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Abstract

The paper deals with the Two-Constraint Binary Knapsack Problem, the special case of Multi-Constraint Knapsack Problem. It is assumed that some of the problem coefficients are realizations of mutually independent random variables. Asymptotical probabilistic properties of selected problem characteristics are investigated for the special cases of the Lagrange multipliers.

1 Introduction

Let us consider a Two-Constraint Binary Knapsack Problem in the following formulation:

$$\begin{aligned} & z_{OPT}(n) = \max \sum_{i=1}^{n} c_i \cdot x_i \\ & \text{subject to} \quad \sum_{i=1}^{n} a_{1i} \cdot x_i \leqslant b_1(n) \\ & \qquad\qquad\quad \sum_{i=1}^{n} a_{2i} \cdot x_i \leqslant b_2(n) \\ & \text{where } \ x_i = 0 \ \text{ or } \ 1, \ i = 1, \dots, n \end{aligned} \tag{1}$$

It is assumed that:

$$c_i > 0, \ a_{ji} > 0, \ 0 < b_j(n) \leqslant \sum_{i=1}^{n} a_{ji}, \ i = 1, \dots, n, \ j = 1, 2.$$

Without restricting the generality of considerations it may be also assumed that:

$$b_1(n) \leqslant b_2(n).$$

The goal of the assumptions that c_i, $a_{ji} > 0$, $0 < b_j(n) \leqslant \sum_{i=1}^{n} a_{ji}$, $i = 1, \ldots, n$, $j = 1, 2$,is to avoid the trivial and degenerated problems. More precisely interpretation of the $a_{ji} = 0$ or $c_i = 0$ is far not obvious. When $b_j(n) > \sum_{i=1}^{n} a_{ji}$ then the corresponding constraint is always fulfilled and therefore it may be removed from the problem formulation, otherwise if $b_j(n) = 0$ then (1) has only the trivial solution i.e. $x_i = 0$, $i = 1, \ldots, n$ and $z_{OPT}(n) = 0$.

Two-Constraint Binary Knapsack Problem is special case of the binary multiconstraint knapsack problem, also known as m-constraint knapsack problem, see Nemhauser and Wolsey [9] and Martello and Toth [7], where in general case there is arbitrary number m of constraints, i.e. $b_j(n)$, $j = 1, \ldots, m$. Another important special case of the multiconstraint knapsack problem is classical (single constraint) or, in other words, Binary Knapsack Problem, which have only one constraint, i.e. $j = 1$ (see Martello and Toth [7]). In the Szkatuła's papers see [12] and [13] probabilistic analysis results of the different cases of the binary multiconstraint knapsack problem were presented. Moreover full case of the classical (single constraint) Binary Knapsack Problem was considered in the paper [13].

The Multi-Constraint Knapsack Problem is well known to be $\mathcal{NP}$ hard and moreover, when $m \geqslant 2$, it is $\mathcal{NP}$ hard in the strong sense (see Garey and Johnson [3]). It does mean that Two-Constraint Binary Knapsack Problem (1) is also $\mathcal{NP}$ hard in the strong sense. Classical (one-constraint) Binary Knapsack Problem is $\mathcal{NP}$ hard combinatorial optimization problem, however not in the strong sense.

The papers by Frieze and Clarke [2], Mamer and Schilling [6], Schilling [10] and [11] investigate the asymptotic value of $z_{OPT}(n)$ for the random model of Multi-Constraint Knapsack Problem, where $b_j(n) = 1$, $j = 1, \ldots, m$. Papers by Szkatuła [12] and [13] were dealing with the random model of the Multi-Constraint Knapsack Problem, where $b_j(n)$ are not restricted to be equal to 1. Papers by Meanti, Rinnooy Kan, Stougie and Vercellis [8], Lee and Oh [4] consider more general random models of Multi-Constraint Knapsack Problem but only for $j = 1, 2$ some partial analytical results describing the growth of $z_{OPT}(n)$ were obtained.

The aim of the present paper is to analyze the growth of the asymptotic value of $z_{OPT}(n)$ for the class of random Two-Constraint Binary Knapsack Problems (1) with possibly full spectrum of the constraints right-hand-sides values. Results of the probabilistic analysis of this important problem may allow to describe asymptotic behavior of the $z_{OPT}(n)$ for practically all combinations of values of $b_1(n)$ and $b_2(n)$ as well as other problem coefficients (considered as realizations of the random variables). Those results may help to better understand number of the theoretical issues related to Two-Constraint Binary Knapsack Problems as well as enable construction of more efficient algorithms for solving the practical instances of the (1).

2 Definitions

The following definitions are necessary for the further presentation:

Definition 1 *We denote $V_n \approx Y_n$, where $n \to \infty$, if*

$$Y_n \cdot (1 - o(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o(1))$$

when V_n, Y_n are sequences of numbers, or

$$\lim_{n \to \infty} P\{Y_n \cdot (1 - o(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o(1))\} = 1$$

when V_n is a sequence of random variables and Y_n is a sequence of numbers or random variables, where $\lim_{n \to \infty} o(1) = 0$ as it is usually presumed.

Definition 2 *We denote $V_n \preceq Y_n (V_n \succeq W_n)$ if*

$$V_n \leqslant (1 + o(1)) \cdot Y_n \ (V_n \geqslant (1 - o(1)) \cdot W_n)$$

when V_n, Y_n (W_n) are sequences of numbers, or

$$\lim_{n \to \infty} P\{V_n \leqslant (1 + o(1)) \cdot Y_n\} = 1 \ (\lim_{n \to \infty} P\{V_n \geqslant (1 - o(1)) \cdot W_n\} = 1)$$

when V_n is a sequence of random variables and Y_n (W_n) is a sequence of numbers or random variables, where $\lim_{n \to \infty} o(1) = 0$.

Definition 3 *We denote $V_n \approxeq Y_n$ if there exist constants $c'' \geqslant c' > 0$ such that*

$$c' \cdot Y_n \preceq V_n \preceq c'' \cdot Y_n$$

where Y_n, V_n are sequences of numbers or random variables.

The following random model of (1) will be considered in the paper:

- $n \to \infty$, $i = 1, \ldots, n$, $j = 1, 2$.
- c_i, a_{ji} are realizations of mutually independent random variables and moreover c_i, a_{ji} are uniformly distributed over $(0, 1]$.
- $0 < \delta \leqslant b_1(n) \leqslant b_2(n) \leqslant n/2$, $b_j(n) \leqslant b_j(n+1)$, for every $n \geqslant 1$ and all $b_j(n)$, $j = 1, 2$, are deterministic, where δ is a constant.

Under the assumptions made about c_i, a_{ji} and $b_j(n)$ the following always hold

$$0 \leqslant z_{OPT}(n) \leqslant \sum_{i=1}^{n} c_i \leqslant n, \ \delta \leqslant b_j(n) \leqslant \sum_{i=1}^{n} a_{ji} \leqslant n, \ j = 1, 2. \tag{2}$$

Moreover, from the strong law of large numbers it follows that

$$\sum_{i=1}^{n} c_i \approx E(c_1) \cdot n = n/2, \ \sum_{i=1}^{n} a_{ji} \approx E(a_{11}) \cdot n = n/2.$$

Therefore, it is justified to enhance formula (2) in the following way:

$$0 \leqslant z_{OPT}(n) \preceq n/2, \ 0 < \delta \leqslant b_1(n) \leqslant b_2(n) \preceq n/2 \tag{3}$$

Formula (3) shows that random model of the Two-Constraint Binary Knapsack Problem (1) is complete in the sense that nearly all possible instances of the problem are considered. In this respect the model where $b_1(n) = b_2(n) = 1$ is just a very special case. Taking into account that $\sum_{i=1}^{n} a_{ji} \approx n/2$ assumption that $b_j(n) \leqslant b_j(n+1)$, $j = 1,2$, for all $n \geqslant 1$, is quite logical.

The growth of $z_{OPT}(n)$ - value of the optimal solution of the problem (1) may be influenced by the problem coefficients, namely:

$$n,\ c_i,\ a_{ji},\ b_1(n),\ b_2(n),\ \text{where } i = 1,\ldots,n.$$

It is assumed that c_i, a_{ji} are realizations of the random variables and therefore their impact on the $z_{OPT}(n)$ growth is in this case indirect. Moreover, we have assumed that $n \to \infty$. The aim of the probabilistic analysis is to investigate asymptotic behavior of $z_{OPT}(n)$ when $n \to \infty$. The impact of the right-hand-side values - $b_1(n), b_2(n)$ - is well illustrated by the Lagrange function and the problem dual to (1), see Averbakh [1], Meanti, Rinnooy Kan, Stougie and Vercellis [8], Szkatuła [12] and [13]. Due to the very complicated formulas, impossible to handle efficiently in the general case, the papers by Szkatuła [12] and [13] investigate only two important special cases of values of constraints right hand sides in the case of Multi-Constraint Knapsack Problem.

3 Lagrange and dual estimations

When the general knapsack type problem, with one or many constraints, is considered then Lagrange function and the corresponding dual problems, see Averbakh [1], Meanti, Rinnooy Kan, Stougie and Vercellis [8], Szkatuła [12] and [13] are very useful tools to perform various kind of analyses of the original problem. In the specific case of the Two-Constraint Binary Knapsack Problem Lagrange function of the problem (1) may be formulated as follows:

$$\begin{aligned} L_n(x) &= \sum_{i=1}^{n} c_i \cdot x_i + \sum_{j=1}^{2} \lambda_j \cdot \left(b_j(n) - \sum_{i=1}^{n} a_{ji} \cdot x_i \right) = \\ &= \lambda_1 \cdot b_1(n) + \lambda_2 \cdot b_2(n) + \sum_{i=1}^{n} (c_i - \lambda_1 \cdot a_{1i} - \lambda_2 \cdot a_{2i}) \cdot x_i \end{aligned}$$

where $x = [x_1, \ldots, x_n]$ and $\Lambda = [\lambda_1, \lambda_2]$ - vector of Lagrange multipliers. Moreover, let for every Λ, $\lambda_j \geq 0$, $j = 1,2$:

$$\phi_n(\Lambda) = \max_{x \in \{0,1\}^n} L_n(x, \Lambda) = \max_{x \in \{0,1\}^n} \left\{ \sum_{j=1}^{2} \lambda_j \cdot b_j(n) + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{2} \lambda_j \cdot a_{ji} \right) x_i \right\}.$$

Using the following notation:

$$x_i(\Lambda) = \begin{cases} 1 & \text{if } c_i - \sum_{j=1}^{2} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

$$c_i(\Lambda) = \begin{cases} c_i & \text{if } c_i - \sum_{j=1}^{2} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$a_{ji}(\Lambda) = \begin{cases} a_{ji} & \text{if } c_i - \sum_{j=1}^{2} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

we have for every Λ, $\lambda_j \geq 0$, $j = 1, 2$:

$$\begin{aligned} \phi_n(\Lambda) &= \sum_{j=1}^{2} \lambda_j \cdot b_j(n) + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{2} \lambda_j \cdot a_{ji} \right) \cdot x_i(\Lambda) = \\ &= \sum_{j=1}^{2} \lambda_j \cdot b_j(n) + \sum_{i=1}^{n} \left(c_i(\Lambda) - \sum_{j=1}^{2} \lambda_j \cdot a_{ji}(\Lambda) \right) \end{aligned}$$

Obviously for $i = 1, \ldots, n,\ j = 1, 2,$

$$c_i(\Lambda) = c_i \cdot x_i(\Lambda), \quad a_{ji}(\Lambda) = a_{ji} \cdot x_i(\Lambda).$$

Dual problem to Two-Constraint Binary Knapsack Problem (1) maybe formulated as follows:

$$\Phi_n^* = \min_{\Lambda \geq 0} \phi_n(\Lambda). \tag{5}$$

For every $\Lambda \geq 0$ the following holds:

$$z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + \sum_{j=1}^{2} \lambda_j (b_j(n) - s_j(\Lambda)). \tag{6}$$

Let us denote:

$$\begin{aligned} z_n(\Lambda) &= \sum_{i=1}^{n} c_i \cdot x_i(\Lambda) = \sum_{i=1}^{n} c_i(\Lambda), s_j(\Lambda) = \sum_{i=1}^{n} a_{ji} \cdot x_i(\Lambda) = \sum_{i=1}^{n} a_{ji}(\Lambda), \\ S_n(\Lambda) &= \sum_{j=1}^{2} \lambda_j \cdot s_j(\Lambda),\ B(\Lambda) = \sum_{j=1}^{2} \lambda_j \cdot b_j(n). \end{aligned}$$

By definition of $c_i(\Lambda)$ and $a_{ji}(\Lambda)$, see (4), we have:

$$c_i(\Lambda) \geq \sum_{j=1}^{2} \lambda_j \cdot a_{ji}(\Lambda),\ i = 1, \ldots, n,$$

and therefore

$$z_n(\Lambda) \geq S_n(\Lambda). \tag{7}$$

For certain Λ, $x_i(\Lambda)$ given by (4) may provide feasible solution of (1), i.e.:

$$s_1(\Lambda) \leq b_1(n) \quad \text{and} \quad s_2(\Lambda) \leq b_2(n). \tag{8}$$

If the above holds then:

$$z_n(\Lambda) \leq z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + B(\Lambda) - S_n(\Lambda). \tag{9}$$

So, if (8) holds, then the below inequality also holds:

$$B(\Lambda) - S_n(\Lambda) \geq 0.$$

From (7) we get:

$$\frac{\phi_n(\Lambda)}{z_n(\Lambda)} = \frac{z_n(\Lambda)}{z_n(\Lambda)} + \frac{B(\Lambda) - S_n(\Lambda)}{z_n(\Lambda)} \leq 1 + \frac{B(\Lambda) - S_n(\Lambda)}{S_n(\Lambda)}.$$

Therefore if (8) holds, then the following inequality also holds:

$$1 \leq \frac{z_{OPT}(n)}{z_n(\Lambda)} \leq \frac{\Phi_n^*}{z_n(\Lambda)} \leq \frac{\phi_n(\Lambda)}{z_n(\Lambda)} \leq \frac{B(\Lambda)}{S_n(\Lambda)}. \tag{10}$$

Formulas (8) and (10) may allow to provide the asymptotical approximation of the $z_{OPT}(n)$ i.e. the optimal solution value of the (1) problem. Namely if there exits such a set of Lagrange multipliers $\Lambda(n)$ asymptotically fulfilling the formulas (8) and (10) then the below conjecture holds:

$$\text{If } \lim_{n\to\infty} \frac{B(\Lambda(n))}{S_n(\Lambda(n))} = 1 \text{ and (8) holds then } \lim_{n\to\infty} \frac{z_{OPT}(n)}{z_n(\Lambda(n))} = 1 \tag{11}$$

Therefore if (11) holds then $x_i(\Lambda(n))$, $i = 1, \ldots, n$, given by (4), provides the asymtotically sub-optimal solution of the Two-Constraint Binary Knapsack Problem (1). Moreover the value of $z_n(\Lambda(n))$ is an asymptotical approximation of the optimal solution value of the Two-Constraint Binary Knapsack Problem i.e. $z_{OPT}(n)$.

4 Probabilistic analysis

In the present section of the paper some probabilistic properties of the Two-Constraint Binary Knapsack Problem (1) will be investigated. It is assumed that c_i, a_{ji} $i = 1, \ldots, n$, $j = 1, 2$ are realizations of mutually independent random variables and moreover c_i, a_{ji} are uniformly distributed over $(0, 1]$. Moreover it is assumed that $0 < \delta \leqslant b_1(n) \leqslant b_2(n) \leqslant n/2$, $b_j(n) \leqslant b_j(n+1)$. In addition it is assumed that Lagrange multipliers λ_1 and λ_2, $\lambda_2 \leq \lambda_1$, $\Lambda = (\lambda_1, \lambda_2)$ are also deterministic. Monotonicity of constraints right hand sides, $b_1(n) \leqslant b_2(n)$, is in this case determining monotonicity of the Lagrange multipliers, i.e. $\lambda_2 \leq \lambda_1$. This is often used in the literature probabilistic model of the general knapsack problems and it suits very well also to the Two-Constraint Binary Knapsack Problem (1).

Let us first observe that due to the assumptions made the following holds, for $i = 1, \ldots, n$, $j = 1, 2$:

$$P(a_{ji} < x) = \begin{cases} 0 & \text{when } x \leqslant 0 \\ x & \text{when } 0 < x \leqslant 1 \\ 1 & \text{when } x \geqslant 1 \end{cases}, \quad P(c_i < x) = \begin{cases} 0 & \text{when } x \leqslant 0 \\ x & \text{when } 0 < x \leqslant 1 \\ 1 & \text{when } x \geqslant 1 \end{cases}. \tag{12}$$

In order to proceed with probabilistic analysis of the Two-Constraint Binary Knapsack Problem (1) it is necessary to consider probabilistic distribution of the following random variables

$$\sum_{j=1}^{k} \lambda_j \cdot a_{ji}, \ k = 1 \text{ or } 2$$

Let $(x)_+ = \frac{|x| + x}{2} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$, $j^* = \begin{cases} 2 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases}$, Then for or $i = 1, \ldots, n$, $j = 1, 2$, the following holds:

$$\begin{aligned} F_1(x, \lambda_j) &= P\{\lambda_j \cdot a_{ji} < x\} = \frac{1}{\lambda_j}((x)_+ - (x - \lambda_j)_+), \ j = 1, 2, \\ F_2(x, \Lambda) &= P\{\lambda_1 \cdot a_{1i} + \lambda_2 \cdot a_{2i} < x\} = \frac{1}{\lambda_j} \int_0^1 F_1(x - \lambda_{j^*} t, \lambda_j) dt = \\ &= \frac{1}{\lambda_1 \cdot \lambda_2} \left((x)_+^2 - (x - \lambda_1)_+^2 - (x - \lambda_2)_+^2 + (x - \lambda_1 - \lambda_2)_+^2\right) \end{aligned} \tag{13}$$

The distribution functions of the random variables $a_{ji}(\Lambda)$, $c_i(\Lambda)$, $i = 1, \ldots, n$, $j = 1, 2$ are:

$$\begin{aligned} G_{ji}(x, \Lambda) &= P\{a_{ji}(\Lambda) < x\} = \\ &= P\left\{a_{ji} < x \bigcup a_{ji} \geq x \bigcap \sum_{k=1}^{2} \lambda_k \cdot a_{ik} \geq c_i\right\} = \\ &= 1 - \int_x^1 \int_0^1 F_1(r - \lambda_j \cdot t, \lambda_{j^*}) dr dt \end{aligned} \tag{14}$$

$$\begin{aligned} H_i(x, \Lambda) &= P\{c_i(\Lambda) < x\} = \\ &= P\left\{c_i < x \bigcup c_i \geq x \bigcap \sum_{k=1}^{2} \lambda_k \cdot a_{ik} \geq c_i\right\} = \\ &= 1 - \int_x^1 F_2(t, \Lambda) dt, \end{aligned} \tag{15}$$

Using above formulas (14) and (15) expectations of the $a_{ji}(\Lambda)$, $c_i(\Lambda)$ could be expressed as follows:

$$E(a_{ji}(\Lambda)) = \int_0^1 x dG_{ji}(x,\Lambda) = \int_0^1 x \int_0^1 F_1(r - \lambda_j \cdot x, \lambda_{j^\bullet}) dr dx = \tag{16}$$
$$= \frac{1}{\lambda_{j^\bullet}} \left(\int_0^1 x \int_0^1 ((r - x\cdot\lambda_j)_+ - (r - x\cdot\lambda_j - \lambda_{j^\bullet})_+) dr dx \right)$$

$$E(c_i(\Lambda)) = \int_0^1 x dH_i(x,\Lambda) = \int_0^1 x \cdot F_2(x,\Lambda) dx = \tag{17}$$
$$= \frac{1}{2\cdot\lambda_1\cdot\lambda_2} \int_0^1 x \cdot \left((x)_+ - (x-\lambda_1)_+^2 - (x-\lambda_2)_+^2 + (x-\lambda_1-\lambda_2)_+^2\right) dx =$$
$$= \frac{1}{2\cdot\lambda_1\cdot\lambda_2} \left(\frac{1}{4} - \int_0^1 x \cdot \left((x-\lambda_1)_+^2 + (x-\lambda_2)_+^2 - (x-\lambda_1-\lambda_2)_+^2\right) dx \right).$$

It is easy to observe that above formulas (16) and (17) may take different formulations, depending on the mutual relations between λ_1, λ_2 and x, r since several items of the above formulas may remain strongly positive or become 0, due to the function $(.)_+$properties. In general 4 specific cases could be distinguished for $i = 1, \ldots, n$, $j = 1, 2$:

1. Case of "large" values of the Lagrange multipliers $1 \leq \lambda_2 \leq \lambda_1$. In this case:

$$E(a_{ji}(\Lambda)) = \frac{1}{\lambda_{j^\bullet}} \int_0^{1/\lambda_j} x \int_{x\cdot\lambda j}^1 (r - x\cdot\lambda_j) dr dx = \frac{1}{24\cdot\lambda_j^2\cdot\lambda_{j^\bullet}} \tag{18}$$
$$E(c_i(\Lambda)) = \frac{1}{2\cdot\lambda_1\cdot\lambda_2} \int_0^1 x^3 dx = \frac{1}{8\cdot\lambda_1\cdot\lambda_2}.$$

2. Case of "mixed" values of the Lagrange multipliers $\lambda_2 \le 1 \le \lambda_1$. In this case:

$$\begin{aligned} E(a_{1i}(\Lambda)) &= \frac{1}{\lambda_2}\left(\int_0^{1/\lambda_1} x \int_{x\cdot\lambda_1}^{1} (r - x\cdot\lambda_1)drdx - \right. \\ &- \left. \int_0^{\frac{1-\lambda_2}{\lambda_1}} x \int_{(x\cdot\lambda_1+\lambda_2)}^{1} (r - x\cdot\lambda_1 - \lambda_2)drdx\right) = \\ &= \frac{1}{24}\frac{-\lambda_2^3 + 4\lambda_2^2 - 6\lambda_2 + 4}{\lambda_1^2}, \\ E(a_{2i}(\Lambda)) &= \frac{1}{\lambda_1}\left(\int_0^1 x \int_{x\cdot\lambda_2}^{1} (r - x\cdot\lambda_2)drdx\right) = \frac{1}{24}\frac{3\lambda_2^2 - 8\lambda_2 + 6}{\lambda_1}, \\ E(c_i(\Lambda)) &= \frac{1}{2\cdot\lambda_1\cdot\lambda_2}\left(\frac{1}{4} - \int_{\lambda_2}^{1} x\cdot(x-\lambda_2)^2 dx\right) = \\ &= \frac{1}{24\lambda_1}\left(\lambda_2^3 - 6\lambda_2 + 8\right). \end{aligned} \tag{19}$$

3. Case of "moderate" values of the Lagrange multipliers $\lambda_2 \le \lambda_1 \le 1$, $\lambda_2 + \lambda_1 \ge 1$. In this case:

$$\begin{aligned} E(a_{ji}(\Lambda)) &= \frac{1}{\lambda_{j^*}}\left(\int_0^1 x \int_{x\cdot\lambda_j}^{1} (r - x\cdot\lambda_j)drdx - \right. \\ &- \left. \int_0^{(1-\lambda_{j^*})/\lambda_j} x \int_{(x\cdot\lambda_j+\lambda_{j^*})}^{1} (r - x\cdot\lambda_j - \lambda_{j^*})drdx\right) = \\ &= \frac{1}{24}\frac{3\lambda_j^4 - 8\lambda_j^3 + 6\lambda_j^2 - 6\lambda_{j^*}^2 + 4\lambda_{j^*} - \lambda_{j^*}^4 + 4\lambda_{j^*}^3 - 1}{\lambda_j^2\lambda_{j^*}}, \\ E(c_i(\Lambda)) &= \frac{1}{2\cdot\lambda_1\cdot\lambda_2}\left(\frac{1}{4} - \int_{\lambda_1}^{1} x\cdot(x-\lambda_1)^2 dx - \int_{\lambda_2}^{1} x\cdot(x-\lambda_2)^2 dx\right) = \\ &= \frac{1}{24\lambda_1\lambda_2}\left(\lambda_1^4 - 6\lambda_1^2 + 8\lambda_1 + \lambda_2^4 - 6\lambda_2^2 + 8\lambda_2 - 3\right). \end{aligned} \tag{20}$$

4. Case of "small" values of the Lagrange multipliers $\lambda_2 \leq \lambda_1 \leq 1$, $\lambda_2 + \lambda_1 \leq 1$. In this case:

$$
\begin{aligned}
E(a_{ji}(\Lambda)) &= \frac{1}{\lambda_{j^*}}\left(\int_0^1 x \int_{x\cdot\lambda_j}^1 (r - x\cdot\lambda_j)drdx - \right. \\
&- \left.\int_0^1 x \int_{(x\cdot\lambda_j+\lambda_{j^*})}^1 (r - x\cdot\lambda_j - \lambda_{j^*})drdx\right) = \\
&= \frac{1}{2} - \frac{1}{3}\lambda_j - \frac{1}{4}\lambda_{j^*}, \\
E(c_i(\Lambda)) &= \frac{1}{2\cdot\lambda_1\cdot\lambda_2}\left(\frac{1}{4} - \int_{\lambda_1}^{1} x\cdot(x-\lambda_1)^2dx - \int_{\lambda_2}^{1} x\cdot(x-\lambda_2)^2dx + \right. \\
&+ \left.\int_{\lambda_1+\lambda_2}^{1} x\cdot(x-\lambda_1-\lambda_2)^2dx)\right) = \\
&= \frac{1}{2} - \frac{1}{6}\lambda_1^2 - \frac{1}{4}\lambda_1\lambda_2 - \frac{1}{6}\lambda_2^2.
\end{aligned}
\tag{21}
$$

Probabilistic, or in other words average case, analysis consists in determining such Lagrange multipliers $\lambda_1(n)$, $\lambda_2(n)$ that when $n \to \infty$, $x_i(\Lambda(n))$, $i = 1, \ldots, n$, defined by (4) will provide solutions of the Two-Constraint Binary Knapsack Problem (1) which are, in the sense of convergence in probability, see Loeve [5], providing solutions which are asymptotically feasible , i.e. $s_j(\Lambda(n))$ is satisfying (8) and moreover if $S_n(\Lambda(n))$ is fulfilling (11) then, due to (10), $\lim_{n\to\infty} \frac{z_{OPT}(n)}{z_n(\Lambda(n))} = 1$ and $z_n(\Lambda(n))$ is suboptimal solution of the (1) and moreover

$$z_{OPT}(n) \approx z_n(\Lambda(n)) \approx E(z_n(\Lambda(n))).$$

The above goal may be achieved by determining $\Lambda(n)$ as the solution of the following system of equations:

$$E(s_1(\Lambda(n))) = b_1'(n), \quad E(s_2(\Lambda(n))) = b_2'(n), \tag{22}$$

where $b_j'(n) = b_j(n) - \epsilon_j(n)$, $\epsilon_j(n) = o(b_j(n))$ and $s_j(\Lambda(n)) \approx b_j'(n) \approx b_j(n)$, $s_j(\Lambda(n)) \leqslant b_j(n)$, $j = 1, 2$ (in the sense of convergence in probability) and therefore $\Lambda(n)$ is fulfilling both (8) and (11).

It may also happen that the system of equations (22) has no solutions, e.g. when difference between $b_1(n)$ and $b_2(n)$ is too large. In this case only $\lambda_1(n) > 0$ and $\lambda_2(n) = 0$ which means that second constraint in the Two-Constraint Binary Knapsack Problem (1) formulation is excessive and could be removed. It does mean that in this situation problem (1) reduces to the classical single constraint knapsack problem. In the Szkatuła paper [13] the following formula summarizing behavior of the optimal solution value was presented

$$z_{OPT}(n) \approx z_n(\Lambda(n)) \approx E(z_n(\Lambda(n))) \approx \begin{cases} \sqrt{\frac{2\cdot n\cdot b_1'(n)}{3}} & \text{if } \delta \leq b_1'(n) \leq \frac{n}{6}, \\ \frac{1}{4}\cdot\left(\frac{n}{2} + 6\cdot b_1'(n)\cdot\left(1 - \frac{b_1'(n)}{n}\right)\right) & \text{if } \frac{n}{6} \leq b_1'(n) \leq \frac{n}{2}. \end{cases} \tag{23}$$

Each of the 4 cases where $\lambda_1(n), \lambda_2(n) > 0$ mentioned above should be considered independently. Let us observe that $E(s_j(\Lambda(n))) = n \cdot E(a_{j1}(\Lambda(n)))$, $E(z_n(\Lambda(n))) = n \cdot E(c_1(\Lambda(n)))$.

Lemma 1 *If c_i, a_{ji} $i = 1, \ldots, n,$ $j = 1, 2,$ are realizations of mutually independent random variables uniformly distributed over $(0,1)$, and if $\delta \le b_1'(n) \le b_2'(n) \le \sqrt[2]{\frac{n \cdot b_1'(n)}{24}}$, where δ is a constant, then*

$$\lambda_1(n) = \frac{1}{b_1'(n)} \sqrt[3]{\frac{n \cdot b_1'(n) \cdot b_2'(n)}{24}}, \quad \lambda_2(n) = \frac{1}{b_2'(n)} \sqrt[3]{\frac{n \cdot b_1'(n) \cdot b_2'(n)}{24}} \tag{24}$$

is the solution of (22) and

$$E(z_n(\Lambda(n))) = 3 \cdot \sqrt[3]{\frac{n \cdot b_1'(n) \cdot b_2'(n)}{24}}. \tag{25}$$

Proof. Above formulas follow immediately from the (18) and (22). From the Lemma 1 assumptions and from (24) it follows that $1 \le \lambda_2(n) \le \lambda_1(n)$. Moreover condition $\delta \le b_1'(n) \le b_2'(n) \le \sqrt[2]{\frac{n \cdot b_1'(n)}{24}}$ holds only when $b_1'(n) \le b_2'(n) \le \frac{n}{24}$ (not vice versa).

Lemma 2 *If c_i, a_{ji} $i = 1, \ldots, n,$ $j = 1, 2,$ are realizations of mutually independent random variables uniformly distributed over $(0,1)$, and if $\delta \le b_1'(n) \le \frac{n}{6}$ and $b_2'(n) > \max\left\{\sqrt[2]{\frac{n \cdot b_1'(n)}{24}}, b_1'(n)\right\}$, where δ is a constant, then*

$$\lambda_1(n) = \sqrt[2]{\frac{n}{6 \cdot b_1'(n)}}, \quad \lambda_2(n) = 0$$

is the optimal set of Lagrange multipliers and

$$E(z_n(\Lambda(n))) = \sqrt{\frac{2 \cdot n \cdot b_1(n)}{3}} \tag{26}$$

Proof. *In this case the system of equations (22) has no solution and therefore Two-Constraint Binary Knapsack Problem (1) is equivalent to the single constraint knapsack problem, refer to Szkatuła [13]; (26) follows immediately from (23).*

Lemma 3 *If c_i, a_{ji} $i = 1, \ldots, n,$ $j = 1, 2,$ are realizations of mutually independent random variables uniformly distributed over $(0,1)$, and if $b_1'(n) \geqslant \max\left\{\frac{n}{6}, \frac{4}{3} \cdot b_2'(n) - \frac{n}{6}\right\}$ then*

$$\begin{aligned} \lambda_1(n) &= \frac{1}{7} \cdot \left(6 - 48 \cdot \frac{b_1'(n)}{n} + 36 \cdot \frac{b_2'(n)}{n}\right), \\ \lambda_2(n) &= \frac{1}{7} \cdot \left(6 + 36 \cdot \frac{b_1'(n)}{n} - 48 \cdot \frac{b_2'(n)}{n}\right) \end{aligned} \tag{27}$$

is the solution of (22) and

$$\begin{aligned} E(z_n(\Lambda(n))) &= \frac{1}{7} \cdot \left(\frac{n}{2} + 6 \cdot b_1'(n) + 6 \cdot b_2'(n) + \right. \\ &+ \left. 36 \cdot \frac{b_1'(n) \cdot b_2'(n)}{n} - 24 \cdot \frac{(b_1'(n))^2}{n} - 24 \cdot \frac{(b_2'(n))^2}{n}\right) \end{aligned} \tag{28}$$

Proof. From the (21) it could be obtained that $\lambda_1(n)$ and $\lambda_2(n)$ given by formula (27) are solving the equation (22) and fulfilling the condition $\lambda_2(n) + \lambda_1(n) \leq 1$. ■

Lemma 4 *If c_i, a_{ji} $i = 1, \ldots, n,$ $j = 1, 2,$ are realizations of mutually independent random variables uniformly distributed over $(0,1)$, and if $\frac{n}{6} \leq b_1'(n) < \min\left\{b_2'(n), \frac{4}{3} \cdot b_2'(n) - \frac{n}{6}\right\}$ then*

$$\lambda_1(n) = 3 \cdot \left(\frac{1}{2} - \frac{b_1'(n)}{n}\right), \quad \lambda_2(n) = 0$$

is the optimal set of Lagrange multipliers and

$$E(z_n(\Lambda(n))) = \frac{1}{4} \cdot \left(\frac{n}{2} + 6 \cdot b_1'(n) \cdot \left(1 - \frac{b_1'(n)}{n}\right)\right)$$

Proof. In this case the system of equations (22) has no solution and therefore Two-Constraint Binary Knapsack Problem (1) is equivalent to the single constraint knapsack problem, refer to Szkatula [13]; (26) follows immediately from (23). Condition $\lambda_2(n) + \lambda_1(n) \leq 1$ holds also in this case. ■

In the cases considered in Lemma 3 and Lemma 4 the condition $\lambda_2(n) + \lambda_1(n) \leq 1$ and formula (3) are providing that following right-hand-sides of the constraints:

$$\frac{5 \cdot n}{12} \preceq b_1'(n) + b_2'(n) \preceq n,$$

are fulfilling assumptions of the Lemma 3.

5 Concluding remarks

In the present paper results describing probabilistic properties of the Two-Constraint Binary Knapsack Problem (1) in the case of smaller values of constraints right-hand-sides (when $1 \leq \lambda_2(n) \leq \lambda_1(n)$, i.e. large) as well as large values of constraints right-hand-sides (when $\lambda_2(n) + \lambda_1(n) \leq 1$, corresponding to small values of $\lambda_1(n)$ and $\lambda_2(n)$) are considered.

In the paper distribution functions of the various random variables representing important problems characteristics are presented.

The future research will be aimed at investigation of 2 remaining cases (mixed and moderate values) of the mutual relations between $\lambda_1(n)$ and $\lambda_2(n)$, feasibility of the received solutions and estimations of the Two-Constraint Binary Knapsack Problem (1) optimal solution values $z_{OPT}(n)$ growth, when $n \to \infty$.

References

[1] I. Averbakh. Probabilistic properties of the dual structure of the multidimensional knapsack problem and fast statistically effcient algorithms. *Mathematical Programming*, 65:311–330, 1994.

[2] A. Frieze and M. Clarke. Approximation algorithms for the m-dimensional 0-1 knapsack problem: Worst case and probabilistic analysis. *European Journal of Operational Research*, 15:100–109, 1984.

[3] M. Garey and D. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Freeman, San Francisco, 1979.

[4] T.-E. Lee and G.-T. Oh. The asymptotic value-to-capacity ratio for the multi-class stochastic knapsack problem. *European Journal of Operational Research*, 103:584–594, 1997.

[5] M. Loève. *Probability Theory I.* Springer Verlag, New York, Heidelberg, Berlin, 1977.

[6] J. Mamer and K. Schilling. On the growth of random knapsacks. *Discrete Applied Mathematics*, 28:223–230, 1990.

[7] S. Martello and P. Toth. *Knapsack Problems: Algorithms and Computer Implementations.* Wiley & Sons, 1990.

[8] M. Meanti, A. R. Kan, L. Stougie, and C. Vercellis. A probabilistic analysis of the multiknapsack value function. *Mathematical Programming*, 46:237–247, 1990.

[9] G. Nemhauser and L. Wolsey. *Integer and Combinatorial Optimization.* John Wiley & Sons Inc., New York, 1988.

[10] K. Schilling. The growth of m-constraint random knapsacks. *European Journal of Operational Research*, 46:109–112, 1990.

[11] K. Schilling. Random knapsacks with many constraints. *Discrete Applied Mathematics*, 48:163–174, 1994.

[12] K. Szkatuła. On the growth of multi-constraint random knapsacks with various right-hand sides of the constraints. *European Journal of Operational Reserch*, 73:199–204, 1994.

[13] K. Szkatuła. The growth of multi-constraint random knapsacks with large right-hand sides of the constraints. *Operations Research Letters*, 21:25–30, 1997.