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On some preliminary results of the probabilistic analysis of the two-constraint binary knapsack problem

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Some preliminary results of the probabilistic analysis of the Two-Constraint Binary Knapsack Problem

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Abstract

The paper deals with the Two-Constraint Binary Knapsack Problem, which is special case of Multi-Constraint Knapsack Problem, with 2 constraints only. It is assumed that some of the problem coefficients are realizations of mutually independent random variables. Asymptotical probabilistic properties of selected problem characteristics are investigated .

1 Introduction

Let us consider a Two-Constraint Binary Knapsack Problem in the following formulation:

$$\begin{array}{ll} z_{OPT}(n) = \max \sum_{i=1}^{n} c_i \cdot x_i & \\ \text{subject to} \qquad \sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant b_j(n) & \\ \text{where} \quad j = 1, 2, \quad x_i = 0 \text{ or } 1 & \end{array} \tag{1}$$

It is assumed that:

$$c_i > 0,\ a_{ji} > 0,\ 0 < b_j(n) \leqslant \sum_{i=1}^{n} a_{ji},\ i = 1, \ldots, n,\ j = 1, 2.$$

Without restricting the generality of considerations it may be also assumed that:

$$b_1(n) \leqslant b_2(n)$$

The assumptions that $c_i > 0,\ a_{ji} > 0, 0 < b_j(n) \leqslant \sum_{i=1}^{n} a_{ji}, i = 1, \ldots, n,\ j = 1, 2$,are supposed to avoid the trivial and degenerated problems. More precisely interpretation of the $a_{ji} = 0$ or $c_i = 0$ is far unobvious. When $b_j(n) > \sum_{i=1}^{n} a_{ji}$ then the j-th constraint is always fulfilled and therefore it may be removed from the problem formulation, otherwise if $b_j(n) = 0$ then (1) has only the trivial solution i.e. $z_{OPT}(n) = 0$.

Two-Constraint Binary Knapsack Problem is special case of the binary multiconstraint knapsack problem, also known as m-constraint knapsack problem, see Nemhauser and Wolsey [10] and Martello and Toth [7], where in general case there is arbitrary number m of constarints, i.e. $j = 1, \ldots, m$. Another important special case is classical (single constraint) or, in other words, Binary Knapsack Problem, which have only one constraint, i.e. $j = 1$ (see Martello and Toth [7]). In the Szkatuła's papers see [13] and [14] probalistic analysis results of the different cases of the binary multiconstraint knapsack problem were presented. Moreover full case of the classical (single constraint) Binary Knapsack Problem was considerd in the paper [14].

The Multi-Constraint Knapsack Problem is well known to be $\mathcal{NP}$ hard, moreover, when $m \geqslant 2$, it is $\mathcal{NP}$ hard in the strong sense (see Garey and Johnson [3]). It does mean that Two-Constraint Binary Knapsack Problem (1) is also $\mathcal{NP}$ hard in the strong sense. Classical (one-constraint) Binary Knapsack Problem is $\mathcal{NP}$ hard combinatorial optimisation problem, however not in the strong sense.

The papers by Frieze and Clarke [2], Mamer and Schilling [6], Schilling [11] and [12] investigate the asymptotic value of $z_{OPT}(n)$ for the random model of Multi-Constraint Knapsack Problem, where $b_j(n) = 1$, $j = 1, \ldots, m$. Papers by Szkatuła [13] and [14] deal with the random model of the Multi-Constraint Knapsack Problem, where $b_j(n)$ are not restricted to be equal to 1. Papers by Meanti, Rinnooy Kan, Stougie and Vercellis [9], Lee and Oh [4] consider more general random models of Multi-Constraint Knapsack Problem but only for $j = 1, 2$ some analytical results describing the growth of $z_{OPT}(n)$ were obtained. Moreover full case of the Binary Knapsack Problem, $j = 1$, was considerd in the Szkatuła [14].

The aim of the present paper is to analyze the growth of the asymptotic value of $z_{OPT}(n)$ for the class of random Two-Constraint Binary Knapsack Problems (1) with full spectrum of the right-hand-sides of the constraints values. Two-Constraint Binary Knapsack Problem is important special case of the general Multi-Constraint Knapsack Problem, see Martello and Toth [8]. It is difficult, $\mathcal{NP}$ hard in the strong sense, combinatorial optimisation problem. Results of the probabilistic analysis may allow to describe asymptotic behavior of the $z_{OPT}(n)$ for practically all combinations of values of $b_1(n)$ and $b_2(n)$ as well as other problem coefficients (considered as realisations of the random variables). Those results may help to better understand the theoretical issues related to Two-Constraint Binary Knapsack Problems as well as enable construction of more efficient algorithms for solving the practical instances of the (1).

2 Definitions

The following definitions are necessary for the further presentation:

Definition 1 *We denote $V_n \approx Y_n$, where $n \to \infty$, if*

$$Y_n \cdot (1 - o(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o(1))$$

when V_n, Y_n are sequences of numbers, or

$$\lim_{n\to\infty} P\{Y_n \cdot (1 - o(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o(1))\} = 1$$

when V_n is a sequence of random variables and Y_n is a sequence of numbers or random variables, where $\lim_{n\to\infty} o(1) = 0$ as it is usually presumed.

Definition 2 *We denote $V_n \preceq Y_n (V_n \succeq W_n)$ if*

$$V_n \leqslant (1 + o(1)) \cdot Y_n \ (V_n \geqslant (1 - o(1)) \cdot W_n)$$

when V_n, Y_n (W_n) are sequences of numbers, or

$$\lim_{n\to\infty} P\{V_n \leqslant (1 + o(1)) \cdot Y_n\} = 1 \ (\lim_{n\to\infty} P\{V_n \geqslant (1 - o(1)) \cdot W_n\} = 1)$$

when V_n is a sequence of random variables and Y_n (W_n) is a sequence of numbers or random variables, where $\lim_{n\to\infty} o(1) = 0$.

Definition 3 *We denote $V_n \approxeq Y_n$ if there exist constants $c'' \geqslant c' > 0$ such that*

$$c' \cdot Y_n \preceq V_n \preceq c'' \cdot Y_n$$

where Y_n, V_n are sequences of numbers or random variables.

The following random model of (1) will be considered in the paper:

- $n \to \infty$, $i = 1, \ldots, n$, $j = 1, 2$.
- c_i, a_{ji} are realizations of mutually independent random variables and moreover c_i, a_{ji} are uniformly distributed over $(0, 1]$.
- $0 < \delta \leqslant b_1(n) \leqslant b_2(n) \leqslant n/2$, $b_j(n) \leqslant b_j(n+1)$, for every $n \geqslant 1$ and all $b_j(n)$, $j = 1, 2$, are deterministic, where δ is a constant.

Under the assumptions made about c_i, a_{ji} and $b_j(n)$ the following always hold

$$0 \leqslant z_{OPT}(n) \leqslant \sum_{i=1}^{n} c_i \leqslant n, \ \delta \leqslant b_j(n) \leqslant \sum_{i=1}^{n} a_{ji} \leqslant n, \ j = 1, 2. \qquad (2)$$

Moreover, from the strong law of large numbers it follows that

$$\sum_{i=1}^{n} c_i \approx E(c_1) \cdot n = n/2, \ \sum_{i=1}^{n} a_{ji} \approx E(a_{11}) \cdot n = n/2.$$

Therefore, it is justified to enhance formula (2) in the following way:

$$0 \leqslant z_{OPT}(n) \preceq n/2, \ 0 < \delta \leqslant b_1(n) \leqslant b_2(n) \preceq n/2 \qquad (3)$$

Formula (3) shows that random model of the Two-Constraint Binary Knapsack Problem (1) is complete in the sense that nearly all possible instances of the problem are considered. In this respect the model where $b_1(n) = b_2(n) = 1$ is just a very special case. Taking into account that $\sum_{i=1}^{n} a_{ji} \approx n/2$ assumption that $b_j(n) \leqslant b_j(n+1)$, for all $n \geqslant 1$, is quite logical.

The growth of $z_{OPT}(n)$ - value of the optimal solution of the problem (1) may be influenced by the problem coefficients, namely:

$$n,\ c_i,\ a_{ji},\ b_j(n),\ \text{where } i = 1,\dots,n,\ j = 1,2.$$

We have assumed that c_i, a_{ji} are realizations of the random variables and therefore their impact on the $z_{OPT}(n)$ growth is in this case indirect. Moreover, we have assumed that $n \to \infty$. The aim of the probabilistic analysis is to investigate asymptotic behavior of $z_{OPT}(n)$ when $n \to \infty$. The impact of the right-hand-side values - $b_1(n), b_2(n)$ - is well illustrated by the Lagrange function and the problem dual to (1), see Averbakh [1], Meanti, Rinnooy Kan, Stougie and Vercellis [9], Szkatuła [13] and [14]. Due to the very complicated formulas, impossible to handle in the general case, the papers by Szkatuła [13] and [14] investigate only two important special cases of values of constraints right hand sides in the case of Multi-Constraint Knapsack Problem.

3 Lagrange and dual estimations

When the general knapsack type problem, with one or many constraints, is considered then Lagrange function and the corresponding dual problems, see Averbakh [1], Meanti, Rinnooy Kan, Stougie and Vercellis [9], Szkatuła [13] and [14] are very useful tools to perform various kind of analyses of the original problem. In the specific case of the Two-Constraint Binary Knapsack Problem Lagrange function of the problem (1) may be formulated as follows:

$$\begin{aligned} L_n(x) &= \sum_{i=1}^{n} c_i \cdot x_i + \sum_{j=1}^{2} \lambda_j \cdot \left(b_j(n) - \sum_{i=1}^{n} a_{ji} \cdot x_i \right) = \\ &= \sum_{j=1}^{2} \lambda_j + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{2} \lambda_j \cdot a_{ji} \right) \cdot x_i \end{aligned}$$

where $x = [x_1, \dots, x_n]$ and $\Lambda = [\lambda_1, \lambda_2]$ - vector of Lagrange multipliers. Moreover, let for every Λ, $\lambda_j \geq 0$, $j = 1,2$:

$$\phi_n(\Lambda) = \max_{x \in \{0,1\}^n} L_n(x, \Lambda) = \max_{x \in \{0,1\}^n} \left\{ \sum_{j=1}^{2} \lambda_j \cdot b_j(n) + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{2} \lambda_j \cdot a_{ji} \right) x_i \right\}.$$

Using the following notation:

$$\begin{aligned} x_i(\Lambda) &= \begin{cases} 1 & \text{if } c_i - \sum\limits_{j=1}^{2} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (4) \\ c_i(\Lambda) &= \begin{cases} c_i & \text{if } c_i - \sum\limits_{j=1}^{2} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \\ a_{ji}(\Lambda) &= \begin{cases} a_{ji} & \text{if } c_i - \sum\limits_{j=1}^{2} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

we have for every Λ, $\lambda_j \geq 0$, $j = 1, 2$:

$$\begin{aligned}
\phi_n(\Lambda) &= \sum_{j=1}^{2} \lambda_j \cdot b_j(n) + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{2} \lambda_j \cdot a_{ji} \right) \cdot x_i(\Lambda) = \\
&= \sum_{j=1}^{2} \lambda_j \cdot b_j(n) + \sum_{i=1}^{n} \left(c_i(\Lambda) - \sum_{j=1}^{2} \lambda_j \cdot a_{ji}(\Lambda) \right)
\end{aligned}$$

Obviously for $i = 1, \ldots, n$, $j = 1, 2$,

$$c_i(\Lambda) = c_i \cdot x_i(\Lambda), \quad a_{ji}(\Lambda) = a_{ji} \cdot x_i(\Lambda).$$

Dual problem to Two-Constraint Binary Knapsack Problem (1) maybe formulated as follows:

$$\Phi_n^* = \min_{\Lambda \geq 0} \phi_n(\Lambda). \tag{5}$$

For every $\Lambda \geq 0$ the following holds:

$$z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + \sum_{j=1}^{2} \lambda_j (b_j(n) - s_j(\Lambda)). \tag{6}$$

Let us denote:

$$\begin{aligned}
z_n(\Lambda) &= \sum_{i=1}^{n} c_i \cdot x_i(\Lambda) = \sum_{i=1}^{n} c_i(\Lambda), s_j(\Lambda) = \sum_{i=1}^{n} a_{ji} \cdot x_i(\Lambda) = \sum_{i=1}^{n} a_{ji}(\Lambda), \\
S_n(\Lambda) &= \sum_{j=1}^{2} \lambda_j \cdot s_j(\Lambda), \ B(\Lambda) = \sum_{j=1}^{2} \lambda_j \cdot b_j(n).
\end{aligned}$$

By definition of $c_i(\Lambda)$ and $a_{ji}(\Lambda)$, see also (4), we have:

$$c_i(\Lambda) \geq \sum_{j=1}^{2} \lambda_j \cdot a_{ji}(\Lambda), \ i = 1, \ldots, n,$$

and therefore

$$z_n(\Lambda) \geq S_n(\Lambda). \tag{7}$$

For certain Λ, $x_i(\Lambda)$ given by (4) may provide feasible solution of (1), i.e.:

$$s_j(\Lambda) \leq b_j(n) \quad \text{for every} \quad j = 1, 2. \tag{8}$$

Then:

$$z_n(\Lambda) \leq z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + B(\Lambda) - S_n(\Lambda). \tag{9}$$

If (8) holds, then the below inequality also holds:

$$B(\Lambda) - S_n(\Lambda) \geq 0.$$

From (7) we get:

$$\frac{\phi_n(\Lambda)}{z_n(\Lambda)} = \frac{z_n(\Lambda)}{z_n(\Lambda)} + \frac{B(\Lambda) - S_n(\Lambda)}{z_n(\Lambda)} \leq 1 + \frac{B(\Lambda) - S_n(\Lambda)}{S_n(\Lambda)}.$$

Therefore if (8) holds, then the following inequality also holds:

$$1 \leq \frac{z_{OPT}(n)}{z_n(\Lambda)} \leq \frac{\Phi_n^*}{z_n(\Lambda)} \leq \frac{\phi_n(\Lambda)}{z_n(\Lambda)} \leq \frac{B(\Lambda)}{S_n(\Lambda)}. \tag{10}$$

Formula (10) shows, that if there exits such a set of Lagrange multipliers $\Lambda(n)$ which is fulfilling the formula (8) and if the formula below holds:

$$\lim_{n\to\infty} \frac{B(\Lambda(n))}{S_n(\Lambda(n))} = 1 \tag{11}$$

then, due to (10), $\lim_{n\to\infty} \frac{z_{OPT}(n)}{z_n(\Lambda(n))} = 1$ and therefore $x_i(\Lambda(n))$, $i = 1, \ldots, n$, given by (4), is the asymtotically sub-optimal solution of the Two-Constraint Binary Knapsack Problem (1). Moreover the value of $z_n(\Lambda(n))$ is an asymptotical approximation of the optimal solution value of the Two-Constraint Binary Knapsack Problem i.e. $z_{OPT}(n)$.

4 Probabilistic analysis

In the present section of the paper some probabilistic properties of the Two-Constraint Binary Knapsack Problem (1) will be investigated. We have assumed that that c_i, a_{ji} $i = 1, \ldots, n$, $j = 1, 2$ are realizations of mutually independent random variables and moreover c_i, a_{ji} are uniformly distributed over $(0, 1]$. Moreover we have assumed that $0 < \delta \leqslant b_1(n) \leqslant b_2(n) \leqslant n/2$, $b_j(n) \leqslant b_j(n+1)$. In addition we will assume that Lagrange multipliers λ_1 and λ_2, $\lambda_2 \leq \lambda_1$, $\Lambda = (\lambda_1, \lambda_2)$ are also deterministic. Monotonicty of constraints right hand sides, $b_1(n) \leqslant b_2(n)$, is in this case determinig montonicity of the Lagrange multipliers, $\lambda_2 \leq \lambda_1$. This is pretty standart probabilistic model of the general knapsack problems and it suits well also to Two-Constraint Binary Knapsack Problem (1).

Let us first observe that due to the assumptions made the following holds, for $i = 1, \ldots, n$, $j = 1, 2$:

$$P(a_{ji} < x) = \begin{cases} 0 & \text{when } x \leqslant 0 \\ x & \text{when } 0 < x \leqslant 1 \\ 1 & \text{when } x \geqslant 1 \end{cases}, \quad P(c_i < x) = \begin{cases} 0 & \text{when } x \leqslant 0 \\ x & \text{when } 0 < x \leqslant 1 \\ 1 & \text{when } x \geqslant 1 \end{cases}. \tag{12}$$

In order to preceed with probabilistic analysis of the Two-Constraint Binary Knapsack Problem (1) it is neccesary to consder probalisitc distribution of the following random variables

$$\sum_{j=1}^{k} \lambda_j \cdot a_{ji}, \; k = 1 \text{ or } 2$$

Let $(x)_+ = \frac{|x|+x}{2} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$, $j^* = \begin{cases} 1 & \text{if } j=2 \\ 2 & \text{if } j=1 \end{cases}$, Then for or $i = 1, \ldots, n$, $j = 1, 2$, the following holds:

$$\begin{aligned} F_1(x, \lambda_j) &= P\{\lambda_j \cdot a_{ji} < x\} = \frac{1}{\lambda_j}((x)_+ - (x-\lambda_j)_+), \\ F_2(x, \Lambda) &= P\{\lambda_1 \cdot a_{1i} + \lambda_2 \cdot a_{2i} < x\} = \frac{1}{\lambda_j}\int_0^1 F_1(x - \lambda_{j^*} t, \Lambda \backslash \lambda_j) dt = \\ &= \frac{1}{\lambda_1 \cdot \lambda_2}\left((x)_+^2 - (x-\lambda_1)_+^2 - (x-\lambda_2)_+^2 + (x-\lambda_1-\lambda_2)_+^2\right) \end{aligned} \tag{13}$$

The distribution functions of the random variables $a_{ji}(\Lambda)$, $c_i(\Lambda)$, $i = 1, \ldots, n$, $j = 1, 2$ are:

$$\begin{aligned} G_{ji}(x, \Lambda) &= P\{a_{ji}(\Lambda) < x\} = \\ &= P\left\{a_{ji} < x \bigcup a_{ji} \geq x \bigcap \sum_{k=1}^{2} \lambda_k \cdot a_{ik} \geq c_i\right\} = \\ &= 1 - \int_x^1 \int_0^1 F_1(r - \lambda_j \cdot t, \Lambda \backslash \lambda_j) dr dt \end{aligned} \tag{14}$$

$$\begin{aligned} H_i(x, \Lambda) &= P\{c_i(\Lambda) < x\} = \\ &= P\left\{c_i < x \bigcup c_i \geq x \bigcap \sum_{k=1}^{2} \lambda_k \cdot a_{ik} \geq c_i\right\} = \\ &= 1 - \int_x^1 F_2(t, \Lambda) dt, \end{aligned} \tag{15}$$

Using above fromulas (14) and (15) expectations of the $a_{ji}(\Lambda)$, $c_i(\Lambda)$ could be expressed as follows:

$$\begin{aligned} E(a_{ji}(\Lambda)) &= \int_0^1 x dG_{ji}(x, \Lambda) = \int_0^1 x \int_0^1 F_1(r - \lambda_j \cdot x, \Lambda \backslash \lambda_j) dr dx = \\ &= \frac{1}{\lambda_{j^*}}\left(\int_0^1 x \int_0^1 ((r - x \cdot \lambda_j)_+ - (r - x \cdot \lambda_j - \lambda_{j^*})_+) dr dx\right) \end{aligned} \tag{16}$$

$$E(c_i(\Lambda)) = \int_0^1 x dH_i(x,\Lambda) = \int_0^1 x \cdot F_2(x,\Lambda)dx = \tag{17}$$

$$= \frac{1}{2\cdot\lambda_1\cdot\lambda_2}\int_0^1 x\cdot\left((x)_+ - (x-\lambda_1)_+^2 - (x-\lambda_2)_+^2 + (x-\lambda_1-\lambda_2)_+^2\right)dx =$$

$$= \frac{1}{2\cdot\lambda_1\cdot\lambda_2}\left(\frac{1}{4} - \int_0^1 x\cdot\left((x-\lambda_1)_+^2 + (x-\lambda_2)_+^2 - (x-\lambda_1-\lambda_2)_+^2\right)dx\right).$$

It is easy to observe that above formulas (16) and (17) may take different values, depending on the mutual relations between λ_1, λ_2 and x, r since several items of the formulas above may become 0 or be strongly postive. 4 specific cases could be distiguished for $i = 1,\ldots,n$, $j = 1,2$:

1. Case of "large" values of the Lagrange multipliers $1 \le \lambda_2 \le \lambda_1$. In this case:

$$E(a_{ji}(\Lambda)) = \frac{1}{\lambda_{j^*}}\int_0^{1/\lambda_j} x\int_{x\cdot\lambda_j}^1 (r - x\cdot\lambda_j)drdx = \frac{1}{24\cdot\lambda_j^2\cdot\lambda_{j^*}} \tag{18}$$

$$E(c_i(\Lambda)) = \frac{1}{2\cdot\lambda_1\cdot\lambda_2}\int_0^1 x^3dx = \frac{1}{8\cdot\lambda_1\cdot\lambda_2}.$$

2. Case of "mixed" values of the Lagrange multipliers $\lambda_2 \le 1 \le \lambda_1$. In this case:

$$E(a_{1i}(\Lambda)) = \frac{1}{\lambda_2}\left(\int_0^{1/\lambda_1} x\int_{x\cdot\lambda_1}^1 (r - x\cdot\lambda_1)drdx - \right. \tag{19}$$

$$\left. - \int_0^{\frac{1-\lambda_2}{\lambda_1}} x\int_{(x\cdot\lambda_1+\lambda_2)}^1 (r - x\cdot\lambda_1 - \lambda_2)drdx\right) =$$

$$= \frac{1}{24}\frac{4 - 6\lambda_2 - \lambda_2^3 + 4\lambda_2^2}{\lambda_1^2},$$

$$E(a_{2i}(\Lambda)) = \frac{1}{\lambda_1}\left(\int_0^1 x\int_{x\cdot\lambda_2}^1 (r - x\cdot\lambda_2)drdx\right) = \frac{1}{24}\frac{3\lambda_2^2 - 8\lambda_2 + 6}{\lambda_1},$$

$$E(c_i(\Lambda)) = \frac{1}{2\cdot\lambda_1\cdot\lambda_2}\left(\frac{1}{4} - \int_{\lambda_2}^1 x\cdot(x-\lambda_2)^2dx\right) =$$

$$= \frac{1}{24\lambda_1}\left(\lambda_2^3 - 6\lambda_2 + 8\right).$$

3. Case of "moderate" values of the Lagrange multipliers $\lambda_2 \le \lambda_1 \le 1$,

$\lambda_2 + \lambda_1 \geq 1$. In this case:

$$E(a_{ji}(\Lambda)) = \frac{1}{\lambda_{j^*}} \left(\int_0^1 x \int_{x\cdot\lambda_j}^1 (r - x\cdot\lambda_j) dr dx - \right. \tag{20}$$

$$\left. - \int_0^{(1-\lambda_{j^*})/\lambda_j} x \int_{(x\cdot\lambda_j+\lambda_{j^*})}^1 (r - x\cdot\lambda_j - \lambda_{j^*}) dr dx \right) =$$

$$= \frac{1}{24} \frac{3\lambda_j^4 - 8\lambda_j^3 + 6\lambda_j^2 - 6\lambda_{j^*}^2 + 4\lambda_{j^*} - \lambda_{j^*}^4 + 4\lambda_{j^*}^3 - 1}{\lambda_j^2 \lambda_{j^*}},$$

$$E(c_i(\Lambda)) = \frac{1}{2\cdot\lambda_1\cdot\lambda_2} \left(\frac{1}{4} - \int_{\lambda_1}^1 x\cdot(x-\lambda_1)^2 dx - \int_{\lambda_2}^1 x\cdot(x-\lambda_2)^2 dx \right) =$$

$$= \frac{1}{24\lambda_1\lambda_2} \left(\lambda_1^4 - 6\lambda_1^2 + 8\lambda_1 + \lambda_2^4 - 6\lambda_2^2 + 8\lambda_2 - 3 \right).$$

4. Case of "small" values of the Lagrange multipliers $\lambda_2 \leq \lambda_1 \leq 1$, $\lambda_2 + \lambda_1 \leq 1$. In this case:

$$E(a_{ji}(\Lambda)) = \frac{1}{\lambda_{j^*}} \left(\int_0^1 x \int_{x\cdot\lambda_j}^1 (r - x\cdot\lambda_j) dr dx - \right. \tag{21}$$

$$\left. - \int_0^1 x \int_{(x\cdot\lambda_j+\lambda_{j^*})}^1 (r - x\cdot\lambda_j - \lambda_{j^*}) dr dx \right) =$$

$$= \frac{1}{2} - \frac{1}{3}\lambda_j - \frac{1}{4}\lambda_{j^*},$$

$$E(c_i(\Lambda)) = \frac{1}{2\cdot\lambda_1\cdot\lambda_2} \left(\frac{1}{4} - \int_{\lambda_1}^1 x\cdot(x-\lambda_1)^2 dx - \int_{\lambda_2}^1 x\cdot(x-\lambda_2)^2 dx + \right.$$

$$\left. + \int_{\lambda_1+\lambda_2}^1 x\cdot(x-\lambda_1-\lambda_2)^2 dx) \right) =$$

$$= \frac{1}{2} - \frac{1}{6}\lambda_1^2 - \frac{1}{4}\lambda_1\lambda_2 - \frac{1}{6}\lambda_2^2.$$

Probablistic, or in other words average case, analysis consists in determining such Lagrange multipiers $= (\lambda_1(n), \lambda_2(n))$ that when $n \to \infty$, $x_i(\Lambda(n))$, $i = 1, \ldots, n$, defined by (4) will provide solutions of the Two-Constraint Binary Knapsack Problem (1) which are, in the sense of convenrgence in probability, see Loeve [5], providing solutions which are asymptotically feasible , i.e. $s_j(\Lambda(n))$ is satifying (8) and moreover if $S_n(\Lambda(n))$ is fulfilling (11) then, due to (10), $\lim_{n\to\infty} \frac{z_{OPT}(n)}{z_n(\Lambda(n))} = 1$ and $z_n(\Lambda(n))$ is suboptimal solution of the (1) and morever

$$z_{OPT}(n) \approx z_n(\Lambda(n)) \approx E(z_n(\Lambda(n))).$$

The above goal may be achived by determing $\Lambda(n)$ as the solution of the following system of equations:

$$E(s_1(\Lambda(n))) = b_1'(n), \quad E(s_2(\Lambda(n))) = b_2'(n), \tag{22}$$

where $b'_1(n) < b_1(n)$ and $b'_2(n) < b_2(n)$ and $\Lambda(n)$ is fulfilling both (8) and (11). Each of the 4 cases mentioned above should be consdiered independently. Let us observe that $E(s_j(\Lambda(n))) = n \cdot E(a_{j1}(\Lambda(n)))$, $E(z_n(\Lambda(n))) = n \cdot E(c_1(\Lambda(n)))$.

Lemma 1 *If c_i, a_{ji} $i = 1, \ldots, n$, $j = 1, 2$, are realizations of mutually independent random variables uniformly distributed over $(0, 1)$, and if $1 \leq \lambda_2 \leq \lambda_1$ then*

$$\lambda_1(n) = \frac{1}{b'_1(n)} \sqrt[3]{\frac{n \cdot b'_1(n) \cdot b'_2(n)}{24}}, \quad \lambda_2(n) = \frac{1}{b'_2(n)} \sqrt[3]{\frac{n \cdot b'_1(n) \cdot b'_2(n)}{24}}$$

is the solution of (22) and

$$E(z_n(\Lambda(n))) = 3 \cdot \sqrt[3]{\frac{n \cdot b'_1(n) \cdot b'_2(n)}{24}}.$$

Proof. Above formulas follow immediately from the (18) and (22). ■

5 Concluding remarks

In the present paper some preliminary results describing probabilistic properties of the Two-Constraint Binary Knapsack Problem (1) are considered.

In the paper distribution functions of the various random variables representing important problems characteristics are presented.

Future research should be devoted to investiagting remaing 3 cases of the mutual relations between $\lambda_1(n)$ and $\lambda_2(n)$, feasibility of the received solutions and estimations of the Two-Constraint Binary Knapsack Problem (1) optimal solution values $z_{OPT}(n)$ growth, when $n \to \infty$

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