5.6.

Theorem 6.1. *If for the operator $L_{\varepsilon h}u_h$ in (4.3)*

(a) the inequalities (6.2) and (6.3) hold for $\delta_2(t) > 0$ and for $\delta_2(t)$ and $-\delta_3(t)$ being bounded and nongrowing functions, where $t \in (0, r)$, $u_h \equiv u_h^n$, $v_h \equiv u_h^*$, $u_h - v_h \equiv z_h^n$ and u_h^n, u_h^* are an approximated and the exactly solution of the problem (4.3), respectively;

(b) $\|z_h^n\|_{U_h} \le r$

(c) $\gamma > 0$ in (6.1),

then for $B_h \equiv -\Delta_h$ the inequality

$$\left\|z_h^n - \gamma B_h^{-1}(L_{\varepsilon h}(u_h^n + z_h^n) - L_{\varepsilon h}u_h^n)\right\|_{U_h} \le \rho(\gamma)\|z_h^n\|_{U_h} \tag{6.5}$$

holds. Here $\rho(\gamma) = (1 - 2\gamma\delta_2(r) + \gamma^2\delta_3(r))^{1/2}$ and such the value γ_0 exists that $\rho(\gamma) < 1$ for $0 < \gamma < \gamma_0$ and $\min\rho(\gamma) = \rho(\gamma^*) = (1 - \delta_2^2(r)\delta_3^{-1}(r))^{1/2}$ where $\gamma^* = \delta_2(r)\delta_3^{-1}(r)$.

The proof of this theorem is given by Djakonov [3]. The truth of (6.5) shows the convergence of the iterative algorithm (6.1).

7. Numerical calculation

As a complement to the theoretical discussion some numerical calculations were done for the boundary value problem representing a model of the molten glass flow in a real tank furnace. To shorten the computing time the model equations (1.1) were taken in a simplified form with the matrix A changed as follows

$$A = \frac{\rho}{\mu}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_v\mu/\lambda \end{bmatrix}.$$

This simplification means physically that the influence of the inertial forces on the liquid motion has been omitted because of the small velocities of the glass flow.

The simplified problem (1.1) can be written in the scalar form

$$\begin{cases} \mu(D_1^2 v_1 + D_2^2 v_1) = D_1 p \\ \mu(D_1^2 v_2 + D_2^2 v_2) = D_2 p - \rho g \beta (T - T_o) \\ \lambda(D_1^2 T + D_2^2 T) = \rho c_v (v_1 D_1 T + v_2 D_2 T) \\ D_1 v_1 + D_2 v_2 = 0 \end{cases} \tag{7.1}$$

with the boundary conditions added which result from the practice and which are: $v_1 = 0$ and $v_2 = 0$ on the bottom and on the side walls of the glass tank (i.e. on Γ_1, Γ_2 and Γ_4 of Ω); $v_1 = v_1(x_1)$ and $v_2 = v_2(x_1)$ on the free surface of the glass mass (i.e. on Γ_3 of Ω) with v_2 beeing quadratic in sections 1 and 3 of the tank; $T = T(x)$ on all boundaries and the function is linear on the side tank walls and it is cubic elsewhere.

The boundary conditions for the function p are unknown and this makes necessary to transform (7.1). It is then advisable to replace the velocities v_1, v_2 by the current function ψ where

$$v_1 = a_o \frac{\partial \psi}{\partial x_2}, \quad v_2 = -a_o \frac{\partial \psi}{\partial x_1} \tag{7.2}$$

with $a_o = \dfrac{\lambda_o}{\rho c_v}$ and $\lambda_o = \lambda(T_o)$. Using (7.2) one can transform (7.1) to the following form

$$\frac{\partial^4 \psi}{\partial x_1^4} + 2\frac{\partial^4 \psi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \psi}{\partial x_2^4} - \frac{\rho g \beta}{\mu a_o}\frac{\partial T}{\partial x_1} = 0 \tag{7.3}$$

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} - \frac{\lambda_o}{\lambda}\left(\frac{\partial \psi}{\partial x_2}\frac{\partial T}{\partial x_1} - \frac{\partial \psi}{\partial x_1}\frac{\partial T}{\partial x_2}\right) = 0 \tag{7.4}$$

that consists of only two equations in contrary to the four ones in (7.1). The reduction of the number of equations shall cause a better convergence of the future iterative algorithm. On the other hand the order of the new equation (7.3) inreased to four and such an order change leads usually to a worse stability of difference quotients.

The boundary conditions for the equations (7.3), (7.4) are

$$\begin{cases} \psi = 0, \quad \dfrac{\partial \psi}{\partial x_2} = 0 & \text{for } x_2 = 0 \\ \psi = 0, \quad \dfrac{\partial \psi}{\partial x_1} = 0 & \text{for } x_1 = 0 \text{ and } x_1 = L \\ \psi = \psi_1(x_1), \quad \dfrac{\partial \psi}{\partial x_2} = 0 & \text{for } x_2 = 0 \text{ and } 0 \le x_1 \le l_1 \\ \psi = \psi_H, \quad \dfrac{\partial^2 \psi}{\partial x_2^2} = 0 & \text{for } x_2 = H \text{ and } l_1 \le x_1 \le L - l_2 \\ \psi = \psi_2(x_1), \quad \dfrac{\partial \psi}{\partial x_2} = 0 & \text{for } x_2 = 0 \text{ and } L - l_2 \le x_1 \le L \\ T = T_1(x_1) = a_1 x_1^3 + b_1 x_1^2 + c_1 x_1 \quad = 0 & \text{for } x_2 = 0 \\ T = T_2(x_2) = a_2 x_2 + b_2 & \text{for } x_1 = 0 \\ T = T_H & \text{for } x_2 = H \text{ and } 0 \le x_1 \le l_1 \\ T = T_3(x_1) = a_3(x_1 - l_1)^3 + b_3(x_1 - l_1)^2 + c_3(x_1 - l_1) + d_3 & \\ & \text{for } x_2 = 0 \text{ and } L - l_2 \le x_1 \le L \\ T = T_4(x_2) = a_4 x_2 + b_4 & \text{for } x_1 = L \end{cases} \tag{7.5}$$

where L and H mean the length and hight of the tank furnace and l_1, l_2 indicate three sections on the surface of the melt (see Fig. 7.1). These relations related to the walls and the bottom of the tank result easy from the boundary conditions concerning the velocities v_1 and v_2 and the temperature T as given generally with the equations (7.1). Some more detailed assumptions connected with the tank surface have to be done. In the first surface section $(0 \le x_1 \le l_1)$ the raw materials are put into the tank and there $v_1 = 0$ holds and a quadratic function for v_2 is assumed [8]. In the second section $(l_1 \le x_1 \le L - l_2)$ $v_2 = 0$ and $D_2 v_1 = 0$ are assumed from which $\psi_H = \text{const}$ results. In the third section $(L - l_2 \le x_1 \le L)$ in which the glass sheet is drawn out of the tank also $v_1 = 0$ holds and a quadratic function for v_2 is assumed. There is stated that the temperature is constant $(T_H = \text{const})$ and equivalent to the melting point of the glass in the first section of the tank surface and it changes according to a third order function in the region $l_1 \le x_1 \le L$ consisting of sections 2 and 3.

All parameters in the equations (7.3), (7.4) are constant with the exception of μ and λ for which the following approximations

$$\begin{cases} \mu = \exp(A_1 + B_1 / (T - C_1)) \\ \lambda = A_2 \exp(B_2 (T - C_2)) \end{cases} \quad (7.6)$$

hold [9].

A discrete approximation of the equations (7.3), (7.4) occurs with the classical difference quotients. They lead, however, in the case of high order derivations to a bad stability at the edge of the difference equations. This is explained by an inaccurate approximation at the edge of the knotted grid [2]. Becuase of that some new central difference quotients have been developed for the fourth order derivations of ψ

$$\begin{cases} \dfrac{\partial^4 \psi}{\partial x_1^4} = -16 \dfrac{\psi_{i+1j} - 2\psi_{ij} + \psi_{i-1j}}{h_1^4} \\ \dfrac{\partial^4 \psi}{\partial x_2^4} = -16 \dfrac{\psi_{ij+1} - 2\psi_{ij} + \psi_{i-1j}}{h_2^4} \end{cases} \quad (7.7)$$

They contain a smaller number of the grid knotted points than the classical difference quotients. This allows to approximate more appropriate the edge region of the grid and to choose some greater discretization steps while calculating the equations.

After the approximation is done the following difference schemes

$$\begin{aligned} \psi_{ij} = {} & (16d^2(\psi_{i+1j} + \psi_{i-1j}) + 16\frac{1}{d^2}(\psi_{ij+1} + \psi_{ij-1}) - \\ & - 2(\psi_{i+1j+1} - 2\psi_{ij+1} + \psi_{i-1j+1} - 2\psi_{i+1j} - 2\psi_{i-1j} + \psi_{i+1j-1} - 2\psi_{ij-1} + \psi_{i-1j-1}) + \\ & + \frac{\rho g \beta \hat{T} \hat{L} \hat{H}^2 h_1 h_2}{2\mu a_0}(T_{i+1j} - T_{i-1j}))/(8(4d^2 + 1 + 4\frac{1}{d^2})) \end{aligned} \quad (7.8)$$

$$\begin{aligned} T_{ij} = {} & (d(T_{i+1j} + T_{i-1j}) + \frac{1}{d}(T_{ij+1} + T_{ij-1}) + \\ & + \frac{\lambda_0}{4\lambda}((T_{ij+1} - T_{ij-1})(\psi_{i+1j} - \psi_{i-1j}) - (T_{i+1j} - T_{i-1j})(\psi_{ij+1} - \psi_{ij-1})))/(2(d + \frac{1}{d})) \end{aligned} \quad (7.9)$$

result from (7.3), (7.4) for i=1,2,...,M and j=1,2,...,N where $\hat{T}$, $\hat{L}$, $\hat{H}$ are some standarization constants; $d = \hat{H}h_2 / \hat{L}h_1$; h_1, h_2 are discretization steps; $h_1 = L/(M+1)$ and $h_2 = H/(N+1)$.

The schemes (7.8), (7.9) are solved by means of the relaxation method using the following iterative algorithm

$$\psi_{ij}^{n+1} = (1 - \omega_1)\psi_{ij}^n + \omega_1 \psi_{ij} \quad (7.10)$$

$$T_{ij}^{n+1} = (1 - \omega_2)T_{ij}^n + \omega_2 T_{ij} \quad (7.11)$$

where ω_1, ω_2 are relaxation coefficients and T_{ij}, ψ_{ij} are calculated from the equations (7.8) and (7.9) in each iteration n=0,1,2,...

For the numerical calculation the values of the physical coefficients and of the space dimensions of the model were chosen according to those of a real tank furnace. The convergence of the iterative algorithm was relatively fast with highly satisfactory accuracy of the calculation obtained. Some examplary results of the temperature and current fields are shown in Figure 7.2. They were obtained for the grid of 600 modes with the accuracy of calculation 10^{-6} for T and 10^{-5} for ψ. The computation was stopped after the total number of 2900 iterations for both the temperature and current schemes. The iteration number for the temperature scheme was 800 and for the current scheme was 2100.

8. Conclusions

In this paper the ε-approximation method for the analysis and the finite differences method for solving of a nonlinear boundary value problem were used. The non-homogenous boundary problem consists of Navier-Stokes and energy equations and they describe the flow of viscous incompressible liquid. For illustrating the theoretical consideration the computation of a glass tank furnace was made. This tank furnace is a practical example of using the Navier-Stokes and energy equations for mathematical modelling.

The theorems of existence and uniqueness of the problem solution and the theorems of the convergence of both methods are presented. The theorems of existence and uniqueness are conditional, i.e. they are true if some conditions are fulfilled. These conditions are the inequalities (3.6) or (4.5) in the case of existence of the solutions of the differential or difference equations and the inequalities (3.15) or (5.5) in the case of uniqueness of these solutions. These conditions are rather strong but in general they depend on the liquid parameters and on the dimensions of the area where the equations are determined.

Analysing the conditions one meets an additional trouble with the existence of an auxiliary function $\overline{w}$ (or its difference approximation $\overline{w}_h$) that is defined implicitly and which helps to transform the primary non-homogenous problem (1.1) into the homogenous one (2.2). Let us take for example the inequality (4.5) that conditions the existence of the solution of the difference problem (4.3). We can show it in the form

$$\|\overline{w}_h\|_{U_h} < \frac{1}{2^{5/4}|A|c_0^{1/2}}(1-|F|c_0) \tag{8.1}$$

where $|A| = \rho\max(c_v/\lambda,\ 1/\mu)$, $|F| = \rho g\beta/\mu$ and $c_0 = m_1 m_2$. From (8.1) results that the existence of the postulated function $\overline{w}_h$ depends on the viscosity μ and on the conductivity λ of the liquid and also on the dimensions m_1, m_2 of Ω. That means that the inequality (8.1) is true if μ and λ are big enough and if Ω is rather small.

The general conclusion is that the solution of the boundary value problem (2.2) exists for big enough values of μ and λ and for small enough values of Ω and of the norm $\|\overline{w}\|_U$ where the function $\overline{w}$ characterizes the inhomogenous boundary conditions ψ. This conclusion holds also for the simplified problem (7.1).

If the problem consists of only the Navier-Stokes equation with the homogenous boundary conditions as well as with the inhomogenous ones the problem solution exists unconditionally. Only if the uniqueness of the solution is required the value of μ should be large enough and - in the case of the inhomogenous problem - the value of an appropriate norm of the boundary function ψ should be small enough [15].

The theoretical results are confirmed with the numerical computation. The mathematical model of the glass mass flow is characterized by the relative large values of the parameters μ and λ. The dimensions of the considered glass tank furnace, i.e. of the area Ω, are also large. There were some troubles at the beginning of the calculation with the convergence of the iterative algorithm (7.10),(7.11) that is a particular form of the algorithm (6.1). These troubles were eliminated after introducing the new difference quotients (7.7).

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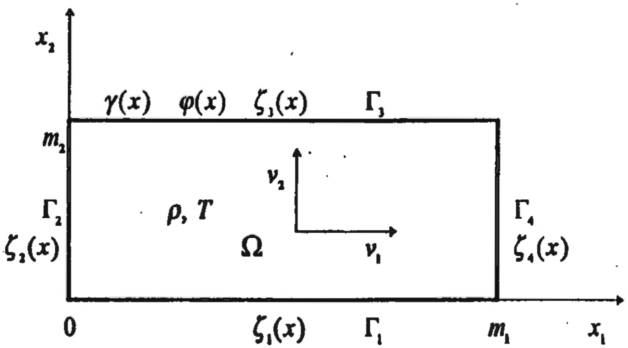

x_2
$\gamma(x)$ $\varphi(x)$ $\zeta_3(x)$ Γ_3
m_2
v_2
Γ_2
ρ, T
Γ_4
$\zeta_2(x)$
Ω
v_1
$\zeta_4(x)$
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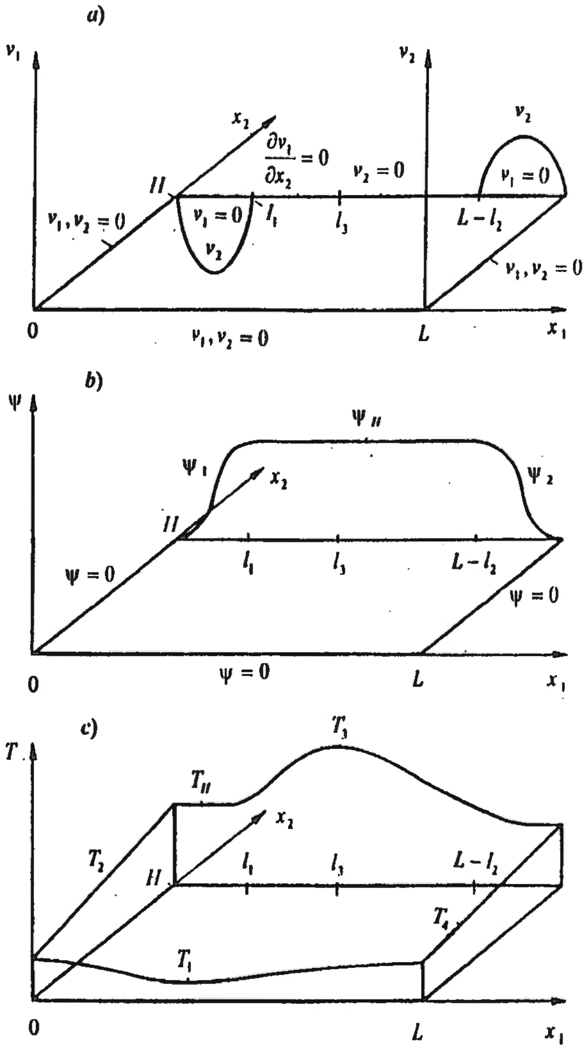
$\zeta_1(x)$ Γ_1
m_1
x_1

Figure 1.1.


a)
v_1
v_2
x_2
$\frac{\partial v_1}{\partial x_2} = 0$
$v_2 = 0$
$v_1 = 0$
H
$v_1, v_2 = 0$
$v_1 = 0$
v_2
l_1
l_3
$L - l_2$
$v_1, v_2 = 0$
0
$v_1, v_2 = 0$
L
x_1
b)
ψ
ψ_H
ψ_1
x_2
ψ_2
H
$\psi = 0$
l_1
l_3
$L - l_2$
$\psi = 0$
0
$\psi = 0$
L
x_1
c)
T
T_3
T_H
x_2
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H
l_1
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$L - l_2$
T_4
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0
L
x_1

Figure 7.1.

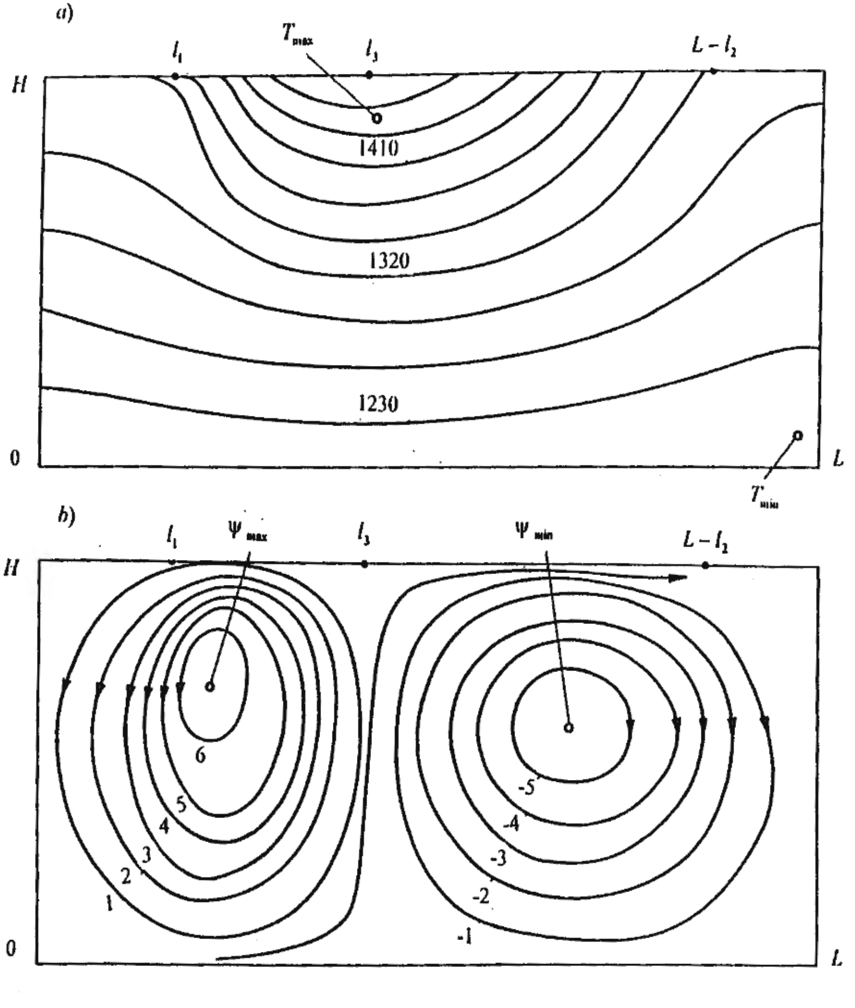
a)
T_{max}
l_1
l_3
$L - l_2$
H
1410
1320
1230
0
L
T_{min}
b)
l_1
Ψ_{max}
l_3
Ψ_{min}
$L - l_2$
H
6
5
4
3
2
1
-5
-4
-3
-2
-1
0
L

Figure 7.2.

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