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Simulating failure-repair process and evaluating reliability parameters for single-source multiple-sink commodity transportation network with stochastically dependent components

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1. Introduction

In the current paper a commodity transfer system composed of nodes and links arranged in a tree structure is considered. Its task is to transfer a commodity (electric power, radio signal, electronic data, water, gas, etc.) from the source (root) node to all destination (leaf) nodes. Let $\{v_0,v_1,...,v_n\}$ and $\{a_1,...,a_n\}$ be the sets of nodes and links respectively, where a_i connects v_i to its parent node, $1 \le i \le n$. Let $\{e_0, e_1,...,e_N\}$ be the set of all network components, $N = 2n$. The components are indexed so that the index of each non-root component is greater than the index of its parent component, e_0 being the root component, i.e. the source node v_0. An exemplary network consisting of 13 nodes and 12 links is presented in Fig. 1.

Each component can be in one of two states – operable and failed; e_0 is always operable. The repair of a failed component begins as soon as a repair teams is available – due to a limited number of such teams the repair may not start immediately after the component's failure. The order in which failed components are chosen for repair depends on the repair policy applied – two of such policies will be considered. The functioning of the component e_i, $1 \le i \le N$, is characterized by three distribution functions: F_i – lifetime d.f. of the operable e_i connected to e_0, G_i – lifetime d.f. of the operable e_i disconnected from e_0, and H_i – repair time d.f. of the failed e_i. It is assumed that F_i and G_i are exponential, unlike H_i which can be arbitrary d.f. on $[0,\infty)$. It is also assumed that $F_i \ge G_i$, which conveys the idea that the components being "under load" are more failure prone, as in many real-life systems. Thus, a component's lifetime depends on the behavior of all „upstream" components but is not influenced by the remaining components (i.e. **not located** between a given component and e_0). However, a component functions independently of the "upstream" components up to the moment when one of them fails. Furthermore, a component's repair time is independent of the states of all other components. Note that e_i directly connected to e_0 is only characterized by F_i and H_i – there is no G_i for such e_i. Also note that $G_i \equiv 0$ if, by assumption, e_i disconnected from e_0 cannot fail.

The above characteristic makes the proposed network model more true-to-life in comparison with many probabilistic models of complex systems which often postulate full independence of components. In our case it would mean that the life and repair times of all components are independent random variables. Obviously, independence of components is mainly taken for granted due to computational simplicity. In reality, however, such independence rarely occurs.

A commodity can be transferred from v_0 to v_i, $1 \leq i \leq n$, if and only if v_i is operable and connected to v_0, i.e. all components between v_0 and v_i (including v_i) are in the operable state. As failures of components occur, the periods of connection between v_0 and the operable v_i are interleaved by the periods during which v_i is failed or disconnected from v_0. The aim pursued in this paper is to determine the mean durations of both time intervals, and the average number of reconnections between v_0 and the operable v_i. in a given time period (e.g. month or year).

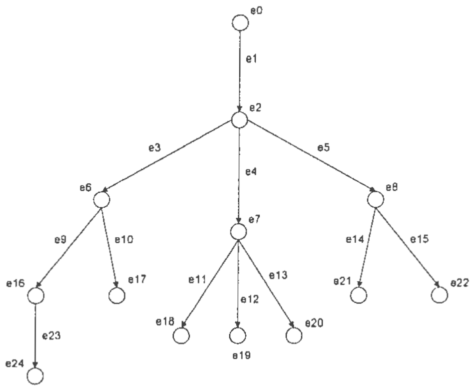

Figure 1. An exemplary system structure

Clearly, connection and disconnection times of individual components depend on two more factors: the number of repair teams assigned to the network maintenance, and the repair policy implemented. As to the first factor, the mean disconnection times increase as the number of repair teams decreases, and vice versa. If there are fewer than N repair teams, a component's repair may not start immediately after its failure, but the average delay decreases along with the increasing number of repair teams, and is equal to zero if this number reaches N. However, such case should be considered only for theoretical purposes, because in practice the number of repair teams is usually considerably smaller than the number of all components. For example, the mean disconnection times computed for N repair teams can be used as lower bounds of respective mean times in case of fewer than N repair teams.

Three repair policies will be studied here. According to the first policy the components are chosen for repair in the order in which they failed, i.e. they form a FIFO queue; if multiple components fail at the same time (**such event occurs with zero probability unless it is a common cause failure**), the one with the largest index is selected as first. This policy will be named "FIFO with largest index priority". If $G_i \equiv 0$, $1 \leq i \leq N$, then for each "linear" subset of components (i.e. all components located between e_0 and a leaf node) it prioritizes the components most distant from e_0. Indeed, as only the components connected to e_0 can fail, if e_y is located below e_x (yielding y>x), then e_y can only fail if e_x is operable, i.e. e_y can only fail before or simultaneously with e_x, hence e_y must precede e_x in the queue for repair.

According to the second policy the components are selected for repair in the order reverse to that in which they failed, i.e they form a LIFO queue; if multiple components fail at the same time, the one with the smallest index is selected as first. This policy will be named "LIFO with smallest index priority". If $G_i \equiv 0$, $1 \leq i \leq N$, then for each "linear" subset of components it prioritizes the components least distant from e_0. Indeed, if e_x is located above e_y (yielding y>x), then e_x can only fail after or simultaneously with e_y, hence e_x must precede e_y in the queue for repair.

The third policy prioritizes the components according to their indexes, i.e. the first component in the queue for repair is the one with the smallest index. This policy will be named "smallest index priority". Note that it does not take into account the order in which the components fail, and is only determined by the numbering of the components. Certainly, in general case, the numbering scheme reflecting a particular repair policy for a multi-component system can be different from the one adopted in this paper.

It is obvious that applying a priority-based repair policy is only necessary if there are less than N repair teams. Otherwise the only feasible strategy is "repair a component upon its failure", as there is always at least one team available when a component fails.

2. Notation and definitions

We will use the following notation:

$L_i^{(1)}$ – random lifetime of operable e_i connected to e_0

$L_i^{(2)}$ – random lifetime of operable e_i disconnected from e_0

R_i – random repair time of failed e_i

F_i, G_i, H_i – distribution functions of $L_i^{(1)}$, $L_i^{(2)}$, and R_i respectively

λ_i – failure intensity of operable e_i connected to e_0

$\varphi_j^{(i)}$ – the moment when, for the j-th time, e_i fails or becomes disconnected from e_0

$\rho_j^{(i)}$ – the moment when, for the j-th time, e_i becomes operable and connected to e_0; by assumption $\rho_0^{(i)} = 0$

$A_j^{(i)}$ – length of time from $\rho_{j-1}^{(i)}$ to $\varphi_j^{(i)}$, i.e. the length of the j-th period during which operable e_i remains connected to e_0

$B_j^{(i)}$ – length of time from $\varphi_j^{(i)}$ to $\rho_j^{(i)}$, i.e. the length of the j-th period during which e_i remains failed or disconnected from e_0

π_1 – the "FIFO with largest index priority" repair policy

π_2 – the "LIFO with smallest index priority" repair policy

π_3 – the "smallest index priority" repair policy

a(i) – the average length of a period during which the operable e_i remains connected to e_0

b(i) – the average length of a period during which e_i remains failed or disconnected from e_0

c(i) – the average number of reconnections between the operable e_i and e_0 per unit time

In order not to complicate the notation, it will be assumed by default that the system is maintained by r repair teams, and the repair policy π_s is applied, s = 1, 2, or 3. By definition

$$(1) \qquad a(i) = \lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} A_j^{(i)}, \quad b(i) = \lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} B_j^{(i)}, \quad c(i) = \lim_{m\to\infty} \frac{m}{\sum_{j=1}^{m} [A_j^{(i)} + B_j^{(i)}]}$$

The formula for a(i) is readily obtained. Indeed, for any j≥1 all components between e_i and e_0 are operational at the instant $\rho_{j-1}^{(i)}$. Due to the exponential distribution's "lack of memory" property the time from $\rho_{j-1}^{(i)}$ to the failure of any e_k located between e_i and e_0 has d.f. F_k, provided that e_k remains connected to e_0 up to the moment of failure. We thus have:

$$(2) \qquad \begin{aligned} \Pr(A_j^{(i)} \le t) &= \Pr(\min(L_k^{(i)} : e_k \lhd e_i) \le t) \\ &= 1 - \exp[-\sum (\lambda_k : e_k \lhd e_i)] \end{aligned}$$

for j≥1, where $e_k \lhd e_i$ denotes that e_k is located above e_i or $e_k = e_i$. From (2) it follows that

$$(3) \qquad E(A_j^{(i)}) = \left[\sum (\lambda_k : e_k \lhd e_i)\right]^{-1}$$

Thus the mean values of $A_j^{(i)}$ are equal for j≥1. The law of large numbers yields immediately that $(A_1^{(i)} + \ldots + A_m^{(i)})/m$ converges in probability to a(i) given by the right-hand side of (3).

In general case b(i) is a random variable, **provided it exists**. Under certain assumptions (see Lemma 2) b(i) is also a constant value to which $(B_1^{(i)} + \ldots + B_m^{(i)})/m$ converges in probability as $m\to\infty$. In turn, from the definition of c(i), we have:

(4) $c(i) = [a(i) + b(i)]^{-1}$

Clearly, the existence of b(i) is necessary for (4) to hold.

3. Theoretical basis for the Monte Carlo estimation of b(i)

In this chapter a theoretical background for evaluating b(i) is presented. Clearly, constructing exact analytical formulas for this parameter is beyond question, as can be concluded from the example of a two-component system analyzed in [7]. Thus, b(i) will be estimated using Monte Carlo simulation. When estimating parameters of a stochastic process the following problem is often encountered: whether statistical data may come from one sample path (realization) of the process or should they be collected from multiple sample paths? Clearly, one sample path is sufficient in case of a recurrent process, i.e. a process $X=\{X(t), t \geq 0\}$ with the following properties:

- the state of X at $t = 0$ is fixed, i.e. X(0) has the one-point distribution,
- with probability one X returns to the state X(0) after finite time, measured from 0,
- if $Y(t)=X(\tau_1+t)$, where τ_1 is the (random) time of the first return of X to its initial state, then $Y=\{Y(t), t \geq 0\}$ and X are stochastically identical processes (X begins anew at $t = \tau_1$).

Sometimes the second property is replaced with the stronger one, i.e. $E(\tau_1)<\infty$. For details see [6].

Let $\underline{X}$ be the vector valued stochastic process $\{[X_t^{(1)},\dots,X_t^{(N)}], t \geq 0\}$, where $X_t^{(i)}$ denotes the state of the component e_i at time t. For recurrent $\underline{X}$ we define:

τ_k – the moment of the k-th return of $\underline{X}$ to the state $\underline{1}$, $k \geq 0$, $\tau_0 = 0$

$C_k^{(i)}$ – the total time within $(\tau_{k-1}, \tau_k]$ during which the operable e_i remains connected to e_0

$D_k^{(i)}$ – the total time within $(\tau_{k-1}, \tau_k]$ during which e_i remains failed or disconnected from e_0.

$Q_k^{(i)}$ – the number of periods in $(\tau_{k-1}, \tau_k]$ during which e_i remains failed or disconnected from e_0

Obviously, $\{C_k^{(i)}, k \geq 1\}$, $\{D_k^{(i)}, k \geq 1\}$, and $\{Q_k^{(i)}, k \geq 1\}$ are sequences of IIDRV. Moreover, $C_k^{(i)} + D_k^{(i)} = \tau_k - \tau_{k-1}$, $k \geq 1$. The following lemma gives some necessary conditions for $\underline{X}$ to be recurrent.

Lemma 1

If $G_i \equiv 0$ for $1 \leq i \leq N$, e_1 is located above all other components and the repair policy π_1 is applied, then $\underline{X}$ is a recurrent process.

Proof

Note that $\underline{X}_0 = \underline{1}$ and $\underline{X}$ begins anew at any moment t when $\underline{X}_t = \underline{1}$, due to the "lack of memory" property of the exponential distribution. Moreover,

(5) $$E(\tau_1) = E(L_1) + E(\tau_1 - L_1),$$

and

(6) $$\begin{aligned} &E(\tau_1 - L_1) = \\ &\quad = E(\tau_1 - L_1 \mid \tau_1 > L_1)\Pr(\tau_1 > L_1) + \\ &\quad + E(\tau_1 - L_1 \mid \tau_1 \leq L_1)\Pr(\tau_1 \leq L_1) \leq \\ &\quad \leq E(\tau_1 - L_1 \mid \tau_1 > L_1) \end{aligned}$$

From (5), (6), and the analysis of the worst case scenario, i.e. "there is only one repair team and all components are failed when e_1 fails", it follows that

(7) $$E(\tau_1) \leq \frac{1}{\lambda_1} + \sum_{i=1}^{N} E(R_i)$$

Thus the stronger version of the second property is fulfilled if $E(R_i)<\infty$, $1 \le i \le N$. For the weaker version it is sufficient that $Pr(R_i<\infty)=1$.

If the policy π_2 or π_3 is applied, then the question whether $\underline{X}$ is recurrent remains open, even for exponentially distributed $L_1,\ldots,L_N$. Most likely, some additional assumptions regarding the distribution functions $H_1,\ldots,H_N$ should be made to ensure that $\underline{X}$ possesses this property.

It follows from (2) that $A_1^{(i)}$, $A_2^{(i)}$,... are independent identically distributed random variables (IIDRV). However, $B_1^{(i)}$, $B_2^{(i)}$,... may not be IIDRV, e.g. if $R_1,\ldots,R_N$ are not exponentially distributed. In consequence, defining b(i) as the average time during which e_i remains failed or disconnected from e_0, one must remember that the successive periods of disconnection may not have one distribution function, therefore in this context "average" is not equivalent to "expected value of". The proper meaning of thus defined b(i) is given by the following lemma.

Lemma 2

If the assumptions of Lemma 1 are fulfilled, and $0 < r_{min} \le R_i \le r_{max} < \infty$ for $1 \le i \le N$, then

$$(8) \qquad \frac{1}{m}\sum_{j=1}^{m} B_j^{(i)} \rightarrow_{prob} \frac{E(D_1^{(i)})}{E(Q_1^{(i)})}$$

as $m \rightarrow \infty$, where $\rightarrow_{prob}$ denotes convergence in probability. Thus b(i) is a constant value equal to the right-hand side of (8).

Proof

We first prove the second part of (8). By virtue of (7)

$$(9) \qquad E(D_1^{(i)}) \le E(\tau_1) \le \frac{1}{\lambda_1} + E(R_1 + \ldots + R_N) \le \frac{1}{\lambda_1} + N \cdot r_{max}$$

i.e. $E(D_1^{(i)})$ is finite, thus we have:

$$E(D_1^{(i)}) = \int_0^{\infty}[1 - H(t)]dt = \sum_{j=1}^{\infty} \int_{(j-1)r_{\min}}^{jr_{\min}} [1 - H(t)]dt \geq$$

(10)

$$\geq r_{\min} \sum_{j=1}^{\infty} [1 - H(j \cdot r_{\min})] = r_{\min} \sum_{j=1}^{\infty} \Pr(D_1^{(i)} \geq j \cdot r_{\min})$$

where H is the distribution function of $D_1^{(i)}$. It is also true that:

(11) $$E(Q_1^{(i)}) = \sum_{j=1}^{\infty} j \cdot \Pr(Q_1^{(i)} = j) = \sum_{j=1}^{\infty} \Pr(Q_1^{(i)} \geq j) \leq \sum_{j=1}^{\infty} \Pr(D_1^{(i)} \geq j \cdot r_{\min})$$

where the last inequality is a consequence of the following implication: if $Q_1^{(i)} \geq j$, then at least j repairs are performed from $\varphi_1^{(i)}$ to τ_1. From (10), (11), and (7) we obtain:

(12) $$E(Q_1^{(i)}) \leq \frac{E(D_1^{(i)})}{r_{\min}} \leq \frac{1}{r_{\min}} \left(\frac{1}{\lambda_1} + N \cdot r_{\max} \right)$$

i.e. $E(Q_1^{(i)})$ is finite.

Let K(i, m), m≥0, be an integer valued random variable equal to k if the interval $(\varphi_m^{(i)}, \rho_m^{(i)}]$ is included in the interval $(\tau_{k-1}, \tau_k]$, i.e. K(i, m) = k if $Q_1^{(i)} + \ldots + Q_{k-1}^{(i)} < m \leq Q_1^{(i)} + \ldots + Q_k^{(i)}$, where $Q_0^{(i)} = 0$. Obviously K(i, m) is non-decreasing in m. As a consequence of (12)

(13) $$\lim_{m \to \infty} K(i, m) = \infty$$

holds with probability 1. For m such that $Q_1^{(i)} + ... + Q_{K(i,m)-1}^{(i)} > 0$ we have:

$$(14) \qquad \frac{D_1^{(i)} + \ldots + D_{K(i,m)-1}^{(i)}}{Q_1^{(i)} + \ldots + Q_{K(i,m)}^{(i)}} \le \frac{1}{m}\sum_{j=1}^{m} B_j^{(i)} \le \frac{D_1^{(i)} + \ldots + D_{K(i,m)}^{(i)}}{Q_1^{(i)} + \ldots + Q_{K(i,m)-1}^{(i)}}$$

$$(15) \qquad \frac{D_1^{(i)} + \ldots + D_{K(i,m)-1}^{(i)}}{Q_1^{(i)} + \ldots + Q_{K(i,m)}^{(i)}} = \frac{D_1^{(i)} + \ldots + D_{K(i,m)}^{(i)}}{Q_1^{(i)} + \ldots + Q_{K(i,m)}^{(i)}} - \frac{D_{K(i,m)}^{(i)}}{Q_1^{(i)} + \ldots + Q_{K(i,m)}^{(i)}}$$

$$(16) \qquad \frac{D_1^{(i)} + \ldots + D_{K(i,m)}^{(i)}}{Q_1^{(i)} + \ldots + Q_{K(i,m)-1}^{(i)}} = \frac{D_1^{(i)} + \ldots + D_{K(i,m)-1}^{(i)}}{Q_1^{(i)} + \ldots + Q_{K(i,m)-1}^{(i)}} + \frac{D_{K(i,m)}^{(i)}}{Q_1^{(i)} + \ldots + Q_{K(i,m)-1}^{(i)}}$$

From (13) – (16) it follows that:

$$(17) \qquad \lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m} B_j^{(i)} = \lim_{m\to\infty}\frac{D_1^{(i)} + \ldots + D_{K(i,m)}^{(i)}}{Q_1^{(i)} + \ldots + Q_{K(i,m)}^{(i)}} = \lim_{k\to\infty}\frac{D_1^{(i)} + \ldots + D_k^{(i)}}{k}\cdot\frac{k}{Q_1^{(i)} + \ldots + Q_k^{(i)}}$$

Now (8) is obtained by applying the Khinchin law of large numbers to (17).

Corollary: Under the assumptions of Lemma 2 we have:

$$(18) \qquad \frac{m}{\sum_{j=1}^{m}[A_j^{(i)} + B_j^{(i)}]} \to_{\text{prob}} \left(a(i) + \frac{E(D_1^{(i)})}{E(Q_1^{(i)})}\right)^{-1}$$

as $m \to \infty$, where a(i) is given by the right hand side of (3).

4. Simulation technique to estimate b(i)

Let us first modify the notation from chapter 3 so that it is better suited to describe the estimation algorithm presented in this section. Thus $C^{(i)}$, $D^{(i)}$, and $Q^{(i)}$ will denote random variables with the same distributions as $C_1^{(i)}$, $D_1^{(i)}$, and $Q_1^{(i)}$ respectively, while $C_k^{(i)}$, $D_k^{(i)}$, or $Q_k^{(i)}$ will denote the k-th element of the random sample from $C^{(i)}$, $D^{(i)}$, or $Q^{(i)}$, $1 \leq k \leq K$.

In order to compute b(i) from (8) we first need to estimate $E(D^{(i)})$ and $E(Q^{(i)})$. For this purpose we will simulate a sample path of the process $X_t^{(i)}$ describing the behavior of the component e_i (see Lemma 1). Obviously, the component e_i can fail, or become disconnected from or reconnected to e_0, solely at the instants T_h, $h \geq 1$, which are the consecutive moments when any component changes its operational state, i.e. either the component fails or its repair is finished. In consequence, the estimation algorithm consists of the following tasks:

1) Generate the sequence $\{T_h, h \geq 1\}$, and the states of all components at the instants T_h. The last in this sequence is the moment when the process $\underline{X}$ returns to its initial state $\underline{1}$.
2) Compute sample values of the variables $D^{(i)}$ and $Q^{(i)}$ from the sequence $\{T_h, h \geq 1\}$.
3) Estimate b(i) in the following way

$$(19) \qquad b(i) \approx \frac{D_1^{(i)} + \ldots + D_K^{(i)}}{Q_1^{(i)} + \ldots + Q_K^{(i)}}$$

i.e. as the quotient of respective sample means from K-element random samples obtained by repeating K times the tasks 1 and 2.

In order to perform Task 1 we need a procedure generating T_h from T_{h-1}, $h \geq 1$. This is accomplished by Procedure 1, outlined below.

Variables used by Procedure 1:

$X_h^{(i)}$: the state of e_i at T_h ; it is assumed that:

$X_h^{(i)} = -q$, if e_i occupies the q-th place in the queue of components awaiting repair,

$X_h^{(i)} = 0$, if e_i is under repair,

$X_h^{(i)} = 1$, if e_i is operable and connected to e_0,

$X_h^{(i)} = 2$, if e_i is operable and disconnected from e_0,

$S_h^{(i)}$: the sojourn time of e_i in the state $X_h^{(i)}$, counted from T_h, on the assumption that all other components do not change their states before e_i does,

q_len : the number of components awaiting repair (queue length),

avl_rt : the number of available repair teams,

sim(2,i), sim(1,i), sim(0,i) : the functions simulating the r.v. $L_i^{(2)}$, $L_i^{(1)}$, and R_i ; the simulation is of Monte Carlo type, therefore it is based on random numbers generation,

PARENT[i] – parent node of component e_i in the tree structure

Procedure 1

setting default values of $X_h^{(i)}$

repeat for i = 1,...,N

$X_h^{(i)} \leftarrow X_{h-1}^{(i)}$;

computing $T_h - T_{h-1}$

(a component awaiting repair at T_{h-1} ($X_{h-1}^{(i)} < 0$) is irrelevant in finding T_h)

$T_h - T_{h-1} \leftarrow \min(S_{h-1}(i) : 1 \leq i \leq N, X_{h-1}(i) \geq 0)$;

```
## adding newly failed components at the end of the queue (repair policy π1)
repeat for i = N,...,1
   if (X_{h-1}^{(i)} ≥ 1) AND (S_{h-1}^{(i)} = T_h − T_{h-1}) then {
      X_h^{(i)} ← −q_len − 1;
      q_len ← q_len + 1;
   }

## releasing repair teams
repeat for i = 1,..., N
   if (X_{h-1}^{(i)} = 0) AND (S_{h-1}^{(i)} = T_h − T_{h-1}) then {
      X_h^{(i)} ← 1;
      avl_rt ← avl_rt + 1;
   }

## taking at most avl_rt components for repair
x ← avl_rt;
repeat for i = 1,..., N {
   if (−x ≤ X_h^{(i)} < 0) then {
      X_h^{(i)} ← 0;
      avl_rt ← avl_rt − 1;
      q_len ← q_len − 1;
   }
   if (X_h^{(i)} < −x) then X_h^{(i)} ← X_h^{(i)} + x;
}
```

```
##updating the states of operable components
## (an operable component may change its state between 1 and 2 at T_h)
repeat for i = 1,...,N {
   if (X_h^(i) ≥ 1) then {
      j ← i ;
      x ← X_h^(i) ;
      repeat while (j > 0 AND x > 0) {
         j ← PARENT[j] ;
         if (j > 0) then x ← X_h^(j) ;
      }
   }
   if (X_h^(i) = 1 AND x ≤ 0) then X_h^(i) ← 2 ;
   if (X_h^(i) = 2 AND x > 0) then X_h^(i) ← 1 ;
}

##simulating the residual sojourn times of components in their states after T_h
repeat for i = 1,...,N {
   if (X_h^(i) < 0) then continue;
   if (X_h^(i) = X_{h-1}^(i))
      then S_h^(i) ← S_{h-1}^(i) – [T_h – T_{h-1}] ;
      else S_h^(i) ← sim(X_h^(i), i) ;
}
```

Remarks:

1. $S_h^{(i)}$ has to be simulated when e_i changes its state to 0, 1 or 2 at the instant T_h, because at such moment e_i "forgets its history". Obviously, if e_i remains in one of these states after T_h, the difference between $S_h^{(i)}$ and $S_{h-1}^{(i)}$ is equal to $T_{h-1} - T_h$. If e_i changes its state to a negative value, or remains in a "negative" state, then e_i is irrelevant in finding T_{h+1}, as neither failure nor repair completion is possible for a component whose state is less than 0.
2. If the repair policy π_2 is in place, then newly failed components have precedence over already failed ones, and those with smaller indexes are favored in case of simultaneous failures. Thus the "old" queue has to be "moved backward" at each T_h, and the following code fragment is used to add failed components to the queue:

```
x = 0;
repeat for i = 1,...,N
   if (X_{h-1}^(i) ≥ 1 AND S_{h-1}^(i) = T_h − T_{h-1}  then {
      x ← x + 1;
      X_h^(i) ← −x ;
      q_len ← q_len + 1;
   }
repeat for i = 1,...,N
   if (X_{h-1}^(i) < 0) then X_h^(i) ← X_{h-1}^(i) − x ;
```

3. If the policy π_3 is applied, then it may be necessary to rearrange the whole queue at each T_h, because a component's place in the queue is solely determined by its index. Thus in case of π_3 the following code fragment is used to add failed components to the queue:

```
x = 0;
repeat for i = 1,...,N
   if (X_{h-1}^{(i)} < 0 OR (X_{h-1}^{(i)} ≥ 1 AND S_{h-1}^{(i)} = T_h – T_{h-1} )) then {
      x ← x + 1;
      X_h^{(i)} ← –x ;
   }
q_len ← x ;
```

As stated in Task 3, b(i) will be estimated using sample means from the random samples $\{D_1^{(i)},\dots,D_K^{(i)}\}$ and $\{Q_1^{(i)},\dots,Q_K^{(i)}\}$ obtained in K simulation cycles, where a cycle corresponds to the time interval between two consecutive returns of the process $\underline{X}$ to the state $\underline{1}$, i.e. one of the intervals $[\tau_{k-1}, \tau_k)$, $k\geq 1$. Thus, the estimation procedure consists in embedding Procedure 1 into Tasks 1 and 2, and repeating these tasks K times. The resulting Procedure 2 is outlined below.

Variables used by Procedure 2:

D, Q : the sample values of $D^{(i)}$ and $Q^{(i)}$

ED, EQ : the sample means of $D^{(i)}$ and $Q^{(i)}$

UD, UQ : the sample means of $(D^{(i)})^2$ and $(Q^{(i)})^2$

VD, VQ : the sample variances of $D^{(i)}$ and $Q^{(i)}$

$Y_h^{(i)}$: a binary variable; $Y_h^{(i)} = 1$ if the operable e_i is connected to e_0 at T_h, otherwise $Y_h^{(i)} = 0$

Z_h : a binary variable; $Z_h = 1$ if $X_h^{(i)} = 1$ for $1 \leq i \leq N$, otherwise $Z_h = 0$

Procedure 2

```
ED = 0;  EQ = 0;  UD = 0;  UQ = 0;  VD = 0;  VQ = 0;
```

```
repeat for k ≥ 1 {

D ← 0;  Q ← 0;  T0 ← 0;  avl_rt ← r;  q_len ← 0;
repeat for i = 1,...,N {
   X0(i) ← 1;
   S0(i) ← sim(1,i);
}

repeat for h ≥ 1 {

   obtain Th and Xh(1),..., Xh(n) using Procedure 1;
   compute Yh(i) from Xh(1),..., Xh(n);

   if (Yh-1(i) = 0) then {
      ## if ei was failed or disconnected from e0 at Th-1,
      ## then D is increased by the time elapsed from Th-1 to Th
      D ← D + (Th – Th-1);
      ## if ei is reconnected to e0 at Th then Q is increased by 1
      if (Yh(i) = 1) then
         Q ← Q + 1;
   }

   compute Zh from Xh(1),..., Xh(n);
   if (Zh = 1) then break;

} ## end of "repeat for h ≥ 1"
```

ED ← ED + (D – ED)/k ; UD ← UD + (D^2 – UD)/k ; VD ← UD – $(ED)^2$;

EQ ← EQ + (Q – EQ)/k ; UQ ← UQ + (Q^2 – UQ)/k ; VQ ← UQ – $(EQ)^2$

if (k fulfills the stopping condition) then break;

} ## end of "repeat for k ≥ 1"

Remarks:

1. The sample means of $D^{(i)}$, $Q^{(i)}$, and their squares are updated in the step k of the outer loop (repeat for k ≥ 1) according to the following formula:

(20) $$\mu_k = \mu_{k-1}\frac{k-1}{k} + \frac{x_k}{k}$$

where

(21) $$\mu_k = \frac{x_1 + \ldots + x_k}{k}$$

To update the sample variances of $D^{(i)}$ and $Q^{(i)}$ the following formula is used:

(22) $$\sigma_k^2 = \frac{x_1^2 + \ldots + x_k^2}{k} - \mu_k^2$$

where

(23) $$\sigma_k^2 = \frac{(x_1 - \mu_k)^2 + \ldots + (x_k - \mu_k)^2}{k}$$

2. Note that in the step h of the inner loop (repeat for $h \geq 1$) only the values $X_{h-1}^{(\cdot)}$, $X_h^{(\cdot)}$, T_{h-1}, T_h are used to update C or D, and possibly Q, while the analogous values obtained up to the step h–2 are irrelevant. In consequence, it is necessary to store only the X's and T's obtained in the current and the previous cycle of the considered loop.

5. Accuracy of the estimation

Clearly, $E[C^{(i)}]/E[Q^{(i)}]$ is estimated by $[D_1^{(i)} + ... + D_K^{(i)}]/[Q_1^{(i)} + ... + Q_K^{(i)}]$ with growing accuracy as the total number of simulation cycles (K) increases. A natural question arises: how to gauge this accuracy? It should be noted that we have to estimate a quotient of two expected values rather than a single expected value. Let us quickly remind how the estimation accuracy is assessed for a random variable X with finite expected value μ and standard deviation σ. The statistical estimation theory states that for sufficiently large k ($k > 30$):

(24) $$\Pr(M_k - z_{1-\alpha/2}\frac{\sqrt{V_k}}{\sqrt{k}} \leq \mu \leq M_k + z_{1-\alpha/2}\frac{\sqrt{V_k}}{\sqrt{k}}) \geq 1-\alpha$$

M_k and V_k are respectively the sample mean and sample variance of X, i.e.

(25) $$M_k = \frac{X_1 + \ldots + X_k}{k}$$

(26) $$V_k = \frac{(X_1 - M_k)^2 + \ldots + (X_k - M_k)^2}{k}$$

where $X_1,...,X_k$ is a random sample from X, and $z_{1-\alpha/2}$ is the $1 - \alpha/2$ quantile of the standardized normal distribution, i.e.

(27) $\Pr(Z \le z_{1-\alpha/2}) = 1 - \frac{\alpha}{2}$

where Z is normally distributed with mean 0 and variance 1. The formula (24) defines the limits of a confidence interval which includes μ with probability $1-\alpha$ known as the confidence level. In consequence, if the confidence level is set to $1-\alpha$ and the estimation accuracy to ε (the half-length of the confidence interval), then the minimal number of samples necessary to obtain this accuracy is given by:

(28) $$n = \left[V_n \left(\frac{z_{1-\alpha/2}}{\varepsilon}\right)^2 \right] + 1$$

where [x] is the integer part of x. Our aim is to derive expressions analogous to (24) and (28), i.e. defining the confidence interval limits and the number of samples yielding the given estimation accuracy for $E[D^{(i)}]/E[Q^{(i)}]$. For that purpose we will need two following lemmas.

Lemma 3

Let $X \ge 0$ and $Y \ge y_{min} > 0$ be random variables with finite means μ_X and μ_Y respectively (y_{min} is constant). Let $M_{X,k}$ and $M_{Y,k}$ be the sample means of size k from X and Y respectively. Under the above assumptions the following formula holds:

(29) $$\Pr\left(\left|\frac{M_{X,k}}{M_{Y,k}} - \frac{\mu_X}{\mu_Y}\right| > \varepsilon\right) \le \Pr\left(|M_{X,k} - \mu_X| > \frac{\varepsilon \cdot y_{min}^2}{2\mu_Y}\right) + \Pr\left(|M_{Y,k} - \mu_Y| > \frac{\varepsilon \cdot y_{min}^2}{2\mu_X}\right)$$

Proof:

We have:

$$\left|\frac{M_{X,k}}{M_{Y,k}}-\frac{\mu_X}{\mu_Y}\right|=\frac{\left|M_{X,k}\mu_Y-M_{Y,k}\mu_X\right|}{M_{Y,k}\mu_Y}=$$

$$=\frac{\left|M_{X,k}\mu_Y+\mu_X\mu_Y-\mu_X\mu_Y-M_{Y,k}\mu_X\right|}{M_{Y,k}\mu_Y}=$$

(30)

$$=\frac{\left|[M_{X,k}-\mu_X]\mu_Y+[\mu_Y-M_{Y,k}]\mu_X\right|}{M_{Y,k}\mu_Y}\leq$$

$$\leq\frac{\left|M_{X,k}-\mu_X\right|\mu_Y+\left|M_{Y,k}-\mu_Y\right|\mu_X}{y_{\min}^2}$$

where the last inequality follows from the fact that $Y \geq y_{min}$ a.s. Let

(31) $$v_k=\frac{\left|M_{X,k}-\mu_X\right|\mu_Y}{y_{\min}^2},\quad w_k=\frac{\left|M_{Y,k}-\mu_Y\right|\mu_X}{y_{\min}^2}$$

On the basis of (30) and (31) we obtain:

$$\Pr(\left|\frac{M_{X,k}}{M_{Y,k}}-\frac{\mu_X}{\mu_Y}\right|>\varepsilon)\leq\Pr(v_k+w_k>\varepsilon)\leq\Pr(2\max(v_k,w_k)>\varepsilon)=$$

(32)

$$=\Pr(\max(v_k,w_k)>\frac{\varepsilon}{2})=\Pr(\{v_k>\frac{\varepsilon}{2}\}\vee\{w_k>\frac{\varepsilon}{2}\})\leq\Pr(v_k>\frac{\varepsilon}{2})+\Pr(w_k>\frac{\varepsilon}{2})$$

which completes the proof.

Lemma 4

Let X and Y be as in Lemma 3 with the additional assumption that they have finite standard deviations σ_X and σ_X. Let

(33) $$\varepsilon_\alpha = \frac{2\, z_{1-\alpha/4}}{y_{\min}^2 \sqrt{k}} \max[\sigma_X \mu_Y, \sigma_Y \mu_X]$$

where $z_{1-\alpha/4}$ is the $1-\alpha/4$ quantile of the standardized normal distribution, i.e.

(34) $$\Pr(Z \le z_{1-\alpha/4}) = 1 - \frac{\alpha}{4}$$

for normally distributed Z with the expected value 0 and variance 1. Then, for sufficiently large k, we have:

(35) $$\Pr(\left| \frac{M_{X,k}}{M_{Y,k}} - \frac{\mu_X}{\mu_Y} \right| > \varepsilon_\alpha) \le \alpha$$

i.e. $[-\varepsilon_\alpha, \varepsilon_\alpha]$ is a $1-\alpha$ confidence interval for μ_X/μ_Y.

Proof:

From (33) it follows that

(36) $$\frac{\varepsilon_\alpha \cdot y_{\min}^2}{2\mu_Y} \ge z_{1-\alpha/4} \frac{\sigma_X}{\sqrt{k}}, \quad \frac{\varepsilon_\alpha \cdot y_{\min}^2}{2\mu_X} \ge z_{1-\alpha/4} \frac{\sigma_Y}{\sqrt{k}}$$

hence

(37) $$\Pr(|M_{X,k}-\mu_X|>\frac{\varepsilon_\alpha\cdot y_{\min}^2}{2\mu_Y})\le\Pr(|M_{X,k}-\mu_X|>z_{1-\alpha/4}\frac{\sigma_X}{\sqrt{k}})$$

and

(38) $$\Pr(|M_{Y,k}-\mu_Y|>\frac{\varepsilon_\alpha\cdot y_{\min}^2}{2\mu_X})\le\Pr(|M_{Y,k}-\mu_Y|>z_{1-\alpha/4}\frac{\sigma_Y}{\sqrt{k}})$$

For sufficiently large n the central limit theorem yields:

(39) $$\Pr(|M_{X,k}-\mu_X|>z_{1-\alpha/4}\frac{\sigma_X}{\sqrt{k}})\le\frac{\alpha}{2},\quad \Pr(|M_{Y,k}-\mu_Y|>z_{1-\alpha/4}\frac{\sigma_Y}{\sqrt{k}})\le\frac{\alpha}{2}$$

so that

(40) $$\Pr(|M_{X,k}-\mu_X|>\frac{\varepsilon_\alpha\cdot y_{\min}^2}{2\mu_Y})\le\frac{\alpha}{2},\quad \Pr(|M_{Y,k}-\mu_Y|>\frac{\varepsilon_\alpha\cdot y_{\min}^2}{2\mu_X})\le\frac{\alpha}{2}$$

Now, (35) is a consequence of (40) and Lemma 3 with $\varepsilon = \varepsilon_\alpha$. The proof is thus completed.

To further simplify the notation the symbols C, D, and Q will be used in place of $C^{(i)}$, $D^{(i)}$, and $Q^{(i)}$, the upper index left out as the default component index. Let D* and Q* be conditional random variables defined as follows: $D^* = D|Q \ge 1$ and $Q^* = Q|Q \ge 1$, the underlying condition being that e_i fails or is disconnected from e_0 at least once in the interval (τ_{k-1}, τ_k). We have:

$$\text{(41)}\quad \frac{ED}{EQ} = \frac{E(D \mid Q \geq 1)\Pr(Q \geq 1) + E(D \mid Q = 0)\Pr(Q = 0)}{E(Q \mid Q \geq 1)\Pr(Q \geq 1) + E(Q \mid Q = 0)\Pr(Q = 0)} = \frac{E(D \mid Q \geq 1)}{E(Q \mid Q \geq 1)} = \frac{E(D^*)}{E(Q^*)}$$

Thus, we can estimate E(D*)/E(Q*) instead of E(D)/E(Q), obtaining the same result.

Note that in Lemmas 3 and 4 no assumption was made about the independence of X and Y, therefore they cover the case of strongly dependent random variables such as D* and Q*. Applying Lemmas 3 and 4 to D* and Q* we obtain that $[-\varepsilon_\alpha, \varepsilon_\alpha]$, where

$$\text{(42)}\quad \varepsilon_\alpha = \frac{2z_{1-\alpha/4}}{\sqrt{k}} \max[\sigma_{D^*}\mu_{Q^*}, \sigma_{Q^*}\mu_{D^*}]$$

is a $1-\alpha$ confidence interval for E(D*)/E(Q*) (note that lemmas 3 and 4 cannot be applied to D and Q, because it is not true that $Q \geq 1$). In consequence, if

$$\text{(43)}\quad \sqrt{k} \geq \frac{2z_{1-\alpha/4}}{\varepsilon} \max(\sigma_{D^*}\mu_{Q^*}, \sigma_{Q^*}\mu_{D^*})$$

which is equivalent to

$$\text{(44)}\quad k \geq \left(\frac{2z_{1-\alpha/4}\sigma_{D^*}\mu_{Q^*}}{\varepsilon}\right)^2 \quad \text{and} \quad k \geq \left(\frac{2z_{1-\alpha/4}\sigma_{Q^*}\mu_{D^*}}{\varepsilon}\right)^2$$

then taking at least k samples from D* and Q* allows to estimate E(D*)/E(Q*) with the given accuracy ε (the half-length of the confidence interval) at the given confidence level $1-\alpha$. For the sake of computational practice, the expected values and standard deviations of D* and Q* are replaced in (44) with the respective sample means or sample standard deviations. Thus the following condition to stop the outer loop (repeat for $k \geq 1$) in Procedure 2 is obtained:

$$\text{if } (k1 > 30 \quad \text{AND}$$

(45) $$k1 \geq 4 \cdot z^2 \cdot VD \cdot (EQ)^2/e^2 \quad \text{AND}$$

$$k1 \geq 4 \cdot z^2 \cdot VQ \cdot (ED)^2/e^2 \;) \text{ then break}$$

where k1 counts all of the outer loop's cycles in which $Q \geq 1$, z is the $1-\alpha/4$ quantile of the standardized normal distribution, and e is the half-length of the confidence interval. It is required that $k1>30$ so that it can be large enough to approximate the Student's t-distribution by the normal distribution (see [4] for details). The respective updating commands (after the end of the inner loop) have to be modified as follows:

```
if (Q ≥ 1) then {
   k1 ← k1 + 1 ;
   ED ← ED + (D – ED)/k1 ; UD ← UD + (D² – UD)/k1 ; VD ← UD – (ED)² ;
   EQ ← EQ + (Q – EQ)/k1 ; UQ ← UQ + (Q² – UQ)/k1 ; VQ ← UQ – (EQ)² ;
}
```

To assess the accuracy with which c(i) is estimated we have to find the limits of a confidence interval for $[a(i) + E(D^*/E(Q^*]^{-1}$, as c(i) is given by (4). These limits are specified in the following lemma:

Lemma 5

Let

(46) $$\varphi_\alpha = \frac{\varepsilon_\alpha}{[a(i) + r_{min}]^2}$$

(42)

where ε_α is given by ~~(33)~~. We then have:

$$(47)\qquad \Pr\left(\left|\frac{1}{a(i)+\frac{\mu_{D^*}}{\mu_{Q^*}}}-\frac{1}{a(i)+\frac{M_{D^*,k}}{M_{Q^*,k}}}\right|>\varphi_\alpha\right)\leq\alpha$$

i.e. $[-\varphi_\alpha, \varphi_\alpha]$ is a $1-\alpha$ confidence interval for $[a(i) + E(D^*)/E(Q^*)]^{-1}$.

Proof:

Let us note that $D^* \geq r_{min}Q^*$, because at least one component has to be repaired during each time interval $(\varphi_j^{(i)}, \rho_j^{(i)}]$, $j \geq 1$.We thus have:

$$\Pr\left(\left|\frac{1}{a(i)+\frac{\mu_{D^*}}{\mu_{Q^*}}}-\frac{1}{a(i)+\frac{M_{D^*,k}}{M_{Q^*,k}}}\right|>\varphi_\alpha\right)=\Pr\left(\frac{\left|\frac{M_{D^*,k}}{M_{Q^*,k}}-\frac{\mu_{D^*}}{\mu_{Q^*}}\right|}{\left(a(i)+\frac{\mu_{D^*}}{\mu_{Q^*}}\right)\left(a(i)+\frac{M_{D^*,k}}{M_{Q^*,k}}\right)}>\varphi_\alpha\right)\leq$$

(48)

$$\leq\Pr\left(\left|\frac{M_{D^*,k}}{M_{Q^*,k}}-\frac{\mu_{D^*}}{\mu_{Q^*}}\right|>\varphi_\alpha[a(i)+r_{min}]^2\right)=\Pr\left(\left|\frac{M_{D^*,k}}{M_{Q^*,k}}-\frac{\mu_{D^*}}{\mu_{Q^*}}\right|>\varepsilon_\alpha\right)\leq\alpha$$

where the last inequality follows directly from Lemma 4.

Lemmas 4 and 5 lead to the following conclusion: if

$$(49)\qquad \sqrt{k}\geq\frac{2z_{1-\alpha/4}}{\varphi\cdot[a(i)+r_{min}]^2}\max(\sigma_{D^*}\mu_{Q^*},\sigma_{Q^*}\mu_{D^*})$$

which is equivalent to

$$(50) \quad k \geq \left(\frac{2z_{1-\alpha/4}\sigma_{D*}\mu_{Q*}}{\varphi \cdot [a(i)+r_{min}]^2} \right)^2 \quad \text{and} \quad k \geq \left(\frac{2z_{1-\alpha/4}\sigma_{Q*}\mu_{D*}}{\varphi \cdot [a(i)+r_{min}]^2} \right)^2$$

then taking at least k samples from D* and Q* allows to estimate $[a(i) + E(D^*)/E(Q^*)]^{-1}$ with the given accuracy φ (the half-length of the confidence interval) at the given confidence level $1 - \alpha$. Thus we have the following stopping condition for the outer loop (repeat for $k \geq 1$) in Procedure 2:

(51)

$$\begin{aligned} &\text{if } (k1 \geq 30 \quad \text{AND} \\ &\quad 4 \cdot z^2 \cdot VD \cdot (EQ)^2 / f^2 (a(i) + r_{min})^4 \quad \text{AND} \\ &\quad k1 \geq 4 \cdot z^2 \cdot VQ \cdot (ED)^2 / f^2 (a(i) + r_{min})^4 \text{) then break} \end{aligned}$$

where f is the half-length of the confidence interval, and z is the same as in (45).

5. Exemplary numerical results

Several results obtained with Procedure 2 for the system in Fig. 1 are presented in Tables 1 and 2. It is assumed that $L_i^{(2)} = \infty$ with probability 1 (i.e.$G_i \equiv 0$), and $L_i^{(1)}$ and R_i are exponentially distributed with $\lambda_i = 0.01$, $\mu_i = 0.1$, $1 \leq i \leq N$. The time unit is one hour. K and t denote the total number of simulation cycles and computing time respectively. The computations were carried out for $i = 8$, $r = 2$, $s = 1, 2$. A PC machine with an Intel Core 2 (2.14 GHz) processor was used.

Table 1. Estimation results for s = 1

α\ε	ε = 0.1	ε = 0.2
α = 0.99	ED*/EQ* = 15.54, EQ* = 1.37 VD* = 426.57, VQ* = 0.56 K ≈ 4,554,000 t ≈ 2'	ED*/EQ* = 15.56, EQ* = 1.36 VD* = 426.75, VQ* = 0.56 K ≈ 1,136,000 t ≈ 35''
α = 0.95	ED*/EQ* = 15.55, EQ* = 1.37 VD* = 426.49, VQ* = 0.55 K ≈ 2,890,000 t ≈ 1'20''	ED*/EQ* = 15.52, EQ* = 1.36 VD* = 421.86, VQ* = 0.56 K ≈ 714,000 t ≈ 20''

Table 2. Estimation results for s = 2

α\ε	ε = 0.1	ε = 0.2
α = 0.99	ED*/EQ* = 14.34, EQ* = 1.49 VD* = 492.36, VQ* = 0.92 K ≈ 6,275,000 t ≈ 3'	ED*/EQ* = 14.33, EQ* = 1.49 VD* = 490.61, VQ* = 0.92 K ≈ 1,562,000 t ≈ 50''
α = 0.95	ED*/EQ* = 14.33, EQ* = 1.49 VD* = 492.72, VQ* = 0.92 K ≈ 3,990,000 t ≈ 2'30''	ED*/EQ* = 14.33, EQ* = 1.49 VD* = 493.11, VQ* = 0.92 K ≈ 1,001,000 t ≈ 40''

It is interesting to see that the variances of D* and Q* depend significantly on the repair policy. These variances are relatively large (in comparison with ED* and EQ*), therefore a suitable variance reduction method should be applied to decrease the computing time. Finding and implementing such a method will be a subject of further research.

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