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A hazard-related probabilistic model of cascading and feedback effects occurring in a multi-process environment

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1. Notation and definitions

1.a General characteristic of the processes and events

$p_1,\ldots,p_n$ – the individual processes as the constitutive elements of the considered environment; each process is (can be) regarded as a phased-mission system; n – the number of these processes

$E_1^{(i)},\ldots,E_{m(i)}^{(i)}$ – different hazardous events that can occur in the process p_i; m(i) – the number of such events

$\lambda_1^{(i)},\ldots,\lambda_{m(i)}^{(i)}$ – the intensities with which $E_1^{(i)},\ldots,E_{m(i)}^{(i)}$ occur as primary events, i.e. not caused by another event in any process (given data)

$X_a^{(i)}$ – the strength of $E_a^{(i)}$; a random variable with values in a finite set S={1,…,s}

$\pi_a^{(i)}(x)$ – the prob. that the strength of an primary $E_a^{(i)}$ is equal to x (given data)

$\pi_a^{(i)}(>x)$ – the prob. that the strength of an primary $E_a^{(i)}$ exceeds x; note that

$\pi_a^{(i)}(>x) = \sum_{y>x}\pi_a^{(i)}(x)$

$N_a^{(i)}(s,t)$ – the number of occurrences of an primary $E_a^{(i)}$ of any strength in the (s, t] time interval; a random variable

$N_a^{(i)}(s,t,x)$ – the number of occurrences of an primary $E_a^{(i)}$ of strength x in the (s, t] time interval; a random variable

1.b The cause-effect probabilities

$\pi_{b,a}^{(j,i)}(y, x)$ – the probability that $E_b^{(j)}$ of strength y directly causes $E_a^{(i)}$ of strength x (given data)

$\pi_{b,a}^{(j,i)}(y, >x)$ – Pr ($E_b^{(j)}$ of strength y directly causes $E_a^{(i)}$ of strength greater than x)

$\pi_{b,a}^{(j,i)}(\cdot, x)$ – Pr ($E_b^{(j)}$ of any strength directly causes $E_a^{(i)}$ of strength x)

$\pi_{b,a}^{(j,i)}(\cdot, >x)$ – Pr ($E_b^{(j)}$ of any strength directly causes $E_a^{(i)}$ of strength greater than x)

$\pi_{b,a}^{(j,i)}(y, x; h)$ – the probability that $E_b^{(j)}$ of strength y causes $E_a^{(i)}$ of strength x as a result of h-step (but not less-than-h-step) cascading effect, $h \geq 2$

$\pi_{b,a}^{(j,i)}(y, >x; h)$, $\pi_{b,a}^{(j,i)}(\cdot, x; h)$, $\pi_{b,a}^{(j,i)}(\cdot, >x; h)$ – the probabilities defined analogously to $\pi_{b,a}^{(j,i)}(y, >x)$, $\pi_{b,a}^{(j,i)}(\cdot, >x)$, $\pi_{b,a}^{(j,i)}(\cdot, >x)$

The probabilities $\pi_{b,a}^{(j,i)}$ with various arguments will be called the cause-effect probabilities, as they quantify the cause-effect relations between the events occurring within the analyzed processes.

1.c Various types of risks

$r_a^{(i)}(s, t, >x)$ – the risk that at least one event $E_a^{(i)}$ of strength >x occurs as a primary event in the (s, t] interval, $1 \leq a \leq m(i)$;

$r_{b,a}^{(j,i)}(s, t, >x)$ – the risk that $E_b^{(j)}$ (an event in p_j) directly causes at least one occurrence of $E_a^{(i)}$ (an event in p_i) of strength > x in the (s, t] interval, where $b \neq a$ for j=i;

$R_a^{(i)}(s, t, >x, 1)$ – the total risk that at least one $E_a^{(i)}$ of strength >x occurs (in p_i) in the (s, t] interval, as a direct effect of any event $E_b^{(j)}$ in any process p_j, $(b,j) \neq (a,i)$. The capital letter R indicates that all the processes rather than one contribute to the risk.

$R_a^{(i)}(s, t, >x, h)$ – the total risk that at least one $E_a^{(i)}$ of strength >x occurs (in p_i) in the (s, t] interval, as a h-step (but not less-than-h-step) cascading effect of any event $E_b^{(j)}$ in any process p_j, $h \geq 2$.

$R_a^{(i)}(s, t, >x, \geq 1)$ – the total risk that at least one $E_a^{(i)}$ of strength >x occurs (in p_i) in the (s, t] interval, as a cascading effect of any step and any event $E_b^{(j)}$ in any process p_j.

2. Introduction

The functioning of practically every technical system can be regarded as a set of interacting, running in parallel processes which represent the individual operations carried out within the considered system. The aim of this work is to construct a model of hazard-related interdependence of these processes. This model should describe the impact of hazardous or harmful events occurring in one process (intrinsic to that process) on the risks of such events adversely affecting the other processes.

The analytical part of the task, apart from defining the interactions between the considered processes, will include the analysis of feedback and cascading effects that can result from the mutual dependencies between the events occurring in different processes. The main analytical result consists in deriving the formulas which on the one hand express the risks of events of different types resulting from other events in the same or other processes, and, on the other hand, quantify the consequences that a given event can entail, in the sense of adverse impact it can have on its own and other processes. Such formulas allow to assess the possibility of the occurrence of a harmful event as a direct or indirect consequence of other events, as well as to assess the harmful impact that a given event has on individual processes. They can be applied to the development and implementation of safeguards protecting against, or mitigating the effects of, hazard-related mutual impacts among the considered processes.

The random variables expressing the strength of events occurring in the processes and the risks of these events can have “crisp” numerical values, or “non-crisp” descriptive values, i.e. fuzzy or linguistic ones, e.g. the strength of an event can be extreme, high, significant, considerable, medium, low, etc. The used quantification approach depends on the degree of accuracy of the intended risk analysis, and the amount and character of the available data. The applicable mathematical tools are the probability/possibility theory, evidence (Dempster-Schafer) theory, and simple arithmetic.

The considered harmful events are divided in four categories: primary (occurring by themselves), directly caused (by another event), indirectly caused by a cascading effect, and indirectly caused by a feedback effect. A cascading effect takes place when the events occur, on a cause-effect basis, in a series whose length exceeds 2; the first event is a primary one, and each other event in the series is directly caused by the preceding one. We will say that an event is a result of a h-step cascading effect if the event's number in the cause-effect series is h+1. A directly caused event can be regarded as a result of a 1-step cascading effect. A feedback effect is a special case of a cascading effect, where the last event in a series is an instance of the first event, i.e. an instance of the primary event $E_a^{(i)}$ causes, by means of a cascading effect of step at least 2, another instance of $E_a^{(i)}$, possibly of different strength. A feedback effect cannot be a 1-step cascading effect, because the natural assumption is adopted that an event cannot be directly caused by itself, i.e.

$$\pi_{a,a}^{(i,i)}(y,x) = 0,\ i\in\{1,\dots,n\},\ a\in\{1,\dots,m(i)\} \qquad (*)$$

However, we admit the possibility of the internal impact, meaning that an event can be directly caused by another event in the same process, i.e.

$$\pi_{b,a}^{(i,i)}(y,x) > 0,\ b\neq a \qquad (**)$$

It is also assumed that the primary events are independent, both within one process, and among all the considered processes, i.e. the instances of $E_1^{(i)},\ldots,E_{m(i)}^{(i)}$, i=1,...,n, as primary events, are mutually independent.

For the sake of computational tractability it is desirable that the sequences of cascading events caused by different primary event be mutually independent. If we assume that the events in a cascade follow each other in a quick succession, i.e. the time of the last event in a cascade triggered by a given primary event always precedes that of the next primary event (or, if there are delays between successive events in a cascade, that no two events can coincide), then this requirement is fulfilled by virtue of the following lemma.

Lemma 1

The sequences of cascading events caused by different triggering events are independent (Clearly, the events in one cascade are not independent).

Proof: the lemma follows directly from the assumption of the mutual independence of instances of primary events, and the impossibility of causing one non-primary event by two primary ones, i.e. the impossibility of the occurrence of a common event in two cause-effect chains (clearly, chains with a common event would be mutually dependent). This impossibility is a consequence of the assumed instantaneousness of a cascading effect.

Practical implementation of the developed model should consist of three phases:

1. Identifying hazards involved in the individual processes, and evaluating/estimating/assessing the internal risks $r_a^{(i)}(s, t, >x)$, $1\leq i \leq n$, $1\leq a \leq m(i)$.

2. Identifying hazard-related interactions between the processes, and evaluating/estimating/assessing the external risks $r_{a,b}^{(i,j)}(s, t, >x)$, $1 \leq a \leq m(i)$, $1 \leq b \leq m(j)$, $j \neq i$; $i,j \in \{1,\ldots,n\}$.

3. Calculating all the risks defined in Section 1.

4. Developing procedures aimed at mitigation, minimization or elimination of possible harmful consequences of the identified hazards, using the risks calculated in step 3.

3. Calculation of the cause-effect probabilities

First, the probabilities $\pi_{b,a}^{(j,i)}(\cdot, x)$ and $\pi_{b,a}^{(j,i)}(\cdot, >x)$, where $b \neq a$ if $j=i$, will be calculated. It holds that:

$$\pi_{b,a}^{(j,i)}(\cdot,\ x) = \Pr\left(E_b^{(j)} of\ any\ strength\ directly\ causes\ E_a^{(i)}\ of\ strength\ x\right) =$$

$$= \Pr\left(\bigcup_{y \in S} E_b^{(j)} has\ strength\ y\ and\ causes\ E_a^{(i)}\ of\ strength\ x\right) =$$

$$= \sum_{y \in S} \Pr\left(X_a^{(i)} = x | X_b^{(j)} = y\right) \Pr\left(X_b^{(j)} = y\right) =$$

$$= \sum_{y \in S} \pi_{b,a}^{(j,i)}(y, x) \pi_b^{(j)}(y) \qquad (1)$$

and

$$\pi_{b,a}^{(j,i)}(\cdot,\ > x) = \Pr\left(E_b^{(j)} of\ any\ strength\ directly\ causes\ E_a^{(i)}\ of\ strength\ > x\right) =$$

$$= \Pr\left(\bigcup_{y \in S} E_b^{(j)} has\ strength\ y\ and\ causes\ E_a^{(i)}\ of\ strength\ > x\right) =$$

$$= \sum_{y \in S} \Pr\left(X_a^{(i)} > x | X_b^{(j)} = y\right) \Pr\left(X_b^{(j)} = y\right) =$$

$$= \sum_{z>x} \sum_{y \in S} \Pr\left(X_a^{(i)} = z | X_b^{(j)} = y\right) \Pr\left(X_b^{(j)} = y\right) =$$

$$= \sum_{z>x} \sum_{y \in S} \pi_{b,a}^{(j,i)}(y,z) \pi_b^{(j)}(y) \tag{2}$$

Now we pass to the calculation of the probabilities related to the cascading effect of degree h≥2. To make the analysis more detailed, different formulas will be obtained depending on whether the internal impact and/or the feedback effect are taken into consideration or not.

Lemma 2.

If both the internal impact and feedback effect are taken into consideration, then $\pi_{b,a}^{(j,i)}(y, >x, h)$, where h≥2, is given by the following recursive formula:

$$\pi_{b,a}^{(j,i)}(y, > x, h) =$$

$$= \coprod_{\substack{k=1,\dots,n \\ c=1,\dots,m(k)}} \sum_{\substack{z \in S \\ z \leq x \ for \ (k,c)=(i,a)}} \pi_{b,c}^{(j,k)}(y,z) \pi_{c,a}^{(k,i)}(z, > x, h-1), \ h \geq 2 \tag{3}$$

where

$$\pi_{c,a}^{(k,i)}(z, > x, 1) = \pi_{c,a}^{(k,i)}(z, > x) = \sum_{u>x} \pi_{c,a}^{(k,i)}(z,u) \tag{4}$$

Using (3) we obtain the following "aggregate" probabilities:

$$\pi_{b,a}^{(j,i)}(\cdot, > x, h) = \sum_{y \in S} \pi_b^{(j)}(y) \pi_{b,a}^{(j,i)}(y, > x, h), \ h \geq 2 \tag{5}$$

and

$$\pi_{b,a}^{(j,i)}(\cdot, \cdot, h) = \pi_{b,a}^{(j,i)}(\cdot, > 0, h), \ h \geq 2 \tag{6}$$

Remark 1: Under the adopted assumptions, (3) also holds for (j,b)=(i,a), and (k,c)=(i,a) is in the range of the "inverted pi" operator (feedback effect). However, it should be remembered that $\pi_{b,c}^{(j,k)}(y,z) = 0$ for (k,c)=(j,b) and $\pi_{c,a}^{(k,i)}(z, >x, 1) = 0$ for (k,c)=(i,a) (see (*)).

Remark 2: If (k,c)=(i,a), then z>x are not in the range of the summation operator, because taking such z into consideration would amount to admitting the possibility that $E_b^{(j)}$ directly causes $E_a^{(i)}$ of strength >x. This would contradict the requirement that $E_a^{(i)}$ of strength >x cannot be a less-than-h-step cascading effect of $E_b^{(j)}$.

Proof of (3):

$$\pi_{b,a}^{(j,i)}(y,\ >x,\ h) =$$

$$= \Pr\left(\begin{matrix} E_a^{(i)}\ of\ strength > x\ occurs\ in\ h\ steps\ as\ a\ cascading\ effect\ of\ E_b^{(j)}, \\ but\ not\ in\ 1,\dots,or\ (h-1)\ steps \end{matrix}\right) =$$

$$= \Pr\left(\bigcup_{\substack{k=1,\dots,n \\ c=1,\dots,m(k)}} \bigcup_{\substack{z\in S \\ z\le x\ for\ (k,c)=(i,a)}} \left\{\begin{matrix} E_c^{(k)}\ of\ strength\ z \\ is\ directly\ caused \\ by\ E_b^{(j)} of\ strength\ y, \\ and\ E_a^{(i)}\ of\ strength\ > x \\ occurs\ in\ (h-1)\ steps \\ as\ a\ cascading\ effect \\ of\ E_c^{(k)} of\ strength\ z, \\ but\ not\ in\ 1,\dots,or\ (h-2)\ steps \end{matrix}\right\}\right) =$$

$$= \coprod_{\substack{k=1,\dots,n \\ c=1,\dots,m(k)}} \sum_{\substack{z\in S \\ z\le x\ for\ (k,c)=(i,a)}} \pi_{b,c}^{(j,k)}(y,z)\pi_{c,a}^{(k,i)}(z,\ >x,\ h-1) \tag{7}$$

The last equality follows from Lemma 1 which states that the sequences of cascading events caused by different triggering events are independent.

Lemma 3

If the internal impact is not taken into consideration, but the feedback effect is, we have:

$$\pi_{b,a}^{(j,i)}(y, >x,\ h) =$$

$$= \coprod_{\substack{k=1,\ldots,n;\ k\neq j \\ c=1,\ldots,m(k)}} \sum_{\substack{z\in S \\ z\leq x\ for\ (k,c)=(i,a)}} \pi_{b,c}^{(j,k)}(y,z)\pi_{c,a}^{(k,i)}(z, >x,\ h-1),\ h\geq 3, \tag{8}$$

Note that k=j is not in the range of the "inverted pi", otherwise an internal impact within p_j would be taken into account. For h=2 (8) changes to:

$$\pi_{b,a}^{(j,i)}(y, >x,\ 2) = \coprod_{\substack{k=1,\ldots,n;\ k\neq j,k\neq i \\ c=1,\ldots,m(k)}} \sum_{z\in S} \pi_{b,c}^{(j,k)}(y,z)\pi_{c,a}^{(k,i)}(z, >x) \tag{9}$$

Note that now k=i and k=j are not in the range of the "inverted pi", otherwise an internal impact in p_i or p_j would be taken into account. Also note that the summation over $z\in S$ is not limited to $z\leq x$ for (k,c)=(i,a), because k=i is not in the range of the "inverted pi".

If no feedback is taken into consideration, then the following two lemmas hold:

Lemma 4

If the feedback effect is not taken into consideration, but the internal impact is, then we have:

$$\pi_{b,a}^{(j,i)}(y, >x,\ h) =$$

$$= \coprod_{\substack{k=1,\ldots,n \\ c=1,\ldots,m(k) \\ (k,c)\neq(i,a)}} \sum_{z\in S} \pi_{b,c}^{(j,k)}(y,z)\pi_{c,a}^{(k,i)}(z, >x,\ h-1),\ h\geq 2, \tag{10}$$

where (j,b)≠(i,a) – otherwise $E_a^{(i)}$ would be a h-step feedback effect of itself. For a similar reason (k,c)=(i,a) is not in the range of the "inverted pi" – otherwise $E_a^{(i)}$ would be a (h–1)-step feedback effect of itself. In consequence, the summation over $z \in S$ is not limited to $z \leq x$ for (k,c)=(i,a) as in (3) or (8).

Lemma 5

If both the feedback effect and internal impact are not taken into consideration, then it holds that

$$\pi_{b,a}^{(j,i)}(y, > x,\ h) =$$

$$= \coprod_{\substack{k=1,\ldots,n;\ k\neq j \\ c=1,\ldots,m(k) \\ (k,c)\neq(i,a)}} \sum_{z\in S} \pi_{b,c}^{(j,k)}(y,z)\pi_{c,a}^{(k,i)}(z, > x,\ h-1)\,,\ h \geq 3\,, \tag{11}$$

where (j,b)≠(i,a). Note that (k,c)=(i,a) is not in the range of the "inverted pi" – otherwise $E_a^{(i)}$ would be a (h–1)-step feedback effect of itself. Also, k=j is not in the range of the "inverted pi" – otherwise an internal impact in p_j would be taken into account. For h=2 (11) changes to:

$$\pi_{b,a}^{(j,i)}(y, > x,\ 2) =$$

$$= \coprod_{\substack{k=1,\ldots,n;\ k\neq j,k\neq i \\ c=1,\ldots,m(k)}} \sum_{z\in S} \pi_{b,c}^{(j,k)}(y,z)\pi_{c,a}^{(k,i)}(z, > x) \tag{12}$$

where (j,b)≠(i,a). Note that k=j and k=i are not in the range of the "inverted pi" – otherwise an internal impact in p_j or p_i would be taken into account.

The proofs of Lemmas 3 through 5 are similar to that of Lemma 2. As to the formulas for "aggregate" probabilities, i.e. (5) and (6), they remain unchanged in all the cases considered in these lemmas.

4. Calculation of the risks of harmful events

Theorem 1

The primary events $E_a^{(i)}$ of strength x or greater than x constitute a Poisson process with the intensity $\lambda_a^{(i)} \cdot \pi_a^{(i)}(x)$ or $\lambda_a^{(i)} \cdot \pi_a^{(i)}(>x)$ respectively.

Proof: It follows from the adopted assumptions that the primary events $E_a^{(i)}$ of any strength constitute a Poisson process with the intensity $\lambda_a^{(i)}$, hence we have:

$$\Pr\left[N_a^{(i)}(s,t,x) = r\right] =$$

$$= \sum_{q=r}^{\infty} \begin{matrix} \Pr\left[X_a^{(i)} = x \textit{ for } r \textit{ of the } q \textit{ occurrences of } E_a^{(i)} \mid N_a^{(i)}(s,t) = q\right] \times \\ \times \Pr\left[N_a^{(i)}(s,t) = q\right] \end{matrix} =$$

$$= \sum_{q=r}^{\infty} \binom{q}{r} \left[\pi_a^{(i)}(x)\right]^r \left[1 - \pi_a^{(i)}(x)\right]^{q-r} \frac{\left[\lambda_a^{(i)} \cdot (t-s)\right]^q}{q!} \exp\left[-\lambda_a^{(i)} \cdot (t-s)\right] =$$

$$= \left[\lambda_a^{(i)} \cdot \pi_a^{(i)}(x) \cdot (t-s)\right]^r \exp\left[-\lambda_a^{(i)} \cdot (t-s)\right] \times$$

$$\times \sum_{q=r}^{\infty} \frac{q!}{r!(q-r)!} \left[1 - \pi_a^{(i)}(x)\right]^{q-r} \frac{\left[\lambda_a^{(i)} \cdot (t-s)\right]^{q-r}}{q!} =$$

$$= \frac{\left[\lambda_a^{(i)} \cdot \pi_a^{(i)}(x) \cdot (t-s)\right]^r}{r!} \exp\left[-\lambda_a^{(i)} \cdot (t-s)\right] \sum_{q=r}^{\infty} \left[1 - \pi_a^{(i)}(x)\right]^{q-r} \frac{\left[\lambda_a^{(i)} \cdot (t-s)\right]^{q-r}}{(q-r)!} =$$

$$= \frac{\left[\lambda_a^{(i)} \cdot \pi_a^{(i)}(x) \cdot (t-s)\right]^r}{r!} \exp\left[-\lambda_a^{(i)} \cdot (t-s)\right] \exp\left(\left[1 - \pi_a^{(i)}(x)\right]\left[\lambda_a^{(i)} \cdot (t-s)\right]\right) =$$

$$= \frac{\left[\lambda_a^{(i)} \cdot \pi_a^{(i)}(x) \cdot (t-s)\right]^r}{r!} \exp\left[-\lambda_a^{(i)} \cdot \pi_a^{(i)}(x) \cdot (t-s)\right] \quad (13)$$

The above equality indicates that the first part of the thesis is true.

In the same way it is proved that the occurrences of $E_a^{(i)}$ of strength >x constitute a Poisson process with the intensity $\lambda_a^{(i)} \cdot \pi_a^{(i)}(>x)$.

Corollary 1:

The "primary" risks $r_a^{(i)}(s, t, x)$ and $r_a^{(i)}(s, t, >x)$ are given by the following formulas:

$$r_a^{(i)}(s,t,x) = 1 - \exp\left[-\pi_a^{(i)}(x) \cdot \lambda_a^{(i)} \cdot (t-s)\right] \quad (14)$$

$$r_a^{(i)}(s,t,>x) = 1 - \exp\left[-\pi_a^{(i)}(>x) \cdot \lambda_a^{(i)} \cdot (t-s)\right] \quad (15)$$

Now we pass to the calculation of the total risk that one or more events $E_a^{(i)}$ of strength x or exceeding x occur on the (s, t] interval, provided that all the processes can contribute to $E_a^{(i)}$.

Theorem 2 (direct impact, no cascading effect)

The events $E_a^{(i)}$ of strength x or greater than x, directly caused by primary events $E_b^{(j)}$ of any strength, constitute a Poisson process with the intensity $\lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot, x)$ or $\lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot, >x)$ respectively, where the probabilities $\pi_{b,a}^{(j,i)}(\cdot, x)$ and $\pi_{b,a}^{(j,i)}(\cdot, >x)$ are given by (1) and (2). Further, the occurrences of $E_a^{(i)}$ of strength x or greater than x, directly caused by any primary event in any process (including p_i), constitute a Poisson process with the intensities given by the following formulas:

$$\Lambda_a^{(i)}(x,1) = \sum_{\substack{j=1,\dots,n \\ b=1,\dots,m(j);\ b\neq a\ for\ j=i}} \lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot,\ x) \tag{16}$$

and

$$\Lambda_a^{(i)}(>x,1) = \sum_{\substack{j=1,\dots,n \\ b=1,\dots,m(j);\ b\neq a\ for\ j=i}} \lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot,\ >x) \tag{17}$$

Proof: Let $N_{b,a}^{(j,i)}(s,t,x)$ be the number of the events $E_a^{(i)}$ of strength x, directly caused by primary events $E_b^{(j)}$ of any strength. We have:

$$\Pr\left[N_{b,a}^{(j,i)}(s,t,x) = r\right] =$$

$$= \sum_{q=r}^{\infty} Pr\left(\begin{matrix} the\ events\ E_b^{(j)}\ directly\ cause \\ r\ events\ E_a^{(i)}\ of\ strength\ x \end{matrix} \middle| N_b^{(j)}(s,t) = q\right) \times$$
$$\times \Pr\left(N_b^{(j)}(s,t) = q\right)$$

$$= \sum_{q=r}^{\infty} \binom{q}{r} \left[\pi_{b,a}^{(j,i)}(\cdot,\ x)\right]^r \left[1 - \pi_{b,a}^{(j,i)}(\cdot,\ x)\right]^{q-r} \frac{\left[\lambda_b^{(j)} \cdot (t-s)\right]^q}{q!} \exp\left[-\lambda_b^{(j)} \cdot (t-s)\right] \tag{18}$$

Proceeding further as in (13) leads to the following formula:

$$\Pr\left[N_{b,a}^{(j,i)}(s,t,x) = r\right] = \frac{\left[\lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot,\ x) \cdot (t-s)\right]^r}{r!} \exp\left[-\lambda_a^{(i)} \cdot \pi_{b,a}^{(j,i)}(\cdot,\ x) \cdot (t-s)\right] \tag{19}$$

In the same way we obtain:

$$\Pr\left[N_{b,a}^{(j,i)}(s,t,>x) = r\right] =$$

$$= \frac{\left[\lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot\,,\ > x) \cdot (t-s)\right]^r}{r!} \exp\left[-\lambda_a^{(i)} \cdot \pi_{b,a}^{(j,i)}(\cdot\,,\ > x) \cdot (t-s)\right] \qquad (20)$$

The first part of the thesis is thus proved. For the proof of the second part let us note that the primary events $E_b^{(j)}$, $j \in \{1,\dots,n\}$, $b \in \{1,\dots,m(j)\}$, $b \neq a$ for $j=i$, occur independently, thus it follows from Lemma 1 that the occurrences of $E_a^{(i)}$ directly caused by these $E_b^{(j)}$-s can be regarded as a superposition of independent Poisson processes, hence their intensity is equal to the sum of the intensities of the individual processes.

Corollary 2:

The "secondary" risks $r_{b,a}^{(j,i)}(s, t, x)$ and $r_{b,a}^{(j,i)}(s, t, >x)$ are given by the following formulas:

$$r_{b,a}^{(j,i)}(s,t,x) = 1 - \exp\left[-\lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot,x)\right] \qquad (21)$$

$$r_{b,a}^{(j,i)}(s,t,>x) = 1 - \exp\left[-\lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot,>x)\right] \qquad (22)$$

Corollary 3:

The total risks $R_a^{(i)}(s, t, x)$ and $R_a^{(i)}(s, t, >x)$ are given by the following formulas:

$$R_a^{(i)}(s,t,x) = 1 - \exp\left[-\Lambda_a^{(i)}(x,1)\right] \qquad (23)$$

$$R_a^{(i)}(s,t,>x) = 1 - \exp\left[-\Lambda_a^{(i)}(>x,1)\right] \qquad (24)$$

where $\Lambda_a^{(i)}(x,1)$ and $\Lambda_a^{(i)}(>x,1)$ are given by (16) and (17).

Theorem 3 (cascading effect of step h≥2)

The events $E_a^{(i)}$ of strength x or greater than x, each of which is a h-step (but not less-than-h-step) cascading effect of a primary event $E_b^{(j)}$ of any strength, constitute a Poisson process with the intensity $\lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot, x, h)$ or $\lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot, >x, h)$ respectively, where the probabilities $\pi_{b,a}^{(j,i)}(\cdot, x, h)$ and $\pi_{b,a}^{(j,i)}(\cdot, >x, h)$ are given by the formulas in Lemmas 2 – 5. We recall that these formulas differ depending on whether the internal impact and/or feedback effect are taken into consideration.

Further, the events $E_a^{(i)}$ of strength x or greater than x, each of which is a h-step (but not less-than-h-step) cascading effect of any primary event in any process (including p_i), constitute a Poisson process with the intensities given by the following formulas:

$$\Lambda_a^{(i)}(x,h) = \sum_{\substack{j=1,\dots,n \\ b=1,\dots,m(j)}} \lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot,\ x,h) \tag{25}$$

and

$$\Lambda_a^{(i)}(>x,h) = \sum_{\substack{j=1,\dots,n \\ b=1,\dots,m(j)}} \lambda_b^{(j)} \cdot \pi_{b,a}^{(j,i)}(\cdot,\ >x,h) \tag{26}$$

If the feedback effect is not taken into consideration, then (j,b)=(i,a) is excluded from the range of the summation operator in (25) and (26) – see lemmas 3 and 4.

Proof: the proof is similar to that of Theorem 2.

Corollary 4:

The total risks $R_a^{(i)}(s, t, x, h)$ and $R_a^{(i)}(s, t, >x, h)$ can be found from the following formulas:

$$R_a^{(i)}(s,t,x) = 1 - \exp\left[-\Lambda_a^{(i)}(x,h)\right] \tag{27}$$

$$R_a^{(i)}(s,t,>x) = 1 - \exp\left[-\Lambda_a^{(i)}(>x,h)\right] \tag{28}$$

where $\Lambda_a^{(i)}(x, h)$ and $\Lambda_a^{(i)}(>x, h)$ are given by (25) and (26).

Theorem 4 (cascading effect of any step)

The events $E_a^{(i)}$ of strength x or greater than x, each of which is a cascading effect of any step of a primary event $E_b^{(j)}$ of any strength, constitute a Poisson process with the intensity $\lambda_b^{(j)} \cdot \sum_{h\geq 1} \pi_{b,a}^{(j,i)}(\cdot, x, h)$ or $\lambda_b^{(j)} \cdot \sum_{h\geq 1} \pi_{b,a}^{(j,i)}(\cdot, >x, h)$ respectively.

Further, the events $E_a^{(i)}$ of strength x or greater than x, each of which is a cascading effect of any step of any primary event in any process (including p_i), constitute a Poisson process with the intensities given by the following formulas:

$$\Lambda_a^{(i)}(x, \geq 1) = \sum_{h\geq 1} \Lambda_a^{(i)}(x, h) \quad (29)$$

and

$$\Lambda_a^{(i)}(> x, \geq 1) = \sum_{h\geq 1} \Lambda_a^{(i)}(> x, h) \quad (30)$$

Proof: The secondary events $E_a^{(i)}$, each of which is a cascading effect of any step of any primary event, constitute a superposition of Poisson processes X_h, h≥1, where the process X_h is a sequence of $E_a^{(i)}$-s, each of which is a h-step (but not less-than-h-step) cascading effect of any primary event. These processes are independent, because, by one of the basic assumptions, the triggering events of the events $E_a^{(i)}$ in the compound process are independent. Thus, (27) and (28) are a consequence of Theorem 3.

Corollary 5:

The total risks $R_a^{(i)}(s, t, x, \geq 1)$ and $R_a^{(i)}(s, t, >x, \geq 1)$ can be found from the following formulas:

$$R_a^{(i)}(s,t,x,\geq 1) = 1 - \exp\left[-\Lambda_a^{(i)}(x, \geq 1)\right] \quad (31)$$

$$R_a^{(i)}(s,t,> x,\geq 1) = 1 - \exp\left[-\Lambda_a^{(i)}(> x, \geq 1)\right] \quad (32)$$

where $\Lambda_a^{(i)}(x, \geq 1)$ and $\Lambda_a^{(i)}(>x, \geq 1)$ are given by (29) and (30).

5. Conclusion

This paper presents the formulas for calculating the risks of various unwanted or harmful events that can occur during the functioning of a technical or industrial system whose hazard-related behavior is modeled by a set of mutually related stochastic point processes. The dependence between the processes follows from fact that events occurring in one process can cause events in the other processes, thus, apart from the primary events (assumed to occur independently), there are also secondary events occurring due to a cascading or feedback effect. The derived formulas are effect-oriented in the sense that they express the total probabilities of the considered events without specifying the degree to which individual primary events contribute to these probabilities. However, the obtained formulas can be modified to the cause-oriented ones, i.e. quantifying the possible effects of individual primary events. This will be the subject of future work.

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