Raport Badawczy
Research Report

RB/38/2012

Maximum likelihood estimators of the Weibull distribution parameters, based on short sequences of dependent lifetimes

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Warszawa 2012

Notation

CDF – cumulative distribution function

MGF – moment generating function

MLE – maximum likelihood estimator

PDF – probability density function

TTF – time-to-failure (a continuous, nonnegative random variable)

r(t) – failure (hazard) rate; i.e. $r(t)=f(t)/[1-F(t)]$, where f(t) and F(t) are the PDF and CDF of the TTF under consideration

i.i.d. – an acronym meaning "independent and identically distributed"

r.v. – an acronym meaning "random variable"

1. Introduction

The current paper deals with the problem of estimating the parameters of the Weibull distribution. Although the topic has been thoroughly investigated by statisticians, a new approach, stemming from the reliability theory, is presented herein. It is a well-known fact that the time-to-failure (TTF) of many technical devices (or their components) is a Weibull distributed random variable. Therefore, in order to estimate its parameters, the usual procedure is to measure the TTF's of a number of test items, and calculate the required estimates from the values of the random sample. This standard procedure has one essential disadvantage – if a failed object is no longer usable then a considerable number of test items is needed in order to achieve high accuracy of the estimation, which may lead to unacceptable cost. However, if the test items are repairable, then another method can be used to reduce this cost. According to this proposed method each item undergoes m–1 minimal repairs, where the i-th repair follows the i-th failure, $1 \leq i \leq m-1$, and is considered unusable after the m-th failure. It is natural for m to be the minimum number of failures that one tested object can survive; usually m is not large. The above procedure is repeated n times, which amounts to destructive testing of n items. Let t_{ij} be the time elapsed between the $(i-1)$-th repair and the i-th failure of the j-th item, $1 \leq i \leq m$, $1 \leq j \leq n$ (the 0-th repair is performed when a new item is put into operation). For each item the times t_{ij}, $1 \leq i \leq m$, are recorded, thus constituting a vector element of the n-sized random sample. Based upon the collected data, and the appropriately constructed estimators, the sought parameters can be evaluated. Following this scheme the desired estimation accuracy may be reached for n significantly smaller than in the case when no repairs are possible, and a test item becomes unusable after the first failure.

The paper is organized as follows. In Section 2 the maximum likelihood estimators of the scale and shape parameters of the Weibull distribution, based on the minimal repairs and failures sequence, are constructed. It is then explained how the expected values of those estimators can be approximated using n independent test units. In Section 3 the sought parameters are expressed in terms of the expected values of the respective estimators. The derived formulas allow to calculate, in a simple way, the biases of the estimators constructed in Section 2.

Standard MLE estimators of the shape and scale parameters are constructed using an i.i.d. sample obtained from a large number of test units. Based on that sample, an equation for the shape parameter, that cannot be solved analytically, is obtained (see [1]). The proposed method allows to express the MLE of both parameters in analytical form. Those estimators occur to be biased; nevertheless, the biases can also be computed analytically.

Thus, a new estimation method for the Weibull distribution, based on analytical formulas alone, has been developed.

2. The maximum likelihood estimators based on a minimal repairs sequence

Lemma 1

Let f(t) be the PDF of the system's TTF. Let the system be subjected to m−1 minimal repairs, where $S_1, S_2, \ldots, S_m$ are the moments of consecutive failures, i.e. the new system is put to work at $S_0=0$, the i-th minimal repair is performed at S_i, $1 \le i \le m-1$, and S_m is the time of the last failure after which no more repairs are performed. Let also $T_i = S_i - S_{i-1}$, $1 \le i \le m$. Under these assumptions the PDF of the m-dimensional random vector $[T_1, \ldots, T_m]^T$, denoted by $f^{(m)}(t_1, \ldots, t_m)$, is given by the following formula:

$$f^{(m)}(t_1, \ldots, t_m) = \prod_{i=1}^{m-1} r(t_1 + \cdots + t_i)\, f(t_1 + \cdots + t_m) = \prod_{i=1}^{m-1} r(s_i)\, f(s_m) \qquad (1)$$

Proof:

For m≥2 it holds that

$$\Pr(T_1 \in (t_1, t_1 + \Delta t_1], \ldots, T_m \in (t_m, t_m + \Delta t_m]) =$$

$$= \Pr(T_1 \in (t_1, t_1 + \Delta t_1]) \times$$

$$\times \prod_{i=2}^{m} \Pr(T_i \in (t_i, t_i + \Delta t_i] | T_{i-1} \in (t_{i-1}, t_{i-1} + \Delta t_{i-1}], \ldots, T_1 \in (t_1, t_1 + \Delta t_1]) \approx$$

$$\approx [F(t_1 + \Delta t_1) - F(t_1)] \prod_{i=2}^{m} \frac{F(t_1 + \cdots + t_{i-1} + t_i + \Delta t_i) - F(t_1 + \cdots + t_{i-1} + t_i)}{1 - F(t_1 + \cdots + t_{i-1})} =$$

$$= \prod_{i=1}^{m-1} \frac{F(t_1 + \cdots + t_i + \Delta t_i) - F(t_1 + \cdots + t_i)}{1 - F(t_1 + \cdots + t_i)} [F(t_1 + \cdots + t_m + \Delta t_m) - F(t_1 + \cdots + t_m)] \qquad (2)$$

The second (approximate) equality follows from the fact that if T_s is the system's residual TTF after a minimal repair completed at the instant s then

$$\Pr(T_s \leq t) = \frac{F(s+t)-F(s)}{1-F(s)} \quad (3)$$

Taking into consideration that

$$\frac{\partial^m F^{(m)}(t_1,\dots,t_m)}{\partial t_1 \dots \partial t_m} = \lim_{\Delta t_1 \to 0,\dots,\Delta t_m \to 0} \frac{\Pr(T_1 \in (t_1,t_1+\Delta t_1],\dots,T_m \in (t_m,t_m+\Delta t_m])}{\Delta t_1 \dots \Delta t_m} \quad (4)$$

where $F^{(m)}(t_1,\dots,t_m)$ is the CDF of $[T_1,\dots,T_m]^T$, the lemma's thesis follows from (2) and the definition of r(t) (see Notation).

In the case of a two-parameter Weibull distribution, i.e.

$$f(t) = \alpha\lambda(\lambda t)^{\alpha-1}\exp[-(\lambda t)^{\alpha}], \qquad r(t) = \alpha\lambda(\lambda t)^{\alpha-1} \quad (5)$$

where α and λ are the shape and scale parameters, we have:

$$f^{(m)}(t_1, \dots, t_m) = \alpha^m \lambda^{\alpha m} \prod_{i=1}^{m} (t_1 + \dots + t_i)^{\alpha-1} \exp\left[-(\lambda)^{\alpha}(t_1 + \dots + t_m)^{\alpha}\right] \quad (6)$$

which is obtained from (1) with the use of the substitutions (4). The above formula also defines the likelihood function of the parameters α and λ, which will be denoted by $L(\alpha,\lambda | t_1,\dots,t_m)$. From (6) it follows that

$$\ln[L(\alpha,\lambda|t_1,\dots,t_m)] = m \cdot \ln(\alpha) + \alpha \cdot m \cdot \ln(\lambda) + (\alpha - 1)\sum_{i=1}^{m} \ln(t_1 + \dots + t_i)$$

$$-\lambda^{\alpha}(t_1 + \dots + t_m)^{\alpha} \quad (7)$$

We have thus derived the expression for the log-likelihood function (the likelihood function's logarithm), which will play fundamental role in finding MLE of the α and λ parameters.

The standard way to find the maximum likelihood estimates of unknown parameters – the arguments of a likelihood function – is to compute the first partial derivatives of the log-likelihood function w.r.t. these parameters, and equate them to zero, while the variables (in this case $t_1,\ldots,t_n$), which constitute a random sample, are considered to be fixed. It should also be checked if the likelihood function actually reaches a maximum where the derivatives are equal to zero, but this check is often omitted. Applying this standard procedure to our case we obtain:

$$\frac{\partial \ln[L(\alpha,\lambda|t_1,\ldots,t_m)]}{\partial\alpha} = \frac{m}{\alpha} + m\cdot\ln(\lambda) + \sum_{i=1}^{m}\ln(t_1+\cdots+t_i)$$

$$-\lambda^{\alpha}(t_1+\cdots+t_m)^{\alpha}\ln[\lambda(t_1+\cdots+t_m)] \qquad (8)$$

$$\frac{\partial \ln[L(\alpha,\lambda|t_1,\ldots,t_m)]}{\partial\lambda} = \frac{\alpha\cdot m}{\lambda} - \alpha\cdot\lambda^{\alpha-1}(t_1+\cdots+t_m)^{\alpha} \qquad (9)$$

In order to find $\hat{\lambda}$ and $\hat{\alpha}$ zeroing the above derivatives we first equate the right-hand side in (9) to zero, which yields:

$$\hat{\lambda} = \frac{m^{1/\hat{\alpha}}}{t_1+\cdots+t_m} \qquad (10)$$

Subsequently, we replace $\hat{\lambda}$ in (8) according to (10), and equate the right-hand side in (8) to zero, obtaining:

$$\frac{m}{\hat{\alpha}} + \frac{m}{\hat{\alpha}}\cdot\ln(m) - m\cdot\ln(t_1+\cdots+t_m) + \sum_{i=1}^{m}\ln(t_1+\cdots+t_i) - \frac{m}{\hat{\alpha}}\cdot\ln(m) = 0 \qquad (11)$$

which yields:

$$\hat{\alpha} = \frac{m}{m\cdot\ln(t_1+\cdots+t_m) - \sum_{i=1}^{m}\ln(t_1+\cdots+t_i)} \qquad (12)$$

Note that in order to eliminate $\hat{\alpha}$ in (10) it remains to substitute it according to (12).

The formulas (10) and (12) define the MLE's based on the minimal repairs sequence. Herein $t_1,\ldots,t_m$ denote the values of a random sample obtained by recording the respective TTF's. Clearly, $\hat{\alpha}$ and $\hat{\lambda}$ are dependent on m, but for simplicity, this is not reflected in the notation. A natural question arises – how "good" are these estimators? Their goodness can be judged by two criteria – the estimators' biases and variances. The first criterion needs no further clarification – we simply have to calculate $\lambda - E(\hat{\lambda})$ and $\alpha - E(\hat{\alpha})$, where the estimators are treated as random variables; the second one is explained in greater detail below. To be more precise, $\ln(\lambda)$ and $1/\alpha$ will be estimated rather than λ and α, the respective biases being $\ln(\lambda) - E[\ln(\hat{\lambda})]$ and $1/\alpha - E(1/\hat{\alpha})$. The reason for this will be given in Section 3.

In the case where the estimation is based on a random sample originating from i.i.d. instances of a random variable, its accuracy is usually determined by the confidence level and the length of the confidence interval, and the sample's size has to be sufficiently large in order to obtain the required accuracy. As is well known, this size is related to the considered random variable's variance. However, in our case the $T_1,\ldots,T_m$ are not independent, have different CDF's, and in practice it is not possible to perform a large number of minimal repairs on one object – often after several such repairs the object becomes unusable. For the above reasons our parameters will be estimated by taking n identical and independent objects, performing m–1 minimal repairs on each of them (m not being large) to obtain n sample values of both $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$, and calculating the respective sample means to approximate $E[\ln(\hat{\lambda})]$ and $E(1/\hat{\alpha})$. Thus, $\widehat{\Lambda}$ and $\widehat{\mathrm{A}}$ defined as follows

$$\widehat{\Lambda} = \frac{\ln(\hat{\lambda}_1)+\cdots+\ln(\hat{\lambda}_n)}{n}, \quad \widehat{\mathrm{A}} = \frac{1/\hat{\alpha}_1+\cdots+1/\hat{\alpha}_n}{n},$$

where $\hat{\lambda}_1, \dots, \hat{\lambda}_n$ and $\hat{\alpha}_1, \dots, \hat{\alpha}_n$ are i.i.d. instances of $\hat{\lambda}$ and $\hat{\alpha}$ respectively, will be used as estimators of $\ln(\lambda)$ and $1/\alpha$. The law of large numbers yields that

$$\widehat{\Lambda} \approx E[\ln(\hat{\lambda})] \text{ and } \widehat{\mathrm{A}} \approx E(1/\hat{\alpha}),$$

thus our estimation task consists in approximating expected values with i.i.d. samples. Clearly, the number n for which the required accuracy is achieved is proportional to $Var(\ln(\hat{\lambda}))$ or $Var(1/\hat{\alpha})$. It should also be remembered that $\widehat{\Lambda}$ and $\widehat{\mathrm{A}}$ are biased estimators of $\ln(\lambda)$ and $1/\alpha$. Let us note that

$$E(\widehat{\Lambda}) = E[\ln(\hat{\lambda})] \text{ and } E(\widehat{\mathrm{A}}) = E(1/\hat{\alpha}),$$

thus computing the biases of $\widehat{\Lambda}$ and $\widehat{\mathrm{A}}$ is equivalent to computing those of $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$. The formulas for the respective biases will be derived in the next section, while finding the variances and confidence intervals will be the subject of further research.

3. Expressing ln(λ) and 1/α in terms of E[ln($\hat{\lambda}$)] and E(1/$\hat{\alpha}$), and finding the respective biases

In the sequel the following two auxiliary lemmas will be necessary.

Lemma 2

For m≥1 it holds that

$$\Pr(T_1 + \cdots + T_m > t) = \exp[-(\lambda t)^{\alpha}] \sum_{k=0}^{m-1} \frac{(\lambda t)^{k\alpha}}{k!} \tag{13}$$

Proof

Clearly, (x+1) holds for m=1. For m≥2 we have:

$$Pr(T_1+\cdots+T_m\le t)=\int_{t_1+\cdots+t_m\le t} r(t_1)\ldots r(t_1+\cdots+t_{m-1})f(t_1+\cdots+t_m)=$$

$$=\int_0^t r(t_1)\ldots\int_0^{t-t_1-\cdots-t_{m-2}} r(t_1+\cdots+t_{m-1})\int_0^{t-t_1-\cdots-t_{m-1}} f(t_1+\cdots+t_m)dt_m\ldots dt_1=$$

$$=\int_0^t r(t_1)\ \ldots\int_0^{t-t_1-\cdots-t_{m-2}} r(t_1+\cdots+t_{m-1})[F(t)-F(t_1+\cdots+t_{m-1})]dt_{m-1}\ldots dt_1=$$

$$=F(t)\int_0^t r(t_1)\ \ldots\int_0^{t-t_1-\cdots-t_{m-2}} r(t_1+\cdots+t_{m-1})dt_{m-1}\ldots dt_1$$

$$+\int_0^t r(t_1)\ \ldots\int_0^{t-t_1-\cdots-t_{m-2}} r(t_1+\cdots+t_{m-1})\,[1-F(t_1+\cdots+t_{m-1})-1]dt_{m-1}\ldots dt_1=$$

$$=\int_0^t r(t_1)\ \ldots\int_0^{t-t_1-\cdots-t_{m-2}} f(t_1+\cdots+t_{m-1})\,dt_{m-1}\ldots dt_1+$$

$$-[1-F(t)]\int_0^t r(t_1)\ \ldots\int_0^{t-t_1-\cdots-t_{m-2}} r(t_1+\cdots+t_{m-1})\,dt_{m-1}\ldots dt_1=$$

$$=Pr(T_1+\cdots+T_{m-1}\le t)+$$

$$-[1-F(t)]\int_0^t r(t_1)\ \ldots\int_0^{t-t_1-\cdots-t_{m-2}} r(t_1+\cdots+t_{m-1})\,dt_{m-1}\ldots dt_1 \qquad (14)$$

We will prove that the integral in the last line satisfies the following equality:

$$\int_0^t r(t_1)\ \ldots\int_0^{t-t_1-\cdots-t_{m-2}} r(t_1+\cdots+t_{m-1})\,dt_{m-1}\ldots dt_1=\frac{1}{(m-1)!}(\lambda t)^{(m-1)\alpha} \qquad (15)$$

For this purpose let

$$s_0=0, s_1=t_1,\ldots,s_{m-1}=t_1+\cdots+t_{m-1}$$

We thus have

$$\int_0^t r(t_1) \dots \int_0^{t-t_1-\dots-t_{m-2}} r(t_1+\dots+t_{m-1})\, dt_{m-1} \dots dt_1 =$$

$$= \int_{s_0}^t r(s_1) \int_{s_1}^t r(s_2) \dots \int_{s_{m-2}}^t r(s_{m-1})\, ds_{m-1} \dots ds_1$$

By the "reverse" induction it can be shown that

$$\int_{s_{k-1}}^t r(s_k) \dots \int_{s_{m-2}}^t r(s_{m-1})\, ds_{m-1} \dots ds_k = \frac{1}{(m-k)!}[(\lambda t)^\alpha - (\lambda s_{k-1})^\alpha]^{m-k} \qquad (16)$$

for k = m−1,...,1. Clearly, (16) holds for k = m−1. The induction step is based on the following derivation:

$$\int_{s_{k-1}}^t r(s_k) \frac{1}{(m-k-1)!}[(\lambda t)^\alpha - (\lambda s_k)^\alpha]^{m-k-1}\, ds_k =$$

$$= \frac{1}{(m-k-1)!} \int_{s_{k-1}}^t [(\lambda t)^\alpha - (\lambda s_k)^\alpha]^{m-k-1} \frac{d(\lambda s_k)^\alpha}{ds_k}\, ds_k =$$

$$= \frac{1}{(m-k-1)!} \int_{u(s_{k-1})}^{u(t)} [(\lambda t)^\alpha - u]^{m-k-1} du =$$

$$= -\frac{1}{(m-k-1)!(m-k)} [(\lambda t)^\alpha - u]^{m-k} \Big|_{u(s_{k-1})}^{u(t)} = \frac{1}{(m-k)!}[(\lambda t)^\alpha - (\lambda s_{k-1})^\alpha]^{m-k}$$

where

$$u(s) = (\lambda s)^\alpha$$

Now (15) is a direct consequence of (16). The formulas (14) and (15) yield the following recursive equation holding true for m ≥ 2:

$$Pr(T_1+\dots+T_m \le t) = Pr(T_1+\dots+T_{m-1} \le t) - exp[-(\lambda t)^\alpha] \frac{1}{(m-1)!} (\lambda t)^{(m-1)\alpha} \qquad (17)$$

from which (13) follows immediately.

Lemma 3

If X is a continuous random variable then

$$E(|X|^r) = r\int_0^\infty x^{r-1}\,\Pr(|X| > x)\,dx \tag{18}$$

The proof can be found in [].

The above lemmas are needed in order to prove the following fact:

Theorem 1

$$E[\ln(T_1 + \cdots + T_i)] = \frac{1}{\alpha}\left[\Gamma'(1) + \sum_{k=1}^{i-1}\frac{1}{k}\right] - \ln(\lambda),\ \ i \geq 1 \tag{19}$$

where Γ is the Euler's gamma function, and Γ' – its first derivative. The sum in brackets is assumed to be equal to zero for i=1.

Proof:

We will first derive an expression for the MGF of $\ln(S_i) = \ln(T_1+\ldots+T_i)$. Let X be a continuous non-negative random variable. Let $G_{\ln(X)}(t)$ denote the MGF of ln(X). We have

$$G_{\ln(X)}(u) = E\left[e^{u\cdot\ln(X)}\right] = E\left[\left(e^{\ln(X)}\right)^u\right] = E[X^u] = u\int_0^\infty x^{u-1}\,\Pr(X > x)\,dx \tag{20}$$

where the last equality follows from (18). Combining (20) and (13) yields

$$G_{\ln(S_i)}(u) = u\int_0^\infty x^{u-1}\exp[-(\lambda x)^\alpha]\sum_{k=0}^{i-1}\frac{(\lambda t)^{k\alpha}}{k!}\,dx \tag{21}$$

If we put $y=(\lambda x)^\alpha$, then

$$x = \frac{1}{\lambda} y^{\frac{1}{\alpha}}, \ \frac{dy}{dx} = \alpha\lambda(\lambda x)^{\alpha-1} = \alpha\lambda y^{1-\frac{1}{\alpha}} \tag{22}$$

and, applying integration by substitution, we obtain

$$u \int_0^{\infty} x^{u-1} \exp[-(\lambda x)^{\alpha}] \frac{(\lambda t)^{k\alpha}}{k!} dx = u \int_0^{\infty} \frac{y^{\frac{1}{\alpha}(u-1)}}{\lambda^{u-1}} \exp(-y) \frac{y^k}{k!} \frac{y^{\frac{1}{\alpha}-1}}{\alpha\lambda} dy =$$

$$= \frac{u}{k!\alpha\lambda^u} \int_0^{\infty} y^{\frac{u}{\alpha}+k-1} \exp(-y) dy = \frac{u}{k!\alpha\lambda^u} \Gamma(\frac{u}{\alpha} + k) \tag{23}$$

On the basis of (21) and (23) it holds that

$$G_{\ln(S_i)}(u) = \frac{u}{\alpha\lambda^u} \sum_{k=0}^{i-1} \frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!} = \frac{1}{\lambda^u} \frac{u}{\alpha} \Gamma\left(\frac{u}{\alpha}\right) + \frac{u}{\alpha\lambda^u} \sum_{k=1}^{i-1} \frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!} =$$

$$= \frac{1}{\lambda^u} \Gamma\left(\frac{u}{\alpha} + 1\right) + \frac{u}{\alpha\lambda^u} \sum_{k=1}^{i-1} \frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!} = G_1(u) + G_2(u) \tag{24}$$

where the third equality is a consequence of the fact that $v\Gamma(v) = \Gamma(v+1)$, and the last sum is equal to zero for i=1. Differentiating $G_1(u)$ and $G_2(u)$ with respect to u yields

$$G_1'(u) = \frac{d\left(\frac{1}{\lambda^u}\right)}{du} \Gamma\left(\frac{u}{\alpha} + 1\right) + \frac{1}{\lambda^u} \Gamma'\left(\frac{u}{\alpha} + 1\right) \frac{1}{\alpha} = \frac{1}{\alpha\lambda^u} \Gamma'\left(\frac{u}{\alpha} + 1\right) - \frac{\lambda^u \ln(\lambda)}{\lambda^{2u}} \Gamma\left(\frac{u}{\alpha} + 1\right) \tag{25}$$

and

$$G_2'(u) = \frac{d\left(\frac{u}{\alpha\lambda^u}\right)}{du} \sum_{k=1}^{i-1} \frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!} + \frac{u}{\alpha\lambda^u} \sum_{k=1}^{i-1} \frac{\Gamma'\left(\frac{u}{\alpha}+k\right)}{k!} =$$

$$= \frac{\alpha\lambda^u - u\alpha\lambda^u \ln(\lambda)}{\alpha^2\lambda^{2u}} \sum_{k=1}^{i-1} \frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!} + \frac{u}{\alpha\lambda^u} \sum_{k=1}^{i-1} \frac{\Gamma'\left(\frac{u}{\alpha}+k\right)}{k!} \tag{26}$$

From the properties of the MGF combined with (25) and (26) we have

$$E[\ln(T_1+\cdots+T_i)] = \left.\frac{dG_{\ln(S_i)}(u)}{du}\right|_{u=0} = \frac{1}{\alpha}\Gamma'(1) - \ln(\lambda)\,\Gamma(1) + \frac{1}{\alpha}\sum_{k=1}^{i-1}\frac{\Gamma(k)}{k!} =$$

$$= \frac{1}{\alpha}\left[\Gamma'(1) + \sum_{k=1}^{i-1}\frac{1}{k}\right] - \ln(\lambda) \qquad (27)$$

The proof is thus completed.

Using Theorem 1 we will express α in terms of $\hat{\alpha}$, and λ in terms of $\hat{\lambda}$, thus solving the problem of finding the unknown parameters of the Weibull distribution. More precisely, for technical reasons, 1/α will be expressed as a function of $E(1/\hat{\alpha})$, and ln(λ) – as a function of $E[\ln(\hat{\lambda})]$. This is not a disadvantage, as $E(1/\hat{\alpha})$ and $E[\ln(\hat{\lambda})]$ are easily approximated using a k-sized random sample of the vector variable $[T_1,\ldots,T_m]$, where k is sufficiently large. From (12) we obtain:

$$E(1/\hat{\alpha}) = \frac{1}{m}\{m\cdot E[\ln(t_1+\cdots+t_m)] - \sum_{i=1}^{m} E[\ln(t_1+\cdots+t_i)]\} =$$

$$= \frac{1}{m}\sum_{i=1}^{m-1}[E[\ln(t_1+\cdots+t_m)] - E[\ln(t_1+\cdots+t_i)]] =$$

$$= \frac{1}{m\cdot\alpha}\sum_{i=1}^{m-1}\left[\sum_{j=1}^{m-1}\frac{1}{j} - \sum_{j=1}^{i-1}\frac{1}{j}\right] = \frac{1}{m\cdot\alpha}\sum_{i=1}^{m-1}\left[\sum_{j=i}^{m-1}\frac{1}{j}\right] \qquad (28)$$

where the last but one equality follows directly from Theorem 1. It holds that

$$\sum_{i=1}^{m-1}\left[\sum_{j=i}^{m-1}\frac{1}{j}\right] = m-1 \qquad (29)$$

which is easily proved by induction. As a consequence of (28) and (29) we have:

$$\frac{1}{\alpha} = \frac{m}{m-1}E\left(\frac{1}{\hat{\alpha}}\right) \qquad (30)$$

hence the bias of $1/\hat{\alpha}$ is given by

$$1/\alpha - E(1/\hat{\alpha}) = \frac{1}{m-1}E(1/\hat{\alpha}) \tag{31}$$

which yields that $1/\hat{\alpha}$ is an asymptotically unbiased estimator of $1/\alpha$ with respect to m. The approximate value of $E(1/\hat{\alpha})$ can be found based on the following formula:

$$E(1/\hat{\alpha}) \approx (\frac{1}{\hat{\alpha}_1} + \cdots + \frac{1}{\hat{\alpha}_n})/n \tag{32}$$

where, according to (12),

$$\hat{\alpha}_j = \frac{m}{m \cdot \ln(t_{1j} + \cdots + t_{mj}) - \sum_{i=1}^{m} \ln(t_{1j} + \cdots + t_{ij})}\ , \ 1 \le j \le n \tag{33}$$

In the above formula $t_{1j},\dots,t_{mj}$ are the TTF's constituting the j-th element of the n-sized random sample of the vector random variable $[T_1,\dots,T_m]$.

Note that $1/\alpha$ and $E(1/\hat{\alpha})$ are used in (31), rather than α and $E(\hat{\alpha})$, as finding the relationship between α and $E(\hat{\alpha})$ would require the analytical computation of $E(m/[m \cdot \ln(s_m) - \sum_{i=1}^{m} \ln(s_i)])$, which is impossible if only the formulas for $E[\ln(s_i)]$ are known. The reason is that knowing the formula for E(X) is not sufficient to compute E(1/X).

Let us now express $\ln(\lambda)$ as a function of $E[\ln(\hat{\lambda})]$. From (10) and (19) we obtain:

$$\ln(\hat{\lambda}) = \frac{1}{\hat{\alpha}}\ln(m) - \ln(t_1 + \cdots + t_m) =$$

$$= \frac{1}{\hat{\alpha}}\ln(m) - \frac{1}{\alpha}\left[\Gamma'(1) + \sum_{j=1}^{m-1} 1/j\right] + \ln(\lambda) \tag{34}$$

Using (30) the above formula is converted to:

$$\ln(\lambda) = E\left[\ln(\hat{\lambda})\right] - \ln(m)\, E\left(\frac{1}{\hat{\alpha}}\right) + \frac{1}{\alpha}\left[\Gamma'(1) + \sum_{k=1}^{m-1}\frac{1}{k}\right] =$$

$$= E\left[\ln(\hat{\lambda})\right] - \ln(m)\, E\left(\frac{1}{\hat{\alpha}}\right) + \frac{m}{m-1}E\left(\frac{1}{\hat{\alpha}}\right)\left[\Gamma'(1) + \sum_{k=1}^{m-1}\frac{1}{k}\right] =$$

$$= E[\ln(\hat{\lambda})] + \frac{m}{m-1} E\left(\frac{1}{\hat{a}}\right)\left[-\ln(\mathrm{m}) + \frac{\ln(\mathrm{m})}{m} + \Gamma'(1) - \frac{1}{m} + \sum_{k=1}^{m} \frac{1}{k}\right] \tag{35}$$

hence the bias of $\ln(\hat{\lambda})$ is given by

$$\ln(\lambda) - E[\ln(\hat{\lambda})] = \frac{m}{m-1} E\left(\frac{1}{\hat{a}}\right)\left[\frac{\ln(\mathrm{m})}{m} - \frac{1}{m} + \Gamma'(1) - \ln(\mathrm{m}) + \sum_{j=1}^{m} 1/j\right] \tag{36}$$

From the special functions theory it is known that $\Gamma'(1) = -\gamma$, where γ is the so-called Euler-Mascheroni constant defined as

$$\gamma = \lim_{m\to\infty}\left[\sum_{j=1}^{m} 1/j - \ln(m)\right] \cong 0.577 \tag{37}$$

We also have:

$$\lim_{m\to\infty} \frac{\ln(m)}{m} = 0, \ \lim_{m\to\infty} \frac{1}{m} = 0, \ \lim_{m\to\infty} \frac{m}{m-1} = 1 \tag{38}$$

thus $\ln(\hat{\lambda})$ is an asymptotically unbiased estimator of $\ln(\lambda)$ with respect to m. The approximate value of $E[\ln(\hat{\lambda})]$ can be found based on the following formula:

$$E[\ln(\hat{\lambda})] \ \approx [\ln(\hat{\lambda}_1) + \cdots + \ln(\hat{\lambda}_n)]/n \tag{39}$$

where, according to (10),

$$\ln(\hat{\lambda}_j) = \frac{1}{\hat{a}_j} \ln(m) - \ln(t_{1j} + \cdots + t_{mj}) \tag{40}$$

and $\hat{a}_j$ is given by (33), $1 \leq j \leq n$.

Note that $\ln(\lambda)$ and $E[\ln(\hat{\lambda})]$ are used in (36), rather than λ and $E(\hat{\lambda})$, as it is easier to operate on logarithms than directly on parameters and their estimators.

The fact that the estimators $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$ are asymptotically unbiased with respect to m has rather theoretical significance, as in practice m is not large enough for these estimators to be close enough to $\ln(\lambda)$ and $1/\alpha$.

4. References

[1] Alkutubi H.S., Ali H.M. "*Maximum likelihood estimators with complete and censored data*" European Journal of Scientific Research, vol. 54, no. 3, pp. 407-410, 2011.

[2] Chikr el-Mezouar Z. "*Estimation of the shape, location and scale parameters of the Weibull distribution*" Reliability and Risk Analysis: Theory and Applications (Electronic journal) vol. 5, no. 4, pp. 36-40, 2010.

[3] Dodson B. "The Weibull analysis handbook. Second edition" American Society for Quality, 2006

[4] Lei Y. "*Evaluation of three methods for estimating the Weibull distribution parameters of Chinese pine*" Journal of Forest Science, vol. 54, no. 12, pp. 566-571, 2008.