Raport Badawczy

Research Report

RB/21/2008

Determining reliability parameters for a commodity transfer system with non-independent components

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Warszawa 2008

1. Introduction

A transmission network composed of n linearly arranged components $e_0,...,e_n$ is considered. For each $i=1,...,n-1$ the component e_i is directly connected to e_{i-1} and e_{i+1}, while e_n is directly connected to e_{n-1} alone; e_0 is the source component from which certain commodity (electric power, radio signal, electronic data, water, gas, etc.) is transferred, via $e_1,...,e_{n-1}$, to e_n. Each component can be in one of two states: 1 – operating, 0 – failed; e_0 is always in operating state. The repair of a failed component is started as soon as one of repair teams is available – due to a limited number of them the repair may not start immediately after the component's failure. The order in which failed components are chosen for repair depends on the repair policy applied – two of such policies will be considered. The time-to-failure and time-to-repair of e_i are random variables with distribution functions F_i and G_i respectively.

A commodity can be transferred from e_0 to e_i, $i=1,...,n$, if and only if e_i is functional and connected to e_0, i.e. $e_1,...,e_i$ are in the operating state. As failures of components occur, the periods during which functional e_i is connected to e_0 are interleaved by the periods during which e_i is failed or disconnected from e_0. The main goal of this paper is to determine the mean durations of these time intervals, i.e. the mean time from the moment when the connection between e_0 and e_i is interrupted to the moment when it is restored, and the mean time of uninterrupted connection between e_0 and e_i.

Most probabilistic models of complex systems assume full independence of their components. In particular, it means that the components' lifetimes and repair times are independent random variables. Obviously, this assumption is made for the sake of computational simplicity. In reality, however, such independence rarely occurs. In order to make the considered network model more true-to-life it is assumed that the functioning of e_i depends on the states of $e_1,...,e_{i-1}$ in the following way: e_i can only fail if $e_1,...,e_{i-1}$ are in the operating state; as long as this condition is fulfilled the time-to-failure of e_i has the distribution function F_i. In consequence, an element cannot fail if it is disconnected from e_0. This conveys the idea that only components being "under load" are failure prone as is the case in many real-life systems. Thus, it can be said that e_i functions independently of $e_1,...,e_{i-1}$ up to the moment when one of $e_1,...,e_{i-1}$ fails. It must be stressed that e_i functions independently of $e_{i+1},...,e_n$, moreover, a component's repair time is independent on the behavior of other components.

It is clear that the network's behavior and, consequently, mean connectedness and non-connectedness times of components depend on two factors: the number of repair teams assigned to the network maintenance, and the repair policy implemented. As to the first factor, the greater the number of repair teams, the shorter the mean non-connectedness times of individual components, and vice versa. If there are fewer than n repair teams, a component's repair may not start immediately after its failure – the average time of delay decreases with the total number of repair teams and is equal to zero if this number reaches n. However, this case should be considered only for theoretical purposes, because in practice the number of repair teams is usually considerably smaller than the number of system's components. As follows from the above argument, the mean non-connectedness times of components in the cases of one and n repair teams are the upper and lower bounds of these mean times in each remaining case (more than one and less than n repair teams).

Passing to the subject of repair policies, two of them will be considered. According to the first policy, the components are chosen for repair in the same order in which they fail. As only the components connected to e_0 can fail, the next component selected for repair (by the first available team) is the one farthest from e_0. This means that the queue of components waiting to be repaired is of the FIFO type. The second policy consists in prioritizing the components which are least distant from e_0, i.e. the next component selected for repair is the one nearest to e_0. Thus the order in which failed components are chosen for repair is reverse to the order in which they fail. This means that the queue of components waiting to be repaired is of the LIFO type.

Obviously, choosing between multiple maintenance policies makes sense only if there are less then n repair teams. Otherwise the only feasible policy to follow is "repair a component upon its failure", as there is always at least one team available when a component fails.

2. Definitions and notation

Throughout the paper the following notation will be used:

L_i – time-to-failure for e_i, provided that $e_1,...,e_{i-1}$ remain in operational state up to the failure of e_i

R_i – time-to-repair for e_i

F_i, G_i – distribution functions of L_i and G_i respectively;

$\varphi_j^{(i)}$ – time of the j-th disconnection of e_i from e_0, $j \geq 1$;

$\rho_j^{(i)}$ – time of the j-th reconnection of e_i to e_0, $j \geq 1$; it is assumed that $\rho_0^{(i)} = 0$;

$\chi_j^{(i)}$ – length of time from $\rho_{j-1}^{(i)}$ to $\varphi_j^{(i)}$, $j \geq 1$, i.e. $\chi_j^{(i)} = \varphi_j^{(i)} - \rho_{j-1}^{(i)}$ is the length of the j-th period during which e_i remains connected to e_0

$\psi_j^{(i)}$ – length of time from $\varphi_j^{(i)}$ to $\rho_j^{(i)}$, $j \geq 1$, i.e. $\psi^{(i)} = \rho_j^{(i)} - \varphi_j^{(i)}$ is the length of the j-th period during which e_i remains disconnected from e_0

π_1 – the "first failed, first selected" policy;

π_2 – the "last failed, first selected" policy;

$C_j(i,n,r,s)$, $D_j(i,n,r,s)$ – the expected values of $\chi_j^{(i)}$ and $\psi_j^{(i)}$, provided that the system is composed of n components, the number of repair teams equals r, and the repair policy π_s is applied, s = 1, 2;

$C(i,n,r,s)$, $D(i,n,r,s)$ – the expected values of limiting means of $\chi_j^{(i)}$ and $\psi_j^{(i)}$, provided the limiting means exist.

We thus have:

$$C_j(i,n,r,s) = E(\chi_j^{(i)})$$
$$D_j(i,n,r,s) = E(\psi_j^{(i)})$$
$$C(i,n,r,s) = E[\lim_{n\to\infty} \sum_{j=1}^{n} \chi_j(i,n,r,s)]$$
$$D(i,n,r,s) = E[\lim_{n\to\infty} \sum_{j=1}^{n} \psi_j(i,n,r,s)]$$

3. Analytical computation of C(i,n,r,s) and D(i,n,r,s) for a two-component system

In this chapter the two basic reliability parameters, i.e. C(i,n,r,s) and D(i,n,r,s) will be computed analytically for a two-component system. The additional assumption, making the computations possible, is that L_1, L_2, R_1, R_2 are exponentially distributed, i.e.

(1) $$F_i(t) = 1 - \exp(-\lambda_i t), \quad G_i(t) = 1 - \exp(-\mu_i t), \quad i = 1, 2$$

The definitions given in chapter 2 imply that every period of time during which both e_1 and e_2 remain connected to e_0 falls within the limits of one of the intervals $[\rho_{j-1}^{(2)}, \varphi_j^{(2)})$, $j\geq 1$, while every period of time when e_1 or e_2 remains disconnected from e_0 falls within the limits of one of the intervals $[\varphi_j^{(2)}, \rho_j^{(2)})$, $j\geq 1$. As the lifetime distributions of e_1 and e_2 are exponential, the system's behavior is stochastically identical on each interval $[\rho_{j-1}^{(2)}, \rho_j^{(2)})$, $j\geq 1$. Thus, by the end of this chapter, let time be measured from any moment $\rho_j^{(2)}$, $j\geq 0$, i.e. from the moment 0 or any other moment when e_2 is reconnected to e_0.

It is very easy to compute C_j(i,n,r,s). Indeed, for any $j\geq 1$ the components $e_1,\ldots,e_i$ are all operational at the instant $\rho_{j-1}^{(i)}$, and the time from $\rho_{j-1}^{(i)}$ to the failure of e_k is exponentially distributed, k=1,...,i (the "lack of memory" property of the exponential distribution). We thus have:

(2) $$P(\chi_j^{(i)} \le t) = P(\min(L_1,...,L_i) \le t) = 1 - \exp[-(\lambda_1 + ... + \lambda_i)]$$

From (2) it follows that

(3) $$C_j(i) = C(i) = \frac{1}{\lambda_1 + ... + \lambda_i}, \quad j \ge 1$$

The analytical computation of $D_j(i,n,r,s)$ is more difficult, however it will be presented for n=2. It should be noted that for a two-component system both repair policies are equivalent. Indeed, if r=1 then the queue of components awaiting repair has maximum length 1, if r=2 then the queue always has zero length. In both cases no selection decision has to be made. First, the case of one repair team will be considered. Let the events A, B, and C be defined as follows:

(4) $$\begin{aligned} A &= \{L_1 < L_2\} \\ B &= \{L_1 > L_2,\ R_2 > L_1 - L_2\} \\ C &= \{L_1 > L_2,\ R_2 < L_1 - L_2\} \end{aligned}$$

Clearly, A, B, and C form a complete system of events. As e_1 can be disconnected from e_0 only in case of the event A or B, the expected time-to-reconnection for e_1, provided that e_1 was disconnected from e_0, is equal to $E(\psi_j^{(1)}|A \cup B)$. The latter symbol denotes the conditional expectation of $\psi_j^{(1)}$, given the event $A \cup B$.

In general, the conditional expectation of a random variable X, given the event E such that P(E)>0, is defined as follows:

(5) $$\int_{-\infty}^{+\infty} t dF_{(X|E)}(t),$$

where

(6) $$F_{(X|E)} = \frac{1}{P(E)} P(\{X \leq t\} \cap E)$$

$F_{(X|E)}$ is called the conditional distribution function of X, given the event E.

Following the above definition, we obtain

(7) $$E(\psi_j^{(1)} \mid A \cup B) = \frac{E(\psi_j^{(1)} \mid A)P(A) + E(\psi_j^{(1)} \mid B)P(B)}{P(A) + P(B)}$$

In case of the event A, i.e. when e_1 fails before e_2, the repair of e_1 starts immediately after its failure, and e_2 cannot fail until the repair of e_1 is completed. The time-to-reconnection for e_1 is thus equal to R_1, therefore

(8) $$E(\psi_j^{(1)} \mid A) = E(R_1) = \frac{1}{\mu_1}$$

with

(9) $$P(A) = \int_0^\infty P(L_2 > x) dF_1(x) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

In case of the event B the failure of e_2 precedes that of e_1, while the repair of e_2, starting immediately after its failure, ends after the failure of e_1, being followed by the repair of e_1. We thus have:

(10) $$E(\psi_j^{(1)} \mid B) = E(R_2^{res} \mid B) + E(R_1)$$

where R_2^{res} is equal to the residual repair time of e_2 , i.e. the time elapsed from the failure of e_1 to the completion of e_2's repair. It is also true that

(11) $$P(\{R_2^{res} \le t\} \cap B) = \int_0^\infty \int_0^x \Pr(x - y < R_2 \le x - y + t) dF_2(y) dF_1(x)$$

and

(12) $$P(B) = \int_0^\infty \int_0^x \Pr(R_2 > x - y) dF_2(y) dF_1(x) = \frac{\lambda_2}{(\lambda_1 + \lambda_2)} \cdot \frac{\lambda_1}{(\lambda_1 + \mu_2)}$$

As

(13) $$\Pr(x - y < R_2 \le x - y + t) = \exp[-\mu_2(x - y)][1 - \exp(-\mu_2 t)] = G_2(t) \Pr(R_2 > x - y),$$

the following equality holds:

(14) $$P(\{R_2^{res} \le t\} \mid B) = G_2(t)$$

meaning that the conditional distribution function of R_2^{res}, given the event B, is equal to the distribution function of R_2. In view of (10) this result yields:

(15) $$E(\psi_j^{(1)} \mid B) = E(R_2) + E(R_1) = \frac{1}{\mu_1} + \frac{1}{\mu_2}$$

Summing up, we obtain the following formula:

(16) $$D(1,2,1) = D_j(1,2,1) = \frac{1}{\mu_1} + \frac{1}{\mu_2} \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2 + \mu_2}$$

Based on a similar argument as in the case of e_1, the expected time-to-reconnection for e_2, provided that e_2 was disconnected from e_0, is equal to $E(\psi_j^{(2)}|A \cup B \cup C)$ – each of the events A, B, and C results in disconnecting e_2 from e_0. Using the total probability law, we obtain:

(17) $$E(\psi_j^{(2)} \mid A \cup B \cup C) = E(\psi_j^{(2)} \mid A)P(A) + E(\psi_j^{(2)} \mid B)P(B) + E(\psi_j^{(2)} \mid C)P(C)$$

In case of the event A, the time-to-reconnection for e_2 is equal to that of e_1. We thus have:

(18) $$E(\psi_j^{(2)} \mid A) = E(\psi_j^{(1)} \mid A) = \frac{1}{\mu_1}$$

In case of the event B, the time-to-reconnection for e_2 is equal to $R_2 + R_1$, therefore

(19) $$E(\psi_j^{(2)} \mid B) = E(R_2) + E(R_1) = \frac{1}{\mu_1} + \frac{1}{\mu_2}$$

In case of the event C, the repair of e_2 starts and finishes before the failure of e_1, hence

$$E(\psi_j^{(2)} \mid C) = E(R_2) = \frac{1}{\mu_2} \tag{20}$$

with

$$P(C) = \int_0^\infty \int_0^x \Pr(R_2 < x - y) dF_2(y) dF_1(x) = \frac{\lambda_2}{(\lambda_1 + \lambda_2)} \cdot \frac{\mu_2}{(\lambda_1 + \mu_2)} \tag{21}$$

Finally the following result is obtained:

$$\begin{aligned} D(2,2,1) = D_j(2,2,1) &= \frac{1}{\mu_1} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} + \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \cdot \frac{\lambda_2}{(\lambda_1 + \lambda_2)} \cdot \frac{\lambda_1}{(\lambda_1 + \mu_2)} \\ &+ \frac{1}{\mu_2} \cdot \frac{\lambda_2}{(\lambda_1 + \lambda_2)} \cdot \frac{\mu_2}{(\lambda_1 + \mu_2)} \end{aligned} \tag{22}$$

Now the case of two repair teams will be considered. As e_1 fails independently of the state of e_2, and one repair team is always available for e1, the failure and repair process of e_1 is an alternating renewal process independent of the state of e_2. The mean time-to-reconnection for e_1 is thus given by $E(R_1)$, i.e:

$$D(1,2,2) = D_j(1,2,2) = \frac{1}{\mu_1} \tag{23}$$

Let $\{Y(t), t \geq 0\}$denote the failure and repair process of e_1, where $Y(t)=1$ if e_1 is operational at the time t, otherwise (e_1 is under repair) $Y(t)=0$. Let $Z(t)$ denote the time elapsing from t to the next state change of e_1. Obviously, $\{Z(t), t>0\}$ is also a stochastic process. As L_1 and R_1 are exponentially distributed, for the process Z we have:

$$\Pr(Z(t) \leq s \mid Y(t) = 1) = 1 - \exp(-\lambda_1 s)$$

(24)

$$\Pr(Z(t) \leq s \mid Y(t) = 0) = 1 - \exp(-\mu_1 s)$$

and for the process Y:

$$\Pr(Y(t+s) = 1 \mid Y(t) = 1) = \frac{\mu_1}{\lambda_1 + \mu_1} + \frac{\lambda_1}{\lambda_1 + \mu_1} \exp[-(\lambda_1 + \mu_1)s]$$

(25)

$$\Pr(Y(t+s) = 0 \mid Y(t) = 1) = \frac{\lambda_1}{\lambda_1 + \mu_1} [1 - \exp[-(\lambda_1 + \mu_1)s]]$$

The latter are the formulas for the transition probabilities of Y which is a Markov process (see [6]). Let the events A, B and C be defined in the following way:

(26)
$$\begin{aligned} A &= \{Z(0) < L_2\} \\ B &= \{Z(0) > L_2, \quad Y(L_2 + R_2) = 0\} \\ C &= \{Z(0) > L_2, \quad Y(L_2 + R_2) = 1\} \end{aligned}$$

Clearly, A, B, and C form a complete system of events. As e_2 is disconnected from e_0 in case of the event A, B or C, the expected time-to-reconnection for e_2, provided that e_2 has been disconnected from e_0, is equal to $E(\psi_j^{(2)} | A \cup B \cup C)$. Using the total probability law, we obtain:

(27)
$$D(2,2,2) = D_j(2,2,2) = E(\psi_j^{(2)} \mid A)P(A) + E(\psi_j^{(2)} \mid B)P(B) + E(\psi_j^{(2)} \mid C)P(C)$$

In the case of A, the time-to-reconnection for e_2 is equal to that of e_1. We thus have:

(28)
$$E(\psi_j^{(2)} \mid A) = \frac{1}{\mu_1}$$

with

(29) $$P(A)=\int_0^\infty P(L_2>x)dF_1(x)=\frac{\lambda_1}{\lambda_1+\lambda_2}$$

In case of the event B the repair of e_1, starting before the failure of e_2, continues for some time after the repair of e_2 has been completed. We thus have:

(30) $$E(\psi_j^{(2)}\mid B)=E(R_2)+E(R_1^{res}\mid B)$$

where R_1^{res} is equal to the residual repair time of e_1, i.e. the time elapsed from the completion of e_1's repair to the completion of e_2's repair. Using (24) we obtain:

(31)
$$P(\{R_1^{res}\le t\}\cap B)=\{Z(0)>L_2,\ \ Y(L_2+R_2)=0,\ \ Z(L_2+R_2)\le t)=$$

$$=\int_0^\infty P(Z(x+y)\le t,\ \ Y(x+y)=0,\ \ Z(0)>x)dG_2(y)dF_2(x)=$$

$$=\int_0^\infty P(Z(x+y)\le t\mid Y(x+y)=0)\times$$
$$\times P(Y(x+y)=0,\ \ Z(0)>x)dG_2(y)dF_2(x)=$$

$$=\exp(-\mu_1 t)\int_0^\infty P(Y(x+y)=0,\ \ Z(0)>x)dG_2(y)dF_2(x)=$$

$$=\exp(-\mu_1 t)P(B)$$

while (25) yields:

$$P(B) = \int_0^\infty P(Y(x+y) = 0 \mid Z(0) > x) P(Z(0) > x) dG_2(y) dF_2(x) =$$

$$= \int_0^\infty P(Y(x+y) = 0 \mid Y(x) = 1) P(Z(0) > x) dG_2(y) dF_2(x) =$$

(32)

$$= \int_0^\infty \frac{\lambda_1}{\lambda_1 + \mu_1} [1 - \exp(-\lambda_1 y - \mu_1 y)] \exp(-\lambda_1 x) dG_2(y) dF_2(x) =$$

$$= \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \mu_1} \cdot \left(1 - \frac{\mu_2}{\lambda_1 + \mu_1 + \mu_2}\right)$$

From (31) it follows that:

(33) $$P(\{R_1^{res} \le t\} \mid B) = G_1(t)$$

meaning that the conditional distribution function of R_1^{res}, given the event B, is equal to the distribution function of R_1. In view of (30) this result yields:

(34) $$E(\psi_j^{(2)} \mid B) = E(R_2) + E(R_1) = \frac{1}{\mu_1} + \frac{1}{\mu_2}$$

In case of the event C, e_1 is operational when the repair of e_2 ends, thus the time-to-reconnection for e_2 is equal to R_2, which means that

(35) $$E(\psi_j^{(2)} \mid C) = E(R_2) = \frac{1}{\mu_2}$$

where, using (25), P(C) is computed as follows:

$$P(C)=\int_0^\infty P(Y(x+y)=1\,|\,Z(0)>x)P(Z(0)>x)dG_2(y)dF_2(x)=$$

$$=\int_0^\infty P(Y(x+y)=1\,|\,Y(x)=1)P(Z(0)>x)dG_2(y)dF_2(x)=$$

(36)

$$=\int_0^\infty\left[\frac{\mu_1}{\lambda_1+\mu_1}+\frac{\lambda_1}{\lambda_1+\mu_1}\exp(-\lambda_1 y-\mu_1 y)\right]\exp(-\lambda_1 x)dG_2(y)dF_2(x)=$$

$$=\frac{\lambda_2}{\lambda_1+\lambda_2}\left(\frac{\mu_1}{\lambda_1+\mu_1}+\frac{\lambda_1}{\lambda_1+\mu_1}\cdot\frac{\mu_2}{\lambda_1+\mu_1+\mu_2}\right)$$

Finally the following result is obtained:

$$D(2,2,2)=D_j(2,2,2)=\frac{1}{\mu_1}\cdot\frac{\lambda_1}{\lambda_1+\lambda_2}+$$

(37)

$$+\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\cdot\frac{\lambda_2}{\lambda_1+\lambda_2}\cdot\frac{\lambda_1}{\lambda_1+\mu_1}\cdot\left(1-\frac{\mu_2}{\lambda_1+\mu_1+\mu_2}\right)+$$

$$+\frac{1}{\mu_2}\cdot\frac{\lambda_2}{\lambda_1+\lambda_2}\cdot\left(\frac{\mu_1}{\lambda_1+\mu_1}+\frac{\lambda_1}{\lambda_1+\mu_1}\cdot\frac{\mu_2}{\lambda_1+\mu_1+\mu_2}\right)$$

.

In conclusion, one remark should be made. For the system under consideration, the analytical method seems to be of minor practical significance, mainly due to the assumption that random variables describing components' behavior are exponentially distributed, but also because of enormously complex formulas that would be derived for n>2. It should be underscored that analytical computation of D(i,n,r,s) for large n is an open problem that can possibly be solved using some recursive method. Nevertheless, the results obtained for n=2 can be helpful in testing the correctness and accuracy of the simulation method presented in the next chapter.

4. Computing C(i,n,r,s) and D(i,n,r,s) by means of Monte Carlo Simulation

In this chapter algorithms for estimating C(i,n,r,s) and D(i,n,r,s), based on Monte Carlo simulation are presented. When estimating the parameters of a stochastic process one often encounters the following problem: whether statistical data may come from one sample path (realization) of the process or should they be collected from multiple sample paths? Clearly, one sample path is sufficient in the case of a recurrent process, i.e. a process $X=\{X(t), t\geq 0\}$ with the following properties:

- the state of X at $t = 0$ is fixed, i.e. X(0) has the one-point distribution,
- with probability one X returns to the state X(0) after finite time, measured from 0,
- if $Y(t)=X(\tau_1+t)$, where τ_1 is the (random) time of the first return of X to its initial state, then $Y=\{Y(t), t\geq 0\}$ and X are stochastically identical processes (X begins anew at $t = \tau_1$).

Sometimes the second property is replaced with the stronger one, i.e. $E(\tau_1)<\infty$. For details see [5].

Lemma 1

If the components' lifetimes (the random variables $L_1,\ldots,L_n$) are exponentially distributed, and the repair policy π_1 is applied, then the vector valued process $\underline{X}=\{[X_1(t),\ldots,X_n(t)], t\geq 0\}$, where $X_i(t)$ denotes the state of e_i at time t, is recurrent.

Proof

Note that $\underline{X}(0)=\underline{1}$ and $\underline{X}$ begins anew at any moment t when $\underline{X}(t)=\underline{1}$, due to the "lack of memory" property of the exponential distribution. Moreover,

$$(38)\qquad \tau_1 = \rho_1^{(n)} = \chi_1^{(n)} + \psi_1^{(n)} \leq \min(L_1,\ldots L_n) + R_1 + \ldots + R_n,$$

The above inequality becomes equality only in the "worst" case, i.e. r=1, e_n is the first failed component, and e_{i-1} fails before e_i has been repaired, $2 \le i \le n$. Otherwise we have strong inequality in (38), provided that $Pr(R_i>0)=1$, $1 \le i \le n$. From (38) it follows that

$$(39) \qquad E(\tau_1) \le \frac{1}{(\lambda_1 + \ldots + \lambda_n)} + \sum_{i=1}^{n} E(R_i),$$

thus the stronger version of the second property is fulfilled if $E(R_i)<\infty$, $1 \le i \le n$. For the weaker version it is sufficient that $Pr(R_i<\infty)=1$.

If the policy π_2 is applied, then the question whether $\underline{X}$ is recurrent remains open, even for exponentially distributed $L_1,\ldots,L_n$. Most likely, some additional assumptions regarding the distribution functions $G_1,\ldots,G_n$ should be made to ensure that $\underline{X}$ has this property.

It follows from Lemma 1 that χ_1, $\chi_2,\ldots$ are independent identically distributed random variables (IIDRV). However, ψ_1, $\psi_2,\ldots$ may not be IIDRV, e.g. if $R_1,\ldots,R_n$ are not exponentially distributed and e_i is reconnected to e_0 more than once while e_n remains disconnected from e_0 (in case of π_1 this may only happen for $r\ge2$). In consequence, defining D(i,n,r,s) as the average time during which e_i remains disconnected from e_0, one must remember that the successive periods of disconnection may not have one distribution function, therefore in this context "average" cannot be mistaken for "expected value of". The proper meaning of thus defined D(i,n,r,s) is given by the following lemma.

Lemma 2

Let $J_k^{(i)}$ be the number of reconnections between e_i and e_0 during the interval $(\varphi_k^{(n)}, \rho_k^{(n)}]$, and let $\theta_k^{(i)}$ be the total time in $[\varphi_k^{(n)}, \rho_k^{(n)})$ during which e_i remains disconnected from e_0, $k\ge1$. If the assumptions of Lemma 1 are fulfilled, and $0 < r_{min} \le R_i \le r_{max} < \infty$ for $i = 1,\ldots,n$, then

(40) $$\lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m}\psi_j^{(i)} \to_{prob} \frac{E(\theta_1^{(i)})}{E(J_1^{(i)})}$$

where $\to_{prob}$ denotes convergence in probability.

Proof

Obviously, $J_k^{(i)}$ and $\theta_k^{(i)}$ are IIDRV for $k\geq 1$. By virtue of (38),

(41) $$E(\psi_1^{(n)}) \leq E(R_1 + \ldots + R_n) \leq n \cdot r_{max}$$

i.e. $E(\psi_1^{(n)})$ is finite, thus we have:

(42) $$E(\psi_1^{(n)}) = \int_0^{\infty}[1 - H(t)]dt = \sum_{j=1}^{\infty}\int_{(j-1)r_{min}}^{jr_{min}}[1 - H(t)]dt \geq$$
$$\geq r_{min}\sum_{j=1}^{\infty}[1 - H(j\cdot r_{min})] = r_{min}\sum_{j=1}^{\infty}\Pr(\psi_1^{(n)} \geq j\cdot r_{min})$$

where H is the distribution function of $\psi_1^{(n)}$. It is also true that:

(43) $$E(J_1^{(i)}) = \sum_{j=1}^{\infty} j\cdot\Pr(J_1^{(i)} = j) = \sum_{j=1}^{\infty}\Pr(J_1^{(i)} \geq j) \leq \sum_{j=1}^{\infty}\Pr(\psi_1^{(n)} \geq j\cdot r_{min})$$

where the last inequality is a consequence of the following implication: if $J_1^{(i)}\geq j$, then at least j repairs are performed from $\varphi_1^{(n)}$ to $\rho_1^{(n)}$. From (42) and (43) we obtain:

(44) $$E(J_1^{(i)}) \leq \frac{n\cdot r_{max}}{r_{min}}$$

i.e. $E(J_1^{(i)})$ is finite. $E(\theta_1^{(i)})$ is also finite, because $\theta_1^{(i)} \leq \psi_1^{(n)}$.

Let K(i,m), m≥0, be an integer valued random variable equal to k if the interval $(\varphi_m^{(i)}, \rho_m^{(i)}]$ is included in the interval $(\varphi_k^{(n)}, \rho_k^{(n)}]$, i.e. K(i,m)=k if $J_1^{(i)}+\ldots+J_{k-1}^{(i)} < m \leq J_1^{(i)}+\ldots+J_k^{(i)}$, where $J_0^{(i)}=0$. In consequence of (41)

$$\lim_{m\to\infty} K(i,m) = \infty \tag{45}$$

holds. For m such that $J_1^{(i)} +\ldots+ J_{K(i,m)-1}^{(i)} > 0$ we have:

$$\frac{\theta_1^{(i)}+\ldots+\theta_{K(i,m)-1}^{(i)}}{J_1^{(i)}+\ldots+J_{K(i,m)}^{(i)}} \leq \frac{1}{m}\sum_{j=1}^{m}\psi_j^{(i)} \leq \frac{\theta_1^{(i)}+\ldots+\theta_{K(i,m)}^{(i)}}{J_1^{(i)}+\ldots+J_{K(i,m)-1}^{(i)}} \tag{46}$$

$$\frac{\theta_1^{(i)}+\ldots+\theta_{K(i,m)-1}^{(i)}}{J_1^{(i)}+\ldots+J_{K(i,m)}^{(i)}} = \frac{\theta_1^{(i)}+\ldots+\theta_{K(i,m)}^{(i)}}{J_1^{(i)}+\ldots+J_{K(i,m)}^{(i)}} - \frac{\theta_{K(i,m)}^{(i)}}{J_1^{(i)}+\ldots+J_{K(i,m)}^{(i)}} \tag{47}$$

$$\frac{\theta_1^{(i)}+\ldots+\theta_{K(i,m)}^{(i)}}{J_1^{(i)}+\ldots+J_{K(i,m)-1}^{(i)}} = \frac{\theta_1^{(i)}+\ldots+\theta_{K(i,m)-1}^{(i)}}{J_1^{(i)}+\ldots+J_{K(i,m)-1}^{(i)}} + \frac{\theta_{K(i,m)}^{(i)}}{J_1^{(i)}+\ldots+J_{K(i,m)-1}^{(i)}} \tag{48}$$

From (45) – (48) it follows that:

$$\lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m}\psi_j^{(i)} = \lim_{m\to\infty}\frac{\theta_1^{(i)}+\ldots+\theta_{K(i,m)}^{(i)}}{J_1^{(i)}+\ldots+J_{K(i,m)}^{(i)}} = \lim_{k\to\infty}\frac{\theta_1^{(i)}+\ldots+\theta_k^{(i)}}{k}\cdot\frac{k}{J_1^{(i)}+\ldots+J_k^{(i)}} \tag{49}$$

Now (40) is obtained by applying the Khinchin law of large numbers to (49).

Let us now pass to the details of our estimation method. It is based on simulating one sample path of the process $X_t(i)$ describing the system's behavior, defined in the sequel. Clearly, the component e_i can be disconnected from or reconnected to e_0 only at the times T_k, $k \geq 1$, which are the consecutive moments when any component changes its operational state, i.e. either the component fails or its repair is finished. Thus, the estimation method consists of the following tasks:

1) generating the sequence $\{T_k, k \geq 1\}$, and the states of all components at the instants T_k,

2) selecting from $\{T_k, k \geq 1\}$ these moments when e_i is disconnected from or reconnected to e_0, i.e. the moments $\varphi_j^{(i)}$ and $\rho_j^{(i)}$, $j \geq 1$,

3) estimating C(i,n,r,s) and D(i,n,r,s) as the sample means computed from $\chi_j^{(i)}$ and $\psi_j^{(i)}$, $1 \leq j \leq L$, where L is the number of samples.

Task 1 is implemented by Procedure 1, outlined below.

Variables used by Procedure 1:

$X_k^{(i)}$: the state of e_i at T_k ; it is assumed that:

$X_k^{(i)} = -q$ if e_i is the q-th component in the queue of components awaiting repair,

$X_k^{(i)} = 0$ if e_i is under repair,

$X_k^{(i)} = 1$ if e_i is operable and connected to e_0,

$X_k^{(i)} = 2$ if e_i is operable and disconnected from e_0,

i*: index of the failed component, located nearest e_0,

$S_k^{(i)}$: the sojourn time of e_i in the state $X_k^{(i)}$, counted from T_k, with the assumption that all other components do not change their states before e_i does,

q_len : the number of components awaiting repair (queue length),

avl_rt : the number of available repair teams,

sim(1,i), sim(0,i) : the functions simulating time-to-failure and time-to-repair for e_i ; the simulation is of Monte Carlo type, therefore it is based on random numbers generation.

Procedure 1

```
T_0 = 0; avl_rt = r; q_len = 0; i*= n + 1;
repeat for i = 1,...,n {
   X_0^(i) = 1; S_0^(i) = sim(1,i);}

repeat for k ≥ 1
{

   T_k= min(S_{k-1}^(i) : 1 ≤ i ≤ n, X_{k-1}^(i) należy do {0,1});
   ##If at T_{k-1} a component was failed and awaiting repair
   ## or it was operable and disconnected from e_0
   ## then the component is irrelevant in determining T_k

   ## adding failed components to the queue (repair policy π_1)
   repeat for i = i*- 1,...,1
      if (X_{k-1}^(i) = 1 AND S_{k-1}^(i) =T_k) then {
         X_k^(i) = -q_len - 1; q_len = q_len + 1; }

   ## releasing repair teams
   repeat for i = 1,..., n
      if (X_{k-1}^(i) = 0 AND S_{k-1}^(i) = T_k) then {
         X_k^(i) = 1; avl_rt = avl_rt + 1; }
```

```
## taking at most avl_rt components for repair
x = avl_rt;
repeat for i = 1,..., n {
    if (–x ≤ X_k^(i) < 0) then {
        X_k^(i) = 0; avl_rt = avl_rt – 1; q_len = q_len – 1; }
    if (X_k^(i) < –x)then X_k^(i)= X_k^(i) + x;
}

##updating i* and the states of operable components
i*= n + 1;
repeat for i = 1,...,n {
    if (X_k^(i) ≤ 0) then i*= i; break; }
repeat for i = 1,..., n {
    if (X_k^(i) = 1 AND i>i*) then X_k^(i)=2;
    if (X_k^(i) = 2 AND i<i*) then X_k^(i)=1; }

##simulating the residual sojourn times of components in their states after T_k
repeat for i = 1,..., n {
    if (X_k^(i) < 0 OR X_k^(i) =2) then continue;
    if (X_k^(i) = 0 AND X_{k-1}^(i) ≠ 0) then S_k^(i) = sim(0,i);
    if (X_k^(i) = 0 AND X_{k-1}^(i) = 0) then S_k^(i) = S_{k-1}^(i) – [T_k – T_{k-1}];
    if (X_k^(i) = 1 AND X_{k-1}^(i) ≠ 1) then S_k^(i) = sim(1,i);
    if (X_k^(i) = 1 AND X_{k-1}^(i) = 1) then S_k^(i) = S_{k-1}^(i) – [T_k – T_{k-1}]; }

} ## end of "repeat for k ≥ 1"
```

Remarks:

1. The time $S_k^{(i)}$ is simulated only if e_i changes its state to 0 or 1 at the instant T_k. Obviously, if e_i remains in the state 0 or 1, the residual sojourn times for e_i at T_{k-1} and T_k differ by the length of time elapsed from T_{k-1} to T_k. If e_i changes its state to 2 or a negative value, or remains in one of those states, it is irrelevant in determining T_{k+1}, as neither failure nor repair completion is possible for a component whose state is not 0 or 1.

2. In case of the repair policy π_2 the newly failed components are placed before those awaiting repair, hence the following code fragment is used to add failed components to the queue:

```
x = 0;
repeat for i=1,...,i*– 1
  if (X_{k–1}^(i) = 1 AND S_{k–1}^(i) = T_k) then {
     x = x+1; X_k^(i) = –x; q_len = q_len + 1; }
repeat for i=i*,...,n
  if (X_{k–1}^(i) < 0) then X_k^(i)=X_{k–1}^(i) – x;
```

As follows from the specification of Task 3, C(i,n,r,s) and D(i,n,r,s) will be estimated by taking sample means from $\chi_j^{(i)}$ and $\psi_j^{(i)}$ over L operating cycles, where a cycle is the time interval between two consecutive reconnections of e_i to e_0, i.e. one of the intervals $[\rho_{j-1}^{(i)}, \rho_j^{(i)})$, $j \geq 1$. Thus, the estimation procedure is constructed by embedding Tasks 2 and 3 into Procedure 1, yielding Procedure 2 outlined below.

Variables used by Procedure 2:

j : the number of the current cycle,

T1 and T0 : the sample values of $\chi_j^{(i)}$ and $\psi_j^{(i)}$

E1 and E0 : the sample means computed from $\chi_h^{(i)}$ and $\psi_h^{(i)}$ over h varying from 1 to j

$Y_k^{(i)}$: the state of connection between e_i and e_0 at T_k, $Y_k^{(i)} = 1$ if e_i is connected, otherwise $Y_k^{(i)} = 0$.

Procedure 2

T1 = 0; T0 = 0; E1 = 0; E0 = 0;

j = 1;

repeat for $k \geq 1$

{

obtain T_k and $X_k^{(1)},\ldots,X_k^{(n)}$ using Procedure 1;

compute $Y_k^{(i)}$ from $X_k^{(1)},\ldots,X_k^{(n)}$

if ($Y_{k-1}^{(i)}$ EQ 1) then {

T1 = T1 + ($T_k - T_{k-1}$);

if ($Y_k^{(i)}$ EQ 0) ## e_i is disconnected from e_0 at T_k

then {

E1 = E1·(j – 1)/j + T1/j; ## updating E1 during the cycle j

T1 = 0; }

}

```
if (Y_{k-1}^(i) EQ 0) {
  T0 = T0 + (T_k − T_{k-1});
  if (Y_k^(i) EQ 1) ## e_i is reconnected to e_0 at T_k
  then {
    E0 = E0·(j − 1)/j + T0/j; ## updating E0 at the end of the cycle j
    T0 = 0;
    j = j+1;
    if (j GT L) then terminate; }
}

} ## end of "repeat for k≥1"
```

Remarks:

1. E1 and E0 are updated based on the following formula:

(50) $$\mu_{n+1} = \mu_n \cdot n/(n+1) + x_{n+1}/(n+1)$$

where

(51) $$\mu_n = (x_1+...+x_n)/n$$

2. In the step k only the values $X_{k-1}^{(\cdot)}$, $X_k^{(\cdot)}$, T_{k-1}, T_k are used to update T0, T1, and possibly E0, E1, while the analogous values obtained in the steps 1,...,k−2 are irrelevant. In consequence, it is necessary to store only the X's and T's obtained in the current and the previous step of Procedure 2.

5. Some numerical results

In Tables 1 and 3 several results obtained using Procedure 2 are presented. It is assumed that L_i and R_i are exponentially distributed with $\lambda_i = 0.01$, $\mu_i = 0.1$, $1 \le i \le n$. The time unit is one hour. The results presented in Table 2 are obtained from (16), (22), (23), and (37).

Tab. 1. Estimated average disconnection periods for a two-component system

D(i,2,r,s)	r=1, s=1	r=1, s=2	r=2, s=1	r=2, s=2
i=1	10.84	10.84	10.01	10.01
i=2	10.46	10.45	10.23	10.24

Tab. 2. Exact average disconnection periods for a two-component system

D(i,2,r,s)	r=1, s=1	r=1, s=2	r=2, s=1	r=2, s=2
i=1	10.8(3)	10.8(3)	10.00	10.00
i=2	10.(45)	10.(45)	10.2381	10.2381

Tab. 3. Estimated average disconnection periods for a ten-component system

D(i,10,r,s)	r=1, s=1	r=1, s=2	r=2, s=1	r=2, s=2
i=1	16.60	15.23	10.43	10.35
i=5	15.20	15.03	11.14	11.12
i=10	13.95	14.11	12.10	12.11

It is interesting to see that if there is one repair team then D(i,n,r,s) is shorter for i=n than for i=1. This observation is confirmed by the exact results for n=2. To explain this fact note that e_n can fail only if $e_1,...,e_{n-1}$ are all in operating state. Thus the repair of e_n begins immediately after its failure, while e_1 has to wait if another component is under repair at the moment of e_1's failure.

6. Bibliography

1. R. Billinton, R. N. Allan: Reliability Evaluation of Engineering Systems, Springer 1992

2. R. Billinton, R. N. Allan: Reliability Evaluation of Power Systems, Plenum Press 1996

3. R. Billinton, W. Li: Reliability Assessment of Electrical Power Systems Using Monte Carlo Methods, Springer 1994

4. R. E. Brown: Electric Power Distribution Reliability, CRC Press 2002

5. W. Feller: An Introduction to Probability Theory and Its Applications, Wiley 1968

6. J. Karpiński, E. Korczak: Metody oceny niezawodności dwustanowych systemów technicznych, Omnitech Press 1990