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# BRIEF NOTES

## Field equations for micropolar current and a heat conducting magnetically-saturated solid

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STARTING from the balance of magnetization and the balance of energy, the equations of motion, boundary conditions and the energy equation are derived. Assuming the existence of internal state variables and postulating the entropy equation, constitutive equations and dissipation inequality are obtained.

### 1. Introduction

THE PURPOSE of the work is to derive basic field equations for an elastic micropolar solid under mechanical, thermal and electromagnetic influence. The thermodiffusion for the studied model, without electromagnetic influences, was analyzed in Paper [6]. The behaviour of deformable media in the electromagnetic field was studied in a number of works, e.g. in [1, 2, 3 and 4].

### 2. External electromagnetic field

The electromagnetic variables satisfy the Maxwell field equations for electric conductors which, in Gaussian units, take the form

$$C\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t}+4\pi C\mathbf{J}, \quad C\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}. \tag{2.1}$$

In addition to Eq. (2.1) the auxiliary Maxwell equations are satisfied identically:

$$\nabla\cdot\mathbf{D} = 4\pi q, \quad \nabla\cdot\mathbf{B} = 0, \tag{2.2}$$

where $C$ is the speed of light. For the electromagnetic field in matter, in the absence of polarization, the following relations exist

$$\mathbf{H} = \mathbf{B}-4\pi\mathbf{M}, \quad \mathbf{D} = \mathbf{E} \tag{2.3}$$

with $M$-magnetic moment per unit volume.

We introduce the following electromagnetic variables defined in points $X^K$ of the moving matter:

$$\boldsymbol{\mu} = \frac{\mathbf{M}}{\ }, \quad \boldsymbol{\epsilon} = \mathbf{E}+C^{-1}(\mathbf{v}\times\mathbf{B}), \quad \mathbf{j} = \mathbf{J}-C^{-1}e\mathbf{v}, \quad e = \frac{q}{\ }. \tag{2.4}$$

These variables are: $\boldsymbol{\mu}$—magnetization density, $\boldsymbol{\epsilon}$—electromotive intensity at a point moving with the particles of the motion, $\mathbf{j}$—conduction current relative to the particles of the motion, $e$—charge density and $\varrho$—mass density.

### 3. Magnetic exchange field

Suppose that the mass of the macroelement remains conserved during the regarded process, and since the material is magnetically saturated, the magnitude of the magnetic moment per unit mass $\boldsymbol{\mu}$ is conserved:

$$\boldsymbol{\mu}\cdot\boldsymbol{\mu} = \mu_s^2 \tag{3.1}$$

with $\mu_s = \text{const}$ when a homogeneous material is assumed. From Eq. (3.1) it follows immediately:

$$\boldsymbol{\mu}\cdot\dot{\boldsymbol{\mu}} = 0 \wedge (\boldsymbol{\mu}\cdot\dot{\boldsymbol{\mu}})_{;k} = 0. \tag{3.2}$$

According to Eq. $(3.2)_1$ we express the rate of magnetization in the following way:

$$\dot{\boldsymbol{\mu}} = \boldsymbol{\omega}\times\boldsymbol{\mu}, \tag{3.3}$$

$\boldsymbol{\omega}$ representing the angular velocity of magnetization.

We postulate the balance of magnetization in the integral form as follows:

$$\frac{d}{dt}\int_v \Gamma^{-1}\varrho\boldsymbol{\mu}\,dv = \int_v \varrho\boldsymbol{\mu}\times(\mathbf{B}+\mathbf{B}_{(L)})\,dv + \int_{\partial v}\varrho\boldsymbol{\mu}\times\mathbf{F}_{(\mu)}\,d\partial v. \tag{3.4}$$

Here $\Gamma$ is the gyromagnetic ratio, $\Gamma^{-1}\boldsymbol{\mu}$ the spin angular momentum and $\mathbf{B}_{(L)}$ the local material magnetic field. The second integral on the *rhs of* Eq. (3.4) represents the contribution of the magnetic exchange field $\mathbf{F}_{(\mu)}$, [3].

The application of the equation of balance (3.4) to an elementary tetrahedron yields the definition of the magnetic exchange tensor $A^{ij}$:

$$F^i_{(\mu)} = A^{ij}n_j. \tag{3.5}$$

Providing

$$(\mu_{j,l}A_k^{\cdot l})_{[j,k]} = 0, \tag{3.6}$$

we obtain the local form of balance of magnetization

$$\varrho\Gamma^{-1}\dot{\mu}^i = e^{ijk}[B_k + B_{(L)k} + \varrho^{-1}(\varrho A_k^{\cdot l})_{,l}]\varrho\mu_j. \tag{3.7}$$

From Eq. (3.7) it may be seen that without loss of generality one may put

$$\mu_j B^j_{(L)} = 0 \wedge \mu_j A^{jl} = 0. \tag{3.8}$$

Multiplying both sides of Eq. (3.7) with $\boldsymbol{\omega}$, we obtain a useful relation:

$$\varrho\dot{\mu}_i[B^i + B^i_{(L)} + \varrho^{-1}(\varrho A^{il})_{,l}] = 0. \tag{3.9}$$

### 4. Balance of energy

We add to the expression of energy balance derived in [6] the contribution due to the existence of the electromagnetic field and magnetic exchange field:

$$Q_{(em)} = -\frac{1}{4\pi}\int_v\left(\mathbf{B}\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}\frac{\partial\mathbf{E}}{\partial t}\right)dv - \frac{C}{4\pi}\oint_{\partial v}(\mathbf{E}\times\mathbf{H})\,d\partial v = \int_v\left(\mathbf{E}\cdot\mathbf{J} - \mathbf{M}\frac{\partial B}{\partial t}\right)dv \tag{4.1}$$

and

$$(4.2)\qquad Q_{(\mu)} = \int_{\partial v} \varrho(\boldsymbol{\mu}\times F_{(\mu)})\cdot \boldsymbol{\omega}\, d\partial v = \int_{\partial v} \varrho\dot{\mu}_i F^i_{(\mu)}\, d\partial v.$$

In that way the equation of energy balance reads

$$(4.3)\qquad \int_v \varrho(\dot{v}^i v_i + \Gamma^{ij}\nu_{ij} + \dot{u})\,dv = \int_v \varrho(f^i v_i + l^{ij}\nu_{ij} + h)\,dv + \int_v \Big(j^i\varepsilon_i + \frac{1}{C} e^{ijk} v_i j_j B_k + \frac{1}{C}\varrho e\varepsilon^i v_i - \varrho\overline{B^i\mu_i}^{\cdot} + \varrho B^i\dot{\mu}_i + \varrho B^{j,i}\mu_j v_i\Big)\,dv + \oint_{\partial v} (T^i v_i + M^{ij}\nu_{ij} + \varrho F^i_{(\mu)}\dot{\mu}_i)\,d\partial v,$$

where

$$(4.4)\qquad \Gamma^{ij} = -\Gamma^{ji} = I^{KL}\dot{\chi}^{[i}_{\cdot K}\chi^{j]}_{\cdot L},$$

represents inertia spin, $I^{KL}$ being inertia coefficients of the macroelement, $v^i$—velocity of the macroelement mass center, $\nu_{ji} = -\nu_{ij}$—gyration tensor; $u$—internal energy density; $T^i$, $H^{ij} = -H^{ji}$—surface force and surface couple per unit area; $f^i$, $l^{ij} = -l^{ji}$—body force and body couple densities; $q$—heat influx per unit area; $h$—heat supply density.

The application of Eq. (4.3) to an elementary tetrahedron yields the definitions of the stress tensor, couple stress tensor and heat flux vector:

$$(4.5)\qquad \begin{aligned} T^i &= t^{ij}n_j, & q &= q^i n_i,\\ M^{ij} &= M^{ijk}n_k, & F^i_{(\mu)} &= A^{ij}n_j, \end{aligned}$$

while Eq. $(4.5)_4$ coincide with Eq. (3.5), $n_i$ being the outwardly directed normal. Relations (4.5) are valid for an interior point. If it is a boundary point, supposing the boundary surface has a continuous tangent plane, the relations (4.5) become boundary conditions for the prescribed values at the *lhs* of Eqs. (4.5).

Taking account of Eqs. (3.9) and (4.5) and using the theorem of divergence, we further obtain

$$(4.6)\qquad \int_v \varrho(\dot{v}^i v_i + \Gamma^{ij}\nu_{ij} + \dot{u})\,dv = \int_v \Big[\varrho\Big(f^i v_i + l^{ij}\nu_{ij} + h + \dot{\mu}_{i,l}A^{il} - B^i_{(L)}\dot{\mu}_i - \overline{B^i\mu_i}^{\cdot} + \mu_j B^{j,i}v_i + \frac{1}{C}e\varepsilon^i v_i\Big) + t^{ij}_{,j}v_i + t^{ij}v_{i,j} + M^{ijk}_{,k}\nu_{ij} + M^{ijk}\nu_{ij,k} + j^i\varepsilon_i + \frac{1}{C}e^{ijk}v_i j_j B_k\Big]\,dv.$$

Requiring the invariance of Eq. (4.6) under the added virtual velocities corresponding to rigid body motions,

$$(4.7)\qquad \begin{aligned} v_i &\to v_i + a_i, & \dot{\mu}_i &\to \dot{\mu}_i + \Omega_{il}\mu^l,\\ v_{i,j} &\to v_{i,j} + \Omega_{ij}, & \dot{\mu}_{i,j} &\to \dot{\mu}_{i,j} + \mu^l_{,j}\Omega_{il},\\ \nu_{ij} &\to \nu_{ij} + \Omega_{ij}, \end{aligned}$$

we obtain the following equations of motion:

$$(4.8)\qquad \begin{aligned} &t^{ij}_{,j} + \frac{1}{C}(e^{ijk}j_j B_k + \varrho e\varepsilon^i) + \varrho\mu_j B^{j,i} = \varrho\dot{u}^i,\\ &t^{[ij]} + M^{ijk}_{,k} + \varrho l^{ij} + \varrho\mu^{[i}B^{j]}_{(L)} + \varrho\mu^{[j}_{,l}A^{i]l} = \varrho\Gamma^{ij}. \end{aligned}$$

The last term at *lhs* of Eq. $(4.8)_2$ vanishes, providing that Eq. (3.6) is satisfied.

Introducing the quantity

$$\chi = u+\mu_i B^i \tag{4.9}$$

having the character of energy density and using Eq. (4.8), we derive the local form of the energy equation as follows:

$$\varrho\dot{\chi} = t^{ij}(v_{i,j}-\nu_{ij})+M^{ijk}\nu_{ij,k}+\varrho A^{ij}(\dot{\mu}_{i,j}-\nu_{il}\mu^l_{,j})-\varrho B^i_{(L)}(\dot{\mu}_i-\nu_{ij}\mu^j)+\varrho h+q^i_{,i}+j^i\varepsilon_i. \tag{4.10}$$

## 5. Thermodynamical restrictions and constitutive equations

We postulate the thermodynamical model with the entropy balance in the form of the following equation:

$$\varrho\dot{\eta} = \frac{1}{\theta}(q^i_{,i}+\varrho h+\mathbf{j}\cdot\boldsymbol{\epsilon}) = \left[\left(\frac{q^i}{\theta}\right)_{,i}+\frac{\varrho h}{\theta}\right]+\frac{q^i\theta_{,i}}{\theta^2}+\frac{j^i\varepsilon_i}{\theta}; \tag{5.1}$$

with $\eta$ being the entropy density. Introducing the free energy density,

$$\psi = \chi-\theta\eta,$$

$$\varrho\dot{\psi} = t^{ij}(v_{i,j}-\nu_{ij})+M^{ijk}\nu_{ij,k}+\varrho A^{lj}(\dot{\mu}_{l,j}-\nu_{li}\mu^i_{,j})-\varrho B^i_{(L)}(\dot{\mu}_i-\nu_{ij}\mu^j)-\varrho\eta\dot{\theta} \tag{5.2}$$

we get the expression for the dissipation function in the form

$$\varrho\phi = \varrho\theta\dot{\eta}-\theta\left(\frac{q^i}{\theta}\right)_{,i}-\varrho h = -\varrho\dot{\psi}+t^{ij}(v_{i,j}-\nu_{ij})+M^{ijk}\nu_{ij,k}$$
$$+\varrho A^{lj}(\dot{\mu}_{l,j}-\nu_{li}\mu^i_{,j})-\varrho B^i_{(L)}(\dot{\mu}_i-\nu_{ij}\mu^j)+\frac{q^i\theta_{,i}}{\theta}+j^i\varepsilon_i-\varrho\eta\dot{\theta}. \tag{5.3}$$

As a sequence of the Clausius-Duhem inequality, we assert that Eq. (5.3) must be non-negative:

$$\varrho\phi \geqq 0. \tag{5.4}$$

We obtain [7] the derivatives at the *rhs* of Eq. (5.3) as follows:

$$\begin{aligned}
v_{i,j}-\nu_{ij} &= \chi^{K}_{\cdot i}X^{L}_{;j}\dot{\varepsilon}_{KL} \quad (\varepsilon_{KL} = x_{l;L}\chi^{l}_{\cdot K}-G_{KL}),\\
\nu_{ij,k} &= \chi^{K}_{\cdot i}\chi^{L}_{\cdot j}X^{M}_{;k}K_{KLM} \quad (K_{KLM} = \chi_{lK}\chi^{l}_{\cdot L;M}),\\
\dot{\mu}_i-\nu_{ij}\mu^i &= \chi^{K}_{\cdot i}\dot{m}_K \quad (m_K = \chi^{k}_{\cdot K}\mu_k),\\
\dot{\mu}_{l,j}-\nu_{li}\mu^i_{,j} &= \chi^{K}_{\cdot l}X^{L}_{;j}\dot{m}_{KL} \quad (m_{KL} = \chi^{k}_{\cdot K}\mu_{k;L}).
\end{aligned} \tag{5.5}$$

From Eqs. (5.3), (5.4) and (5.5) we obtain

$$-\varrho\dot{\psi}+t^{ij}\chi^{L}_{\cdot i}X^{K}_{;j}\dot{\varepsilon}_{KL}+M^{ijk}\chi^{K}_{\cdot i}\chi^{L}_{\cdot j}X^{M}_{;k}\dot{K}_{KLM}+\varrho A^{lj}\chi^{K}_{\cdot l}X^{L}_{;j}\dot{m}_{KL}-\varrho B^i_{(L)}\chi^{K}_{\cdot i}\dot{m}_K$$
$$-\varrho\eta\dot{\theta}+\frac{q^i\theta_{,i}}{\theta}+j^i\varepsilon_i \geqq 0. \tag{5.6}$$

Now we assume that there exists a set of $n$-internal state variables $\alpha_{(\nu)}$ having an influence on the dissipation of energy. Let by that assumption and by the expression (5.6), we come to the following list of arguments of the internal energy density:

$$\text{(5.7)} \qquad \varepsilon_{KL}, \quad K_{KLM}, \quad m_K, \quad m_{KL}, \quad \theta, \quad \frac{\theta_{,i}}{\theta}, \quad \varepsilon_i, \quad \alpha_{(\nu)}, \qquad \nu = 1, 2, \ldots, n.$$

Following the Truesdell's principle of equipresence, we suppose that all other response functions in Eq. (5.3) also depend on the arguments (5.7). The rate of change of $\alpha_{(\nu)}$ may also be governed by a generally nonlinear function $f_{(\nu)}$ of the arguments (5.7). Hence:

$$\text{(5.8)} \qquad \begin{aligned} \dot{\alpha}_{(\nu)} &= f_{(\nu)}\left(\varepsilon_{KL}, K_{KLM}, m_K, m_{KL}, \theta, \frac{\theta_{,i}}{\theta}, \varepsilon_i, \alpha_{(\nu)}\right), \\ q^k &= q^k\left(\varepsilon_{KL}, K_{KLM}, m_K, m_{KL}, \theta, \frac{\theta_{,i}}{\theta}, \varepsilon_i, \alpha_{(\nu)}\right), \\ j^k &= j^k\left(\varepsilon_{KL}, K_{KLM}, m_K, m_{KL}, \theta, \frac{\theta_{,i}}{\theta}, \varepsilon_i, \alpha_{(\nu)}\right). \end{aligned}$$

Returning to the conditions of the constraints in Eqs. (3.2), introducing four Lagrangian multipliers $N$ and $N^K$ and proceeding in the usual manner, we obtain from Eq. (5.6) the following:

$$\text{(5.9)} \qquad -\left(\varrho\frac{\partial\psi}{\partial\varepsilon_{KL}} - t^{ij}\chi^{L}_{\cdot i}X^{K}_{;j}\right)\dot{\varepsilon}_{KL} - \left(\varrho\frac{\partial\psi}{\partial K_{KLM}} - M^{ijk}\chi^{K}_{\cdot i}\chi^{L}_{\cdot j}X^{M}_{;k}\right)\dot{K}_{KLM} - \left[\varrho\frac{\partial\psi}{\partial m_K} + \varrho(B^i_{(L)} + N\mu^i + N^l\mu^i_{;L})\chi^{K}_{\cdot i}\right]\dot{m}_K - \left[\varrho\frac{\partial\psi}{\partial m_{KL}} - \varrho(A^{lj} - N^P\mu^l x^j_{;P})\chi^{K}_{\cdot l}X^{L}_{;j}\right]\dot{m}_{KL} - \varrho\left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} - \varrho\frac{\partial\psi}{\partial\left(\frac{\theta_{,i}}{\theta}\right)}\overline{\left(\frac{\theta_{,i}}{\theta}\right)}^{\bullet} - \varrho\frac{\partial\psi}{\partial\varepsilon_i}\dot{\varepsilon}_i - \varrho\frac{\partial\psi}{\partial\alpha_{(\nu)}}\dot{\alpha}_{(\nu)} + \frac{q^i\theta_{,i}}{\theta} + \varepsilon_i j^i \geqq 0.$$

In order to have Eq. (5.4) satisfied for any independent thermodynamical process, the following equations must take place:

$$\text{(5.10)} \qquad \begin{aligned} &t^{ij} = \varrho\frac{\partial\psi}{\partial\varepsilon_{KL}}\chi^{i}_{\cdot K}x^{j}_{;L}, \qquad M^{ijk} = \varrho\frac{\partial\psi}{\partial K_{KLM}}\chi^{i}_{\cdot K}\chi^{j}_{\cdot L}x^{k}_{;M}, \qquad \eta = -\frac{\partial\psi}{\partial\theta}, \\ &B^i_{(L)} = -\frac{\partial\psi}{\partial m_K}\chi^{i}_{\cdot K} - N\mu^i - N^K\mu^i_{;K}, \qquad A^{ij} = \left(\frac{\partial\psi}{\partial m_{KL}}\chi^{i}_{\cdot K} + N^L\mu^i\right)x^{j}_{;L}; \\ &\partial\psi/\partial\left(\frac{\theta_{,k}}{\theta}\right) = 0, \qquad \partial\psi/\partial\varepsilon_k = 0. \end{aligned}$$

The last two equations imply that $\psi$, $t^{ij}$, $M^{ijk}$, $B^i_{(L)}$, $A^{ij}$ and $\eta$ are independent of $\left(\frac{\theta_{,k}}{\theta}\right)$ and $\varepsilon_k$. Hence these six response functions depend only on

$$\text{(5.11)} \qquad \varepsilon_{KL}, K_{KLM}, m_K, m_{KL}, \theta, \alpha_{(\nu)}, \qquad \nu = 1, 2, \ldots, n.$$

So the list of arguments (5.7) with the response functions in Eqs. (5.8) and the list of arguments (5.11) with the response functions in Eqs. (5.10) characterize the behaviour of the model in question. The non-vanishing part remaining in *lhs* of Eq. (5.9) represents the thermodynamical restriction imposed upon $\psi, f_{(\nu)}, q^i$ and $j^i$ which must be satisfied, together with Eqs. (5.8) and (5.10), in every thermodynamically admissible process of the body. This restriction is found in the form of the dissipation inequality [5]:

$$\varrho \frac{\partial \psi}{\partial \alpha_{(\nu)}} f_{(\nu)} - q^i \left( \frac{\theta_{,i}}{\theta} \right) - j^i \varepsilon_i \leqq 0. \tag{5.12}$$

The Lagrangian multipliers in Eqs. (5.10) may be found from Eqs. (3.8) in the form

$$\mu_s^2 N = -\frac{\partial \psi}{\partial m_K} m_K, \qquad \mu_s^2 N^L = -\frac{\partial \psi}{\partial m_{KL}} m_K. \tag{5.13}$$

We finally remark that the expression for $\psi$ satisfies the principle of material objectivity: if $\psi$ is a single-valued function of the arguments (5.11), it is a scalar invariant under rigid rotation of the body, as may be easily checked. The condition (3.6), assuring the invariance of the exchange field in a rigid rotation of the spin system with respect to the material lattice, is thus also satisfied.

## References

1. C. TRUESDELL and R. A. TOUPIN, *The classical field theories*, Handbuch der Physik, S. Flügge, Bd. III/1, Springer-Verlag, Berlin 1960.
2. G. A. MAUGIN and A. C. ERINGER, J. Math. Phys. **13**, 143–155, 1334–1347, 1972.
3. H. F. TIERSTEN and C. F. TSAI, J. Math. Phys., **13**, 361–378, 1972.
4. H. PARKUS, Arch. Mech., **24**, 819–827, 1972.
5. B. COLEMAN and M. GURTIN, J. Chem. Phys., **47**, 2, 1967.
6. M. PLAVŠIĆ and N. NAERLOVIČ, Arch. Mech., **28**, 4, 1976.
7. M. PLAVŠIĆ and J. JARIĆ, Rozpr. Inżyn., **22**, 181–211, 1974.

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