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On the non-standard formulation of mechanics

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IN THE known formulations of mechanics, real bodies are represented either by discrete sets of material points or by material continua. The purpose of the present paper is to introduce a more general model of the real bodies, which includes both discrete and continuous material systems as special cases. The approach is based on the reinterpretation of the relations of classical mechanics from the point of view of non-standard analysis [1].

W znanych sformułowaniach mechaniki ciała rzeczywiste występują pod postacią albo dyskretnych zbiorów punktów materialnych albo jako kontinua materialne. Głównym celem niniejszej pracy jest wprowadzenie bardziej ogólnego modelu ciał rzeczywistych w mechanice, który obejmuje m.in. zarówno dyskretne zbiory punktów materialnych jak i kontinua materialne jako przypadki szczególne. Zastosowane podejście jest oparte na zinterpretowaniu znanych aksjomatów mechaniki klasycznej w ramach niestandardowej analizy matematycznej [1].

В известных формулировках механики реальные тела выступают в виде или дискретных множеств материальных точек, или как материальные континуума. Главной целью настоящей работы является введение более общей модели реальных тел в механике, которая охватывает, между прочим, так дискретные множества материальных точек, как и материальные континуума как частные случаи. Примененный подход опирается на интерпретации известных аксиом классической механики в рамках нестандартного математического анализа, [1].

Introduction

REAL bodies are in classical mechanics represented either by discrete sets of material points or by material continua. The main purpose of the present paper is to construct a new model of a real body from which both discrete and continuous material systems can be derived as special cases. The approach will be based on the non-standard reinterpretation of physical space from the point of view of Q-topology (cf. [1], pp. 99, 106). Notions of Q-topology related to physical space and to objects embedded in that space, enable us to represent real bodies by systems of material points in which, roughly speaking, the distances between points are greater than a certain positive "infinitesimal" real number. Such systems will be called Q-discrete material systems. To formulate the mechanics of Q-discrete material systems, we shall extend the known relations of Newtonian mechanics. It will be shown that from such a non-standard development of mechanics, we can derive not only the mechanics of known discrete systems and material continua but also other new models of real bodies.

The reader is assumed to be familiar with the fundamental concepts of the non-standard analysis given in [1]; some additional information can be found in the Appendix. Throughout the paper, we use small thin-face letters to denote scalars, small bold-face letters to denote vectors, and bold face capital letters to denote second order tensors or matrices.

1. Q-discrete material systems

In the Gallilean space-time M, let there be given an arbitrary but fixed inertial coordinate system $\xi: M \to E^3 \times R$, where E^3 stands for the (standard) physical space and R stands for the (standard) time axis. By $^*E^3$, $^*R$ we shall denote extensions of the sets E^3, R, respectively, $E^3 \subset {}^*E^3$, $R \subset {}^*R$, and refer to $^*E^3$ as the physical space and to $^*R$ as the time axis (cf. [1], p. 36).

Throughout the paper, we consider a fixed Q-finite set S. This is a set of elements which is in one-to-one correspondence with the Q-finite sequence $\{1, 2, \dots, \nu\}$ of natural numbers in $^*N$—i.e., we can put $S = \{P_1, P_2, \dots, P_\nu\}$, where ν can be a finite or infinite positive integer (cf. [1], p. 123).

The elements $P \in S$ (or P_γ, $\gamma = 1, 2, \dots, \nu$) will be called material points, provided that there is given a mapping $m: S \to {}^*R^+$, where $m(P) > 0$ is finite for each $P \in S$. The positive finite real numbers $m(P)$ will be called the masses of material points. By definition, each mass $m(P)$ has a uniquely defined standard part $stm(P)$; if $stm(P) = 0$, then the mass of the point is infinitesimal.

By the configuration of S, we shall mean an injection $\varkappa: S \to {}^*E^3$, where $\varkappa(S)$ is a Q-discrete internal set in the principal galaxy of the physical space $^*E^3$ (cf. Appendix). The one-parameter family $\varkappa_t$ of configurations, $\varkappa_t(P) = \varkappa(P, t)$, $P \in S$, $t \in (t_0, t_1) \subset R^*$, t_0, t_1—standard, will be called the motion of the system S. Throughout the paper, we confine ourselves to motions which can be represented by internal functions having the form:

$$(1.1) \qquad \varkappa(P, t) = {}_0\varkappa(P, t) + \sum_{\pi=1}^{\infty} {}_\pi\varkappa(P, t)|d\varepsilon(P)|^\pi; \quad P \in S, \quad t \in (t_0, t_1) \subset {}^*R,$$

where $d\varepsilon(P)$ is the infinitesimal positive real number and the functions ${}_\pi\varkappa(P, \cdot)$, $\pi \in {}^*N$ (π ranges over all natural numbers, finite and infinite) are standard, ${}_\pi\varkappa = st_\pi\varkappa$ for each standard $t \in (t_0, t_1)$, and have continuous time derivatives up to the second order in (t_0, t_1). The function ${}_0\varkappa(P, t)$ will be called the basic motion and the function ${}_n\varkappa(P, t)|d\varepsilon(P)|^n$, $n \geqslant 1$ is said to be the relative superimposed motion of n-th order. We assume that all infinite sums under consideration exist (cf. [1], p. 123).

The velocity and acceleration in an arbitrary motion of S will be defined by:

$$(1.2) \qquad \begin{aligned} \dot{\varkappa}(P, t) &\equiv {}_0\dot{\varkappa}(P, t) + \sum_{\pi=1}^{\infty} {}_\pi\dot{\varkappa}(P, t)|d\varepsilon(P)|^\pi, \\ \ddot{\varkappa}(P, t) &\equiv {}_0\ddot{\varkappa}(P, t) + \sum_{\pi=1}^{\infty} {}_\pi\ddot{\varkappa}(P, t)|d\varepsilon(P)|^\pi, \end{aligned}$$

respectively, where π runs over all natural numbers of $^*N$.

The set S of material points related to the physical space $^*E^3$ by means of its motion $\varkappa_t: S \times (t_0, t_1) \to {}^*E^3$, will be called the Q-discrete material system.

In this paper, we assume that the motion of the material system can be restricted by contraints having the form:

$$(1.3) \qquad \alpha_\mu\big(t, \varkappa(P_{\mu_1}, t), \dots, \varkappa(P_{\mu_n}, t)\big) = 0, \quad \mu = 1, \dots, r, \quad t \in (t_0, t_1) \subset {}^*R,$$

where $\{P_{\mu_1}, \ldots, P_{\mu_n}\}$ is an absolutely finite subset of S (i.e., n is a standard positive integer). α_μ are known differentiable standard functions and r is a natural number (finite or infinite). The virtual displacements $\delta\boldsymbol{\varkappa}(P_\gamma, t)$ will be defined in the known way:

$$\sum_{\gamma=1}^{\nu} \frac{\partial\alpha_\mu}{\partial\boldsymbol{\varkappa}(P_\gamma, t)} \cdot \delta\boldsymbol{\varkappa}(P_\gamma, t) = 0, \quad \mu = 1, 2, \ldots, r, \tag{1.4}$$

where the derivatives $\partial\alpha_\mu/\partial\boldsymbol{\varkappa}(P, t)$ are interpreted as extensions of the derivatives of the standard functions α_μ. A more general form of constraints than that given by the Eqs. (1.3) can be also taken into account.

We postulate that to any motion of the material system S is assigned a vector field $\mathbf{f}(P, t)$, $P \in S$, $t \in (t_0, t_1) \subset {}^*R$, with finite values in ${}^*R^3$. For a fixed P and t, the vector $\mathbf{f}(P, t)$ is said to be the resultant force acting at the material point P at the time instant t. Moreover, we assume that the function $\mathbf{f}(P, \cdot)$ is internal for every $P \in S$. Throughout the paper we shall confine ourselves to the internal functions $\mathbf{f}(P, \cdot)$ which can be given by:

$$\mathbf{f}(P, t) = {}_0\mathbf{f}(P, t) + \sum_{\pi=1}^{\infty} {}_\pi\mathbf{f}(P, t)|d\varepsilon(P)|^\pi, \quad t \in (t_0, t_1) \subset {}^*R, \tag{1.5}$$

where ${}_\pi\mathbf{f}(P, \cdot)$ are, for each $\pi \in {}^*N$, continuous standard functions, and the positive integer π in Eq. (1.5) runs over all natural numbers (finite and infinite).

Making use of primitive concepts introduced in this Section, we can formulate foundations of dynamics of Q-discrete material systems.

2. Dynamics of Q-discrete material systems

Let P be an arbitrary material point of the mass $m(P)$, belonging to a Q-discrete material system. We postulate that the motion of this point is governed by the known relation:

$$m(P)\ddot{\boldsymbol{\varkappa}}(P, t) = \mathbf{f}(P, t); \quad P \in S, \quad t \in (t_0, t_1) \subset {}^*R, \tag{2.1}$$

where $\ddot{\boldsymbol{\varkappa}}(P, \cdot)$, $\mathbf{f}(P, \cdot)$ are, for each $P \in S$, the internal functions defined on $(t_0, t_1) \subset {}^*R$ (in this paper they are assumed to have the form given by the Eqs. $(1.2)_2$, (1.5)). Moreover, we have to postulate, in each problem under consideration, suitable defining equations for the resultant forces $\mathbf{f}(P_\gamma, t)$, $\gamma = 1, 2, \ldots, \nu$, where ν is, as usual, the number (finite or infinite) of points in the system S. We confine ourselves here to defining equations of the "elastic type", which have the form:

$$\mathbf{f}(P, t) = \mathbf{h}\big(P, t, \boldsymbol{\varkappa}(P, t)\big) + \underset{\overline{P} \in S_P}{\mathbf{g}}\big(P, t|\boldsymbol{\varkappa}(P, t) - \boldsymbol{\varkappa}(\overline{P}, t)|\big) + \mathbf{r}(P, t), \quad P \in S, \tag{2.2}$$

where $\mathbf{h}(P, \cdot)$ is the interaction of the material point P with external fields, $\mathbf{g}(P, \cdot)$ is the interaction between P and other points (S_P is a known absolutely finite set, $S_P \subset S - \{P\}$) and $\mathbf{r}(P, t)$ is the reaction force due to the constraints. The functions $\mathbf{h}(P, \cdot)$, $\mathbf{g}(P, \cdot)$ are assumed to be known and standard, $\mathbf{r}(P, \cdot)$ is an internal function (for each $P \in S$), and the following identities must hold:

$$\sum_{\gamma=1}^{\nu} \mathbf{g}(P_\gamma, \cdot) = 0, \quad \sum_{\gamma=1}^{\nu} \boldsymbol{\varkappa}(P_\gamma, \cdot) \times \mathbf{g}(P_\gamma, \cdot) = 0. \tag{2.3}$$

Finally, we postulate that the reaction forces $\mathbf{r}(P, t)$ are related to the constraints (1.3) by the known ideality condition:

$$\sum_{\gamma=1}^{r} \mathbf{r}(P_\gamma, t) \cdot \delta \boldsymbol{\varkappa}(P_\gamma, t) = 0, \tag{2.4}$$

where the virtual displacements $\delta\boldsymbol{\varkappa}(P, t)$ are arbitrary internal solutions of the Eqs. (1.4).

The Eqs. (2.1), (2.2), (1.3), (1.4), (2.4) describe the elastodynamics of Q-discrete material systems. All relations mentioned here are assertions transferred from classical discrete mechanics (i.e., from mechanics of S-discrete systems, cf. [1], p. 113 and Appendix) to the mechanics of Q-discrete material systems. We shall see, in what follows, that such an approach leads to new results which cannot be deduced from the known classical discrete mechanics (cf. [1], p. 50). In general, the Q-discrete material systems represent a wider class of models of real bodies then the classical—i.e., S-discrete material systems—since Q-topology (our main mathematical tool) is finer than S-topology, which is applied in the problems of classical discrete or continuous material systems.

3. Absolutely finite material systems

Throughout this Section, we assume that S is an absolutely finite set, $S = \{P_1, P_2, \ldots, P_r\}$, r being the standard positive integer. From now on, we shall use the denotations $\alpha_\gamma(\cdot) \equiv \alpha(P_\gamma, \cdot)$ for an arbitrary function $\alpha(P, \cdot)$, putting $m_\gamma \equiv m_\gamma(P_\gamma)$, $\mathbf{h}_\gamma(\cdot) \equiv \mathbf{h}(P_\gamma, \cdot)$ etc. Let us define the following vector functions with values in the three dimensional space:

$$\begin{gathered} {}_\pi\mathbf{q}(t) \equiv \big({}_\pi\boldsymbol{\varkappa}(P_\gamma, t)\big), \quad \pi = 0, 1, 2, \ldots, \quad \gamma = 1, 2, \ldots, r, \\ \mathbf{b}\big(t, q(t)\big) \equiv \Big(\mathbf{h}_\gamma\big(t, \boldsymbol{\varkappa}(P_\gamma, t)\big)/m_\gamma + \mathbf{g}_\gamma\big(t, |\boldsymbol{\varkappa}(P_\gamma, t) - \boldsymbol{\varkappa}(P, t)|\big)/m_\gamma\Big). \end{gathered} \tag{3.1}$$

From (1.1) we obtain $\mathbf{q}(t) \equiv \sum_{\pi=0} {}_\pi\mathbf{q}(t) d\varepsilon^\pi$, assuming that $d\varepsilon(P_\gamma) = d\varepsilon = \text{const}$. From (2.1), (2.2), (1.2) and (3.1), assuming that there are no constraints, we arrive at:

$$ {}_0\ddot{\mathbf{q}}(t) + d_0\ddot{\mathbf{q}}(t) = \mathbf{b}\big(t, {}_0\mathbf{q}(t) + d_0\mathbf{q}(t)\big), \quad d_0\mathbf{q}(t) \equiv \sum_{\pi=1}^{\infty} {}_\pi\mathbf{q}(t) d\varepsilon^\pi. \tag{3.2}$$

Let the function $\mathbf{b}(t, \cdot)$ by analytical in the neighborhood of the basic motion ${}_0\mathbf{q}(t)$; then,

$$\mathbf{b}\big(t, {}_0\mathbf{q}(t) + d_0\mathbf{q}(t)\big) = \mathbf{b}\big(t, {}_0\mathbf{q}(t)\big) + \mathbf{B}\big(t, {}_0\mathbf{q}(t)\big) d_0\mathbf{q}(t) + \ldots, \tag{3.3}$$

where $\mathbf{B} \equiv \partial\mathbf{b}/\partial\mathbf{q}$. Moreover, let the masses m_γ be standard for the time being. Then, combining the Eqs. (3.2) and (3.3), we arrive after simple calculations at the following system of standard differential vector equations:

$$\begin{aligned} {}_0\ddot{\mathbf{q}} &= \mathbf{b}(t, {}_0\mathbf{q}), \\ {}_1\ddot{\mathbf{q}} &= \mathbf{B}(t, {}_0\mathbf{q})\,{}_1\mathbf{q}, \\ {}_2\ddot{\mathbf{q}} &= \mathbf{B}(t, {}_0\mathbf{q})\,{}_2\mathbf{q} + \mathbf{p}^1(t, {}_0\mathbf{q}; {}_1\mathbf{q}), \\ {}_n\ddot{\mathbf{q}} &= \mathbf{B}(t, {}_0\mathbf{q})\,{}_n\mathbf{q} + \mathbf{p}^{n-1}(t, {}_0\mathbf{q}; {}_1\mathbf{q}, \ldots, {}_{n-1}\mathbf{q}), \quad n = 3, 4, 5, \ldots, \end{aligned} \tag{3.3'}$$

where $\mathbf{p}^m(t_0, {}_0\mathbf{q}; {}_1\mathbf{q}, \ldots, {}_m\mathbf{q})$, $m = 1, 2, \ldots$, are the polynomials of the $m+1$ order in arguments ${}_1\mathbf{q}, {}_2\mathbf{q}, \ldots, {}_m\mathbf{q}$, such that $\mathbf{p}^m(t, {}_0\mathbf{q}; \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) \equiv \mathbf{0}$.

Let the initial condition for the vector function $\mathbf{q}(t)$ be given in the form $\mathbf{q}(t_0) = \bar{\mathbf{q}}$, $\dot{\mathbf{q}}(t_0) = \bar{\mathbf{v}}$, where $\bar{\mathbf{q}} = st\bar{\mathbf{q}}$, $\bar{\mathbf{v}} = st\bar{\mathbf{v}}$ are known standard vectors in R^{3r}. It follows that ${}_n\mathbf{q}(t_0) = 0$, ${}_n\dot{\mathbf{q}}(t_0) = 0$ for $n \geqslant 1$. From the Eq. $(3.3)_2$ we obtain now ${}_1\mathbf{q}(t) = 0$ and then from the Eqs. $(3.3)_{3,4,\ldots}$ it succesively follows that ${}_2\mathbf{q}(t) = 0$, ${}_3\mathbf{q}(t) = 0$ etc. Hence, $\mathbf{q}(t) = {}_0\mathbf{q}(t)$, and we arrive at the mechanics of classical discrete systems of material points, governed by the Eq. $(3.3)_1$.

If the vector ${}_0\mathbf{q}$ represents the equilibrium position of the system—i.e., if $\mathbf{b}(t_0, {}_0\mathbf{q}) = 0$ for each t—then the Eq. $(3.3)_2$ describes the problem of small oscillations. If the initial conditions have the form $\mathbf{q}(t_0) = \mathbf{q}_0 + {}_1\bar{\mathbf{q}}\,d\varepsilon + {}_2\bar{\mathbf{q}}\,d\varepsilon^2$, $\dot{\mathbf{q}}(t_0) = {}_0\bar{\mathbf{v}} + {}_1\bar{\mathbf{v}}\,d\varepsilon + {}_2\bar{\mathbf{v}}\,d\varepsilon^2$, where ${}_0\bar{\mathbf{q}}$, ${}_1\bar{\mathbf{q}}$, ${}_2\bar{\mathbf{q}}$, ${}_0\bar{\mathbf{v}}$, ${}_1\bar{\mathbf{v}}$, ${}_2\bar{\mathbf{v}}$ are standard vectors, then the Eq. $(3.3)_3$ characterizes small motions superimposed on a basic motion ${}_0\mathbf{q}(t)$. Note that, in dealing with the non-standard approach to mechanics, no linearisation procedure is needed to obtain the governing equations for small oscillations or small superimposed motion.

It may be observed that not all absolutely finite Q-discrete material systems reduce to the classical—i.e. S-discrete material systems. As an example, let us take the absolutely finite system of points $Z = \{P_1, P_2, \ldots, P_r\}$, $r \geqslant 4$, motion of which is restricted to homogeneous deformations only, given by:

$$\varkappa(P_\gamma, t) = \boldsymbol{\chi}(t) + \mathbf{F}(t)\mathbf{d}_\gamma, \quad \gamma = 1, \ldots, r, \quad P_\gamma \in Z, \tag{3.4}$$

where $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r\}$ is a see of r known infinitesimal non-complanar vectors, and $\boldsymbol{\chi}(t)$, $\mathbf{F}(t)$ are arbitrary finite vector and tensor functions, respectively. We assume that $\det\mathbf{F}(t) > 0$ for $t \in (t_0, t_1)$ and that the functions $\boldsymbol{\chi}$, $\mathbf{F}$ and the vectors $\mathbf{d}_\gamma$ have representations in the form of infinite series (cf. (1.1), (1.2) or (1.5)). We also assume that $\Sigma\, m_\gamma \mathbf{d}_\gamma = 0$—i.e., that $\boldsymbol{\chi}(t)$ is the motion of the mass center of the system Z. Under some further restrictions, which will be mentioned in what follows, the system Z will be referred to as a simple particle.

As subject of our investigations, we shall take now an absolutely finite set, $\Omega = \{Z_1, \ldots, Z_k\}$ of mutually interactig simple particles. To obtain the appropriate system of governing equations, let us combine the Eqs. (2.1), (3.4), (2.2), (2.4). The resulting relation has the form

$$\sum_{\gamma=1}^{r} \{m_\gamma(Z)[\ddot{\boldsymbol{\chi}}(Z, t) + \ddot{\mathbf{F}}(Z, t)\mathbf{d}_\gamma(Z)] - \mathbf{h}_\gamma(Z, t, \cdot)$$
$$- \mathbf{g}_\gamma(Z, t, \cdot)\} \cdot [\delta\boldsymbol{\chi}(Z, t) + \delta\mathbf{F}(Z, t)\mathbf{d}_\gamma(Z)] = 0, \quad Z \in \Omega,$$

where we have used denotations according to the scheme $\alpha_\gamma(Z, \cdot) \equiv \alpha(P_\gamma(Z), \cdot)$, $\gamma = 1, \ldots, r$, $P_\gamma(Z)$, being an arbitrary material point belonging to the particle Z. From the foregoing, by virtue of $\sum m_\gamma \mathbf{d}_\gamma = 0$, and $\sum \mathbf{g}_\gamma = 0$, we obtain

$$\begin{aligned} &\sum_{\gamma=1}^{r} m_\gamma(Z)\ddot{\boldsymbol{\chi}}(Z, t) = \sum_{\gamma=1}^{r} \mathbf{h}_\gamma(Z, \cdot), \\ &\sum_{\gamma=1}^{r} m_\gamma(Z)\mathbf{d}_\gamma(Z) \otimes \mathbf{d}_\gamma(Z)\ddot{\mathbf{F}}^T(Z, t) = \sum_{\gamma=1}^{r} [\mathbf{h}_\gamma(Z, \cdot) + \mathbf{g}_\gamma(Z, \cdot)] \otimes \mathbf{d}_\gamma(Z), \quad Z \in \Omega. \end{aligned} \tag{3.5}$$

Let us assume that the interactions within Ω have the form:

$$\mathbf{h}_\gamma(Z,\cdot) = \mathbf{e}_\gamma\big(Z, t; \boldsymbol{\chi}(Z,t)\big)+\mathbf{s}_\gamma(Z,t),$$
$$\mathbf{g}_\gamma(Z,\cdot) = -\frac{\partial\bar{\pi}\big(Z, |\boldsymbol{\varkappa}(P_\gamma(Z),t)-\boldsymbol{\chi}(Z,t)|\big)}{\partial\boldsymbol{\varkappa}\big(P_\gamma(Z),t\big)}, \quad Z\in\Omega,$$

where $\bar{\pi}(Z,\cdot)$ is the potential of interaction within the particle $Z = \{P_1(Z), \ldots, P_\gamma(Z)\}$, $\mathbf{s}_\gamma(Z,t)$ is the interaction of the point $P_\gamma(Z)$ with the points which do not belong to the particle Z, and $\mathbf{e}_\gamma(Z,\cdot)$ is the interaction of the point $P_\gamma(Z)$ with the fields which are external with respect to the whole system Ω. The conditions (2.3) now become identities. Let us also introduce the following denotations:

$$\begin{gathered} m(Z) \equiv \sum_{\gamma=1}^{r} m_\gamma(Z), \quad \mathbf{J}(Z) \equiv \sum_{\gamma=1}^{r} m_\gamma(Z)\mathbf{d}_\gamma(Z)\otimes\mathbf{d}_\gamma(Z), \\ \pi(Z, \mathbf{F}^T\mathbf{F}) \equiv \bar{\pi}\big(Z, |\mathbf{F}(Z,t)\mathbf{d}_\gamma(Z)|\big), \quad \mathbf{e}(Z,\cdot) \equiv \sum_{\gamma=1}^{r} \mathbf{e}_\gamma(Z,\cdot), \\ \mathbf{s}(Z,t) \equiv \sum_{\gamma=1}^{r} \mathbf{s}_\gamma(Z,t), \quad \mathbf{S}(Z,t) \equiv \sum_{\gamma=1}^{r} \mathbf{s}_\gamma(Z,t)\otimes\mathbf{d}_\gamma(Z). \end{gathered} \tag{3.6}$$

From $(3.6)_6$ it follows that

$$\frac{\partial\pi}{\partial\mathbf{F}(Z,t)} = \sum_{\gamma=1}^{r} \frac{\partial\bar{\pi}}{\partial\boldsymbol{\varkappa}\big(P_\gamma(Z),t\big)} \otimes \mathbf{d}_\gamma(Z) = -\sum_{\gamma=1}^{r} \mathbf{g}_\gamma(Z,\cdot)\otimes\mathbf{d}_\gamma(Z). \tag{3.7}$$

Taking into account all the foregoing relations and denotations, and assuming that $m_\gamma = \mathrm{const}$, $\mathbf{e}_\gamma = \mathrm{const}$ in each particle Z, we rewrite the Eqs. (3.5) in the final form:

$$\begin{gathered} m(Z)\ddot{\boldsymbol{\chi}}(Z,t) = \mathbf{e}(Z, t; \boldsymbol{\chi})+\mathbf{s}(Z,t), \\ \mathbf{S}(Z,t) = \frac{\partial\pi(Z, \mathbf{F}^T\mathbf{F})}{\partial\mathbf{F}(Z,t)} + \mathbf{J}(Z)\ddot{\mathbf{F}}^T(Z,t); \quad Z\in\Omega, \quad t\in(t_0, t_1). \end{gathered} \tag{3.8}$$

The Eqs. (3.8) will be called the equations of motion of the simple particle.

If nothing is known about the forces $\mathbf{s}_\gamma(Z,t)$ of interactions among the material points belonging to the different particles, nothing can be said about the motion of the system Ω of particles, since $\mathbf{s}(Z,t)$ and $\mathbf{S}(Z,t)$ are unknowns in the Eqs. (3.8). In what follows, we assume that the motion of the system $\Omega = \{Z_1, \ldots, Z_k\}$ of simple particles is restricted by the constraints:

$$\underset{Z\in\Omega}{\boldsymbol{\gamma}}\big(t, \boldsymbol{\chi}(Z,t), \mathbf{F}(Z,t)\big) = 0, \quad t\in(t_0, t_1), \tag{3.9}$$

where $\boldsymbol{\gamma}$ is a known differentiable standard function with values in a certain finite dimensional vector space. At the same time, we interpret the interactions $\mathbf{s}_\gamma(Z,t)$ as the reaction forces due to the constaints (3.9). Assuming that the constraints are ideal, we obtain the following ideality conditions:

$$\sum_{Z\in\Omega}\sum_{\gamma=1}^{r} \mathbf{s}_\gamma(Z,t)\cdot[\delta\boldsymbol{\chi}(Z,t)+\delta\mathbf{F}(Z,t)\mathbf{d}_\gamma(Z)] = 0.$$

Using the denotations (3.6), we can write:

$$\sum_{Z\in\Omega} [\mathbf{s}(Z,t)\cdot\delta\boldsymbol{\chi}(Z,t)+\mathbf{S}(Z,t)\cdot\delta\mathbf{F}(Z,t)] = 0. \tag{3.10}$$

The foregoing ideality condition must hold for any $\delta\boldsymbol{\chi}$, $\delta\mathbf{F}$, such that

$$\sum_{Z\in\Omega}\left[\frac{\partial\Upsilon}{\partial\boldsymbol{\chi}(Z,t)}\cdot\delta\boldsymbol{\chi}(Z,t)+\frac{\partial\Upsilon}{\partial\mathbf{F}(Z,t)}/\partial\mathbf{F}(Z,t)\right] = 0. \tag{3.11}$$

The Eqs. (3.8)–(3.11) are governing relations of the elastodynamics of an absolutely finite system $\Omega\equiv\{Z_1, Z_2, \ldots, Z_k\}$ of simple material particles. Such systems can be used as models of real bodies instead of the discrete systems of material points.

Let us now give an example of a system of simple particles which constitute a discrete representation of the elastic material system. To this end, we introduce the reference configuration $\varkappa_R$ of Ω, putting $\varkappa_R\colon \Omega\to\Lambda$, where Λ is a lattice in the physical space ${}^*E^3$ with the vector basis $\Delta_\alpha(\mathbf{Z})$ (i.e., if $\mathbf{Z}\in\Lambda$ then $\mathbf{Z}+\Delta_\alpha(\mathbf{Z})\in\Lambda$, $\alpha = 1, 2, 3$). Let the Eq. (3.9) be postulated in the form(1):

$$\Delta_\alpha\chi_k(\mathbf{Z},t)-\mu_\alpha^\beta F_{k\beta}(\mathbf{Z},t) = 0, \quad \mathbf{Z} = \varkappa_R(Z), \tag{3.12}$$

where $\Delta_\alpha f(\mathbf{Z})\equiv [f(\mathbf{Z}+\Delta_\alpha(\mathbf{Z}))-f(\mathbf{Z})]/|\Delta_\alpha(\mathbf{Z})|$, $\mu_\beta^\alpha f_\beta(\mathbf{Z})\equiv\frac{1}{2}[f_\alpha(\mathbf{Z}+\Delta_\alpha(\mathbf{Z}))+f_\alpha(\mathbf{Z})]$ are finite difference and mean value operators, respectively, and f, f_α are arbitrary real valued functions defined on Λ. The Eqs. (3.12) are assumed to hold for any $\mathbf{Z} = \varkappa_R(Z)$ such that $\varkappa_R^{-1}((\mathbf{Z}+\Delta_\alpha(\mathbf{Z}))\in\Omega$ (for the meaning of the Eqs. (3.12) cf. [2]). We assume that the domain of the definition of the Eqs. (3.12) is not empty. From (3.12) and (3.8)–(3.11), we obtain after some calculations:

$$\begin{aligned} &\bar{\Delta}_\alpha T^{k\alpha}(\mathbf{Z},t)+e^k(\mathbf{Z},t,\boldsymbol{\chi}) = m(\mathbf{Z})\ddot{\boldsymbol{\chi}}^k(\mathbf{Z},t),\\ &\bar{\mu}_\beta^\alpha T^{k\beta}(\mathbf{Z},t) = \frac{\partial\pi(\mathbf{Z},\mathbf{F}^T\mathbf{F})}{\partial F_{k\alpha}(\mathbf{Z},t)}+J^{\alpha\beta}(\mathbf{Z})\ddot{F}_\beta^k(\mathbf{Z},t),\end{aligned} \tag{3.13}$$

where $\bar{\Delta}_\alpha f(\mathbf{Z})\equiv [f(\mathbf{Z}-\Delta_\alpha(\mathbf{Z}))-f(\mathbf{Z})]/|\Delta_\alpha \mathbf{r}(\mathbf{Z})|$, $\bar{\mu}_\beta^\alpha f^\beta(\mathbf{Z})\equiv\frac{1}{2}[f^\alpha(\mathbf{Z}-\Delta_\alpha(\mathbf{Z})+f^\alpha(\mathbf{Z})]$ and where $\bar{\mu}_\beta^\alpha T^{k\beta} = S^{k\alpha}$. The Eqs. (3.12) and (3.13) constitute a non-standard form of governing equations of the system of simple particles under consideration. To obtain the standard form of the foregoing equations, we assume that $m_\gamma(\mathbf{Z})$, $\mathbf{h}_\gamma(\mathbf{Z},\cdot)$, $\mathbf{g}_\gamma(\mathbf{Z},\cdot)\in 0(1)$; then in view of $(3.5)_2$ and (3.6), we arrive at $\mathbf{J}\in o(\mathbf{T}, \partial\pi/\partial\mathbf{F})$. Making use of the series $\boldsymbol{\chi}(\mathbf{Z},t) = \sum_\pi \boldsymbol{\chi}(\mathbf{Z},t)d\varepsilon^\pi$, $\mathbf{F}(\mathbf{Z},t) = \sum_\pi\mathbf{F}(\mathbf{Z},t)d\varepsilon^\pi$ and $\mathbf{T}(\mathbf{Z},t) = \sum_\pi\mathbf{T}(\mathbf{Z},t)d\varepsilon^\pi$, we replace (3.12) and (3.13) by the infinite sequence of standard equations for standard functions ${}_\pi\boldsymbol{\chi}(\mathbf{Z},t)$, ${}_\pi\mathbf{F}(\mathbf{Z},t)$, ${}_\pi\mathbf{T}(\mathbf{Z},t)$, $\pi = 0, 1, \ldots$. The standard equations for ${}_0\boldsymbol{\chi}$, ${}_0\mathbf{F}$ and ${}_0\mathbf{T}$ have the form:

$$\begin{aligned} &\Delta_\alpha({}_0\chi_k)-\mu_\alpha^\beta({}_0F_{k\beta}) = 0, \quad \bar{\Delta}_\alpha({}_0T^{k\alpha})+e^k(Z,t,{}_0\chi) = m_0\ddot{\chi}^k,\\ &\bar{\mu}_\beta^\alpha({}_0T^{k\beta}) = \frac{\partial\pi(Z,{}_0\mathbf{F}^T{}_0\mathbf{F})}{\partial_0F_{k\alpha}},\end{aligned} \tag{3.14}$$

(1) We use here the index notation; indices k, l, α, β run over the sequence 1, 2, 3; the summation convention holds.

and do not include the term with the inertia tensor $J^{\alpha\beta}$. The standard equations for ${}_1\boldsymbol{\chi}$, ${}_1\mathbf{F}$ and ${}_1\mathbf{T}$ have a slighty different form:

$$(3.15)\qquad \begin{gathered} \Delta_\alpha({}_1\chi_k)-\mu^\beta_\alpha({}_1F_{k\beta}) = 0, \quad \bar{\Delta}_\alpha({}_1T^{k\alpha})+\frac{\partial e^k(Z, t, {}_0\chi)}{\partial_0\chi_l}({}^1\chi_l) = m_1\ddot{\chi}^k, \\ \bar{\mu}^\alpha_\beta({}_1T^{k\beta}) = \frac{\partial^2\pi(Z, {}_0\mathbf{F}^T{}_0\mathbf{F})}{\partial_0F_{k\alpha}\,\partial_0F_{l\beta}}({}_1F_{l\beta})+{}_1J^{\alpha\beta}{}_0\ddot{F}^k_\beta, \qquad {}_1J^{\alpha\beta} \equiv st\frac{J^{\alpha\beta}}{d\varepsilon}. \end{gathered}$$

It can be observed that, from the formal point of view, the Eqs. (3.14) are finite difference equations of non-linear elasticity. If the functions ${}_0\boldsymbol{\chi}$, ${}_0\mathbf{F}$ are known, then the Eqs. (3.15) constitute the linear system of equations for ${}_1\boldsymbol{\chi}$, ${}_1\mathbf{F}$, ${}_1\mathbf{T}$. Note that the Eqs. (3.15) cannot be treated as finite difference equations of the linear elasticity because of the new term containing the inertia tensor $J^{\alpha\beta}$. Thus, the approach to finite difference formulation of the linear elastodynamics based on the elastodynamics of a finite system of simple particles leads to more general results than that based on the formal discretization procedure of equations of linear elasticity.

4. Smooth systems of simple particles

Let $\bar{\Omega}$ be a Q-finite set of simple particles which is not absolutely finite—i.e., $\bar{\Omega} = \{Z_1, \ldots, Z_\omega\}$, where ω is a fixed infinite natural number. Let there be given the injection $\boldsymbol{\varkappa}_R\colon \bar{\Omega} \to {}^*E^3$, such that the set $\bar{D}_R \equiv \boldsymbol{\varkappa}_R(\Omega)$ is internal, Q-discrete and S-continuous in ${}^*E^3$; the mapping $\boldsymbol{\varkappa}_R$ will be called the reference configuration of $\bar{\Omega}$, and we shall denote $\mathbf{Z} = \boldsymbol{\varkappa}_R(Z)$, $Z \in \bar{\Omega}$. Moreover, we shall take into account only such a reference configuration for which the set of standard points $\mathbf{X}$ in ${}^*E^3$, given by (cf. [1] p. 101)

$${}^0\bar{D}_R \equiv \{\mathbf{X}|\mu(\mathbf{X}) \cap \bar{D}_R \neq \phi\},$$

constitutes a closed region $\bar{B}_R$ in E^3 (if $\bar{D}_R$ is internal, then ${}^0\bar{D}_R$ is closed, cf. [1] p. 101)—i.e., ${}^0\bar{D}_R = \bar{B}_R$, $B_R \subset E^3$. We also introduce the subsets $D_R \equiv \bar{D}_R \cap {}^*B_R$, $\partial D_R \equiv \bar{D}_R \cap {}^*\partial B_R$ in ${}^*E^3$, and the subsets $\Omega \equiv \boldsymbol{\varkappa}_R^{-1}(D_R \cap \partial D_R)$, $\partial\Omega \equiv \boldsymbol{\varkappa}_R^{-1}(\partial D_R)$ of the set $\bar{\Omega}$. Elements $Z \in \partial\Omega$ will be called boundary particles.

The results obtained in Sect. 3 for the absolutely finite set of simple particles will be now applied to the non-absolutely finite set Ω, $\Omega \subset \bar{\Omega}$. The relations (3.8) hold, but the ideality condition (3.10) is no longer valid, since the interactions $\mathbf{s}_\gamma(Z, t)$ now represent not only the reaction forces due to the constraints, but also the forces acting on Ω from $\bar{\Omega}-\Omega$. Let us assume that the latter are distributed only on the boundary particles. It follows that

$$\mathbf{s}_\gamma(Z, t) = \mathring{\mathbf{s}}_\gamma(Z, t)+\hat{\mathbf{s}}_\gamma(Z, t), \qquad Z \in \partial\Omega,$$

where $\bar{\mathbf{s}}_\gamma$ are reaction forces due to the constraints, and $\mathring{\mathbf{s}}_\gamma$ are interactions between $\bar{\Omega}-\Omega$ and Ω. Thus we have to modify the condition (3.10) to the following form:

$$(4.1)\qquad \sum_{Z\in\Omega}\sum_\gamma \mathbf{s}_\gamma(Z, t)\cdot[\delta\boldsymbol{\chi}(Z, t)+\delta\mathbf{F}(Z, t)\mathbf{d}_\gamma(Z)] - \sum_{Z\in\partial\Omega}\sum_\gamma \mathring{\mathbf{s}}_\gamma(Z, t)\cdot[\delta\boldsymbol{\chi}(Z, t)+\delta\mathbf{F}(Z, t)\mathbf{d}_\gamma(Z)] = 0.$$

After denotation

$$\mathring{\mathbf{s}}(Z, t) \equiv \sum_{\gamma} \mathring{\mathbf{s}}_{\gamma}(Z, t),$$

we obtain the kinetic condition on $\partial\Omega$:

$$\mathring{\mathbf{s}}(Z, t) + \mathbf{r}(Z, t) = 0. \tag{4.2}$$

Assuming that $\mathring{\mathbf{s}}_{\gamma} = \text{const.}$ for each particle Z, by virtue of the Eqs. $(3.6)_{5,6}$ we rewrite the ideality condition (4.1) in the form

$$\sum_{Z\in\Omega} [\mathbf{s}(Z, t) \cdot \delta\boldsymbol{\chi}(Z, t) + \mathbf{S}(Z, t) \cdot \delta\mathbf{F}(Z, t)] + \sum_{Z\in\partial\Omega} \mathbf{r}(Z, t) \cdot \delta\boldsymbol{\chi}(Z, t) = 0. \tag{4.3}$$

Thus the field $\mathbf{r}(Z, t)$ represents the boundary reaction force due to the constraints.

The equations of motion (3.8), the equations of constraints (3.9), the ideality conditions (4.3) and the kinetic boundary condition (4.2) constitute the basis for further analysis.

In what follows, we shall confine ourselves to what will be called smooth systems of simple particles. To that end, we introduce the following assumptions:

1. There exists an internal fine partition (cf. [1], p. 71) of the region $^*B_R$ into volume elements $dv_R = dv_R(\mathbf{Z})$ with centres $\mathbf{Z} \in D_R$. There also exists an internal fine partition of the surface $^*\partial B_R$ into surface elements $ds_R = ds_R(\mathbf{Z})$ with centres $\mathbf{Z} \in \partial D_R$. We postulate that the mass and the force distributions within D_R and ∂D_R are given by the formulae([2]):

$$\begin{aligned}
&m(\mathbf{Z}) = \varrho_R(\mathbf{Z}) dv_R, \quad \mathbf{J}(\mathbf{Z}) = J_R(\mathbf{Z}) dv_R d\varepsilon^2, \\
&\mathbf{e}(\mathbf{Z}, \cdot) = \mathbf{b}_R(\mathbf{Z}, \cdot) dv_R, && \mathbf{r}(\mathbf{Z}, t) = \mathbf{t}_R(\mathbf{Z}, t) ds_R, \\
&\mathbf{s}(\mathbf{Z}, t) = \mathbf{d}_R(\mathbf{Z}, t) dv_R, && \mathring{\mathbf{s}}(\mathbf{Z}, t) = \mathbf{p}_R(\mathbf{Z}, t) ds_R; \ \mathbf{Z} \in \partial D_R, \\
&\mathbf{S}(\mathbf{Z}, t) = \mathbf{T}_R(\mathbf{Z}, t) dv_R, \\
&\pi(\mathbf{Z}, \cdot) = \sigma(\mathbf{Z}, \cdot) dv_R; \quad \mathbf{Z} \in D_R,
\end{aligned} \tag{4.4}$$

where $\varrho_R(\mathbf{Z}), \ldots, \sigma_R(\mathbf{Z}, \mathbf{F}^T\mathbf{F})$ are internal finite functions defined and S-continuous on D_R, and $\mathrm{t}_R(\mathbf{Z}, t)$, $\mathbf{p}_R(\mathbf{Z}, t)$ are internal finite functions defined and S-continuous on ∂D_R (cf. Appendix), provided that all arguments except $\mathbf{Z}$ are fixed.

2. We have assumed already that the internal functions under consideration have the form:

$$\begin{aligned}
\mathbf{d}_R(\mathbf{Z}, t) &= \sum_{\pi=0}^{\infty} {}_{\pi}\mathbf{d}_R(\mathbf{Z}, t) d\varepsilon^{\pi}, & \varrho_R(\mathbf{Z}) &= \sum_{\pi=0}^{\infty} {}_{\pi}\varrho_R(\mathbf{Z}) d\varepsilon^{\pi}, \\
\mathbf{T}_R(\mathbf{Z}, t) &= \sum_{\pi=0}^{\infty} {}_{\pi}\mathbf{T}_R(\mathbf{Z}, t) d\varepsilon^{\pi}, & \mathbf{J}_R(\mathbf{Z}) &= \sum_{\pi=0}^{\infty} {}_{\pi}\mathbf{J}_R(\mathbf{Z}) d\varepsilon^{\pi}, \\
\boldsymbol{\chi}(\mathbf{Z}, t) &= \sum_{\pi=0}^{\infty} {}_{\pi}\boldsymbol{\chi}(\mathbf{Z}, t) d\varepsilon^{\pi}, & \mathrm{t}_R(\mathbf{Z}, t) &= \sum_{\pi=0}^{\infty} {}_{\pi}\mathrm{t}_R(\mathbf{Z}, t) d\varepsilon_{\pi}, \\
\mathbf{F}(\mathbf{Z}, t) &= \sum_{\pi=0}^{\infty} {}_{\pi}\mathbf{F}(\mathbf{Z}, t) d\varepsilon^{\pi}, & \mathbf{p}_R(\mathbf{Z}, t) &= \sum_{\pi=0}^{\infty} {}_{\pi}\mathbf{p}_R(\mathbf{Z}, t) d\varepsilon^{\pi},
\end{aligned} \tag{4.5}$$

([2]) We assume here that $\mathbf{d}_{\gamma} \in 0(d\varepsilon)$, $\mathrm{s}_{\gamma} \in 0(d\varepsilon^2)$ and $\sum s_{\gamma} \in 0(d\varepsilon^3)$ (cf. [1], p. 79 for the meaning of the symbol $O(\cdot)$).

where ${}_{\pi}\mathbf{d}_R(\mathbf{Z}, t), \ldots, {}_{\pi}\mathbf{p}_R(\mathbf{Z}, t)$ are standard in $(t_0, t_1) \subset {}^*R$ for any fixed $\mathbf{Z} \in D_R \subset {}^*B_R$ or $\mathbf{Z} \in \partial D_R \subset {}^*\partial B_R$ (cf. Sect. 2). Now, we postulate that there exist the smooth standard functions ${}_{\pi}\mathbf{d}_R(\mathbf{X}, t), \ldots, {}_{\pi}\mathbf{F}(\mathbf{X}, t)$ defined on ${}^*B_R \times (t_0, t_1)$, ${}_{\pi}\varrho_R(\mathbf{X}), {}_{\pi}\mathbf{J}_R(\mathbf{X})$ defined on ${}^*B_R$, and ${}_{\pi}\mathbf{t}_R(\mathbf{X}, t)$, ${}_{\pi}\mathbf{p}_R(\mathbf{X}, t)$ defined on ${}^*\partial B_R \times (t_0, t_1)$, which for every $\mathbf{X} \equiv \mathbf{Z} \in D_R \subset {}^*B_R$ or $\mathbf{X} = \mathbf{Z} \in \partial D_R \subset {}^*\partial B_r$ coincide with the corresponding functions on the right-hand sides of the Eqs. (4.5).

The system of simple particles under consideration, for which the foregoing assumptions hold, will be called a smooth system of simple particles.

Suppose ${}^0\mathbf{Z} = st\mathbf{Z} \in \overline{D}_R$ for each $\mathbf{Z} \in \overline{D}_R$; this means that the Eqs. $(4.5)_{1-6}$ hold for $\mathbf{Z} = {}^0\mathbf{Z} \in B_R$ and the Eqs. $(4.5)_{7,8}$ hold for $\mathbf{Z} = {}^0\mathbf{Z} \in \partial B_R$. Now, recall that the standard functions ${}_{\pi}\mathbf{d}_R(\mathbf{X}, t), \ldots, {}_{\pi}\mathbf{p}_R(\mathbf{X}, t)$ are uniquely determined by their values ${}_{\pi}\mathbf{d}_R({}^0\mathbf{Z}, t)$, $\ldots, {}_{\pi}\mathbf{p}_R({}^0\mathbf{Z}, t)$, respectively, in the standard points $\mathbf{X} = {}^0\mathbf{Z}$. Hence we see that, in each smooth system of simple particles there exists a uniquely determined set of standard functions ${}_{\pi}\mathbf{d}_R(\mathbf{X}, t), \ldots, {}_{\pi}\mathbf{p}_R(\mathbf{X}, t)$. Thus, for each such system, the set of internal functions defined by

$$(4.6) \qquad \mathbf{d}_R(\mathbf{X}, t) \equiv \sum_{\pi=0}^{\infty} {}_{\pi}\mathbf{d}_R(\mathbf{X}, t)\, d\varepsilon^{\pi}, \ \ldots, \ \mathbf{p}_R(\mathbf{X}, t) \equiv \sum_{\pi=0}^{\infty} {}_{\pi}\mathbf{p}_R(\mathbf{X}, t)\, d\varepsilon^{\pi},$$

is also uniquely determined.

Now we shall write all basic relations describing the smooth system of simple particles. To simplify further analysis, we confine ourselves to local constraints, i.e. to constraints which can be expressed in the form of differential equations. We assume that the equations of constraints are given by

$$(4.7) \qquad \boldsymbol{\gamma}\big(\mathbf{X}, t, \boldsymbol{\chi}(\mathbf{X}, t), \nabla\boldsymbol{\chi}(\mathbf{X}, t), \ldots, \nabla^k\boldsymbol{\chi}(\mathbf{X}, t), \mathbf{F}(\mathbf{X}, t), \nabla\mathbf{F}(\mathbf{X}, t), \ldots, \nabla^l\mathbf{F}(\mathbf{X}, t)\big) = 0,$$
$$\mathbf{X} \in {}^*\mathbf{B}_R, \quad t \in (t_0, t_1),$$

where $\boldsymbol{\gamma}$ is the known differentiable vector function(3). The ideality condition (4.3) can, after taking into account the internal fine partition of ${}^*B_R$ and ${}^*\partial B_R$, now be written in the form:

$$(4.8) \qquad \int_{B_R} [\mathbf{T}_R(\mathbf{X}, t) \cdot \delta\mathbf{F}(\mathbf{X}, t) + \mathbf{d}_R(\mathbf{X}, t) \cdot \delta\boldsymbol{\chi}(\mathbf{X}, t)]\, dv_R + \oint_{\partial B_R} \mathbf{t}_R(\mathbf{X}, t) \cdot \delta\boldsymbol{\chi}(\mathbf{X}, t)\, ds_R = 0,$$

where the integrands have to be represented in terms of standard functions using (4.6). The relation (4.8) must hold for any virtual increments $\delta\boldsymbol{\chi}(\mathbf{X}, t)$, $\delta\mathbf{F}(\mathbf{X}, t)$, which are smooth solutions of the vector equation:

$$(4.9) \qquad \sum_{n=0}^{k} \frac{\partial\boldsymbol{\gamma}}{\partial\nabla^n\boldsymbol{\chi}} \cdot \nabla^n \delta\boldsymbol{\chi}(\mathbf{X}, t) + \sum_{n=0}^{l} \frac{\partial\boldsymbol{\gamma}}{\partial\nabla^n\mathbf{F}} \cdot \nabla^n \delta\mathbf{F}(\mathbf{X}, t) = 0.$$

Substituting the right-hand sides of the Eqs. (4.4) into the Eqs. (3.8), putting $\mathbf{Z} = {}^0\mathbf{Z} \in B_R$,

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(3) We use here the notation $\nabla f(\mathbf{X}) \equiv \sum_{\pi=0}^{\infty} \nabla_{\pi} f(\mathbf{X})\, d\varepsilon^{\pi}$, $\nabla^{n+1} \equiv \nabla\nabla^n$ where ${}_{\pi}f(\mathbf{X})$ is a sufficiently smooth standard function defind on ${}^*B_R$ for each $\pi \in {}^*N$, provided that the corresponding infinite sum exists.

and then extending the relations obtained to $^*B_R$ (this can be done in the class of internal functions given by (4.6)), we arrive at the equations of motion:

$$(4.10)\qquad \begin{gathered} \varrho_R(\mathbf{X})\ddot{\boldsymbol{\chi}}(\mathbf{X},t) = \mathbf{b}_R(\mathbf{X},t,\boldsymbol{\chi}(\mathbf{X},t))+\mathbf{d}_R(\mathbf{X},t),\\ \mathbf{T}_R(\mathbf{X},t) = \frac{\partial\sigma_R(\mathbf{X},\mathbf{F}^T\mathbf{F})}{\partial\mathbf{F}}+d\varepsilon^2\mathbf{J}_R(\mathbf{X})\ddot{\mathbf{F}}^T(\mathbf{X},t), \quad \mathbf{X}\in{}^*B_R, \quad t\in(t_0,t_1). \end{gathered}$$

Analogously, from the Eqs. (4.2) and (4.6), we arrive at the following kinetic boundary condition:

$$(4.11)\qquad \mathbf{t}_R(\mathbf{X},t)+\mathbf{p}_R(\mathbf{X},t) = \mathbf{0}, \quad \mathbf{X}\in{}^*\partial B_R, \quad t\in(t_0,t_1)\subset{}^*R.$$

The Eqs. (4.7)–(4.11), in which all functions are internal and represented by the formulae (4.6), describe the mechanics of the smooth system of simple particles.

As an example of the system under consideration, let us take the system in which the Eqs. (4.7) are assumed in the form $\nabla\boldsymbol{\chi}-\mathbf{F} = \mathbf{0}$. Then, from (4.8) it follows that $\mathbf{d}_R = \operatorname{div}\mathbf{T}_R$ in $^*B_R$ and $\mathbf{t}_R = -\mathbf{T}_R\mathbf{n}_R$ on $^*\partial B_R$ (in the class of internal functions defined by the Eqs. (4.6)), where $\mathbf{n}_R$ is the unit vector normal to $^*\partial B_R$. Using the Eqs. (4.6) and assuming that $\varrho_R = {}_0\varrho_R$, $\mathbf{J}_R = {}_0\mathbf{J}_R$, we arrive at the infinite series of standard relations, which, in tensor notations, have the form (cf. Sect. 3, Eqs. (3.12)–(3.15)):

$$(4.12)\qquad \begin{gathered} {}_0T_R^{k\alpha}{}_{,\alpha}+{}_0b_R^k = \varrho_R\,{}_0\ddot{\chi}^k, \quad {}_0T_R^{k\alpha} = \frac{\partial_0\sigma_R}{\partial_0\chi_{k,\alpha}}, \quad \mathbf{X}\in B,\\ {}_0T_R^{k\alpha}\eta_{R\alpha} = {}_0p_R^k, \quad \mathbf{X}\in\partial B_R,\\ {}_1T_R{}^{k\alpha}{}_{,\alpha}+\frac{\partial_0b_R^k}{\partial_0\chi^l}\,{}_1\chi^l = \varrho_R\,{}_1\chi^k, \quad {}_1T_R^{k\alpha} = \frac{\partial^2{}_0\sigma_R}{\partial_0\chi_{k,\alpha}\partial_0\chi_{l,\beta}}\,{}_1\chi_{l,\beta}, \quad \mathbf{X}\in B_R,\\ {}_1T_R^{k\alpha}n_{R\alpha} = {}_1p_R^k, \quad \mathbf{X}\in\partial B_R,\\ {}_nT_R^{k\alpha}{}_{,\alpha}+\frac{\partial_0b_R^k}{\partial_0\chi^l}\,{}_n\chi^l+p_{(n-1)R}^k = \varrho_R\,{}_n\ddot{\chi}^k,\\ {}_nT_R^{k\alpha} = \frac{\partial^2{}_0\sigma_R}{\partial_0\chi_{k,\alpha}\partial_0\chi_{l,\beta}}\,{}_n\chi_{l,\beta}+P_{(n-1)R}^{k\alpha}+J_R^{\alpha\beta}{}_{(n-2)}\ddot{\chi}^k{}_{,\beta}, \quad \mathbf{X}\in B_R,\\ {}_nT_R^{k\alpha}n_{R\alpha} = {}_np_R^k, \quad \mathbf{X}\in\partial B_R, \quad n = 2,3,4,\ldots,\\ {}_0b_R^k \equiv b_R^k(\mathbf{X},t;{}_0\boldsymbol{\chi}), \quad {}_0\sigma_R \equiv \sigma_R(\mathbf{X},\nabla_0\boldsymbol{\chi}^T\nabla_0\boldsymbol{\chi}), \quad t\in(t_0,t_1)\subset R, \end{gathered}$$

where $p_{mR}^k(\mathbf{X},t,{}_0\boldsymbol{\chi};{}_1\boldsymbol{\chi},\ldots,{}_m\boldsymbol{\chi})$, $P_{mR}^{k\alpha}(\mathbf{X},t,\nabla_0\boldsymbol{\chi};\nabla_1\boldsymbol{\chi},\ldots,\nabla_m\boldsymbol{\chi})$, $m = 1,2,\ldots$, are polynomials of mth order in arguments ${}_1\boldsymbol{\chi},\ldots,{}_m\boldsymbol{\chi}$ and $\nabla_1\boldsymbol{\chi},\ldots,\nabla_m\boldsymbol{\chi}$, respectively, such that $p_{mR}^k(\mathbf{X},t,{}_0\boldsymbol{\chi};\mathbf{0},\ldots,\mathbf{0}) = 0$ and $P_{mR}^{k\alpha}(\mathbf{X},t,\nabla_0\boldsymbol{\chi};\mathbf{0},\ldots,\mathbf{0}) = 0$.

The Eqs. $(4.12)_{1-3}$ are known equations of classical nonlinear elasticity, and the Eqs. $(4.12)_{4-6}$ are known equations of small motions superimposed on the finite motion of the hyperelastic body. The Eqs. (4.12) for $n > 2$ have a form resembling equations met with in the theory of succesive approximations (cf. [3] Sect. 5), but they also contain the new inertial terms $\mathbf{J}_R(\mathbf{t}_{n-2}\nabla\ddot{\boldsymbol{\chi}})^T$. Moreover, treating the equations for ${}_\pi\mathbf{T}_R$, $\pi = 0,1,2,\ldots$ as the stress relations of hyperelastic bodies, we can see that they do not satisfy the principle of material frame indifference when $\pi \geqslant 2$.

5. Final remarks

In Sect. 4, we have shown that Q-discrete material systems may represent not only classical discrete systems of material points but also material continua. However, Q-discrete mechanics enables us also to formulate models of real bodies in mechanics which are neither discrete nor continuous in the standard sense. A trivial example of such a system has been given by a simple particle. Apart from simple particles, we can also construct particles with a larger or smaller number of degrees of freedom (for example the Cosserat particles—i.e., simple particles with the constraints $\mathbf{F}^T\mathbf{F} = \mathbf{1}$). Here, we shall give a more complex example of the Q-discrete system, which is neither discrete nor continuous in the standard interpretation. To this end, we take into consideration the system $\{Z_1, Z_2, \ldots, Z_\omega\}$ of simple particles which is not absolutely finite and is not smooth (ω is a fixed infinite positive integer). Suppose that this system is governed by the Eqs. (3.12), (3.13), where $Z = Z_1, Z_2, \ldots, Z_\omega$ and Λ is the lattice in $^*E^3$ with infinitesimal vector basis. The standard Eqs. (3.14), (3.15) are still valid, but they represent now infinite systems of ordinary differential or algebraic equations, respectively. Thus we arrive at a new model of real bodies in mechanics, which cannot be obtained by the standard approach.

The characteristic feature of the mechanics of Q-discrete systems is the very simple form of governing relations (2.1), (2.2), (1.3), (1.4) and (2.4), which seems to contrast with the variety of different models that can be obtained from these relations. It is also of interest to note that the different linear theories in Q-discrete mechanics are derived directly from the axioms of mechanics taking into account the class of internal functions we have to deal with and do not constitute formal approximations of these axioms. Moreover, in some linear equations, new terms appear which cannot be obtained by the formal linearization procedure (cf. Eqs. $(3.15)_3$ or $(4.12)_8$).

The results of the present paper can be extended and formulated in a form which includes also non-elastic material systems and non-holonomic or even non-local in the case of smooth systems constraints. The analysis of a more general class of internal functions including, for example, the possibility of infinite forces and accelerations should be also taken into account. The investigation and application of special Q-discrete systems introduced in this paper will, not being strictly connected with the non-standard approach to mechanics, be treated separately. We may mention here that the approach to continuum mechanics based on the Eqs. (4.7)–(4.11) for a smooth system of simple particles, has found application as a certain generalization of the concept of material continuum with constraints, [4].

Appendix. Some concepts of Q-topology

Following [1], we denote by T a metric space with a distance function $\varrho(x, y)$; $x, y \in T$. By $^*T$ we denote the fixed extension of T, $^*T \supset T$. The distance function $\varrho(x, y)$ can now take not only positive standard values $_0\varrho \in R^+$, but also positive near standard values $_0\varrho + d\varrho$ (where $d\varrho$ is an infinitesimal real number—i.e., $|d\varrho| < \delta$ for all standard positive real numbers δ), and infinite positive values (such that $|\varrho| > m$ for all standard real numbers m). Points of $^*T$ belonging to T will be called standard. By a finite point in $^*T$ we shall mean a point x for which there exists a standard point ${}^0x$, such that $\varrho(x, {}^0x)$ is a finite positive number. The set of all finite points in $*T$ is said to be the principal galaxy of $*T$. In this paper, we confine ourselves only to Euclidean metric spaces in which the finite point is near standard—i.e., it posseses a uniquely defined standard part ${}^0x = stx$. The set $\mu(y)$ of points $x \in *T$, such that $\varrho(x, y)$ is an infinitesimal positive number, is said to be the monad of the point $y \in *T$.

By the Q-topology in $*T$, we understand the topology in which as a basis is taken a set of all open Q-balls $B(x, r)$ with a center $x \in *T$ and a radius $r \in *R^+$—i.e., $B(x, r) = \{y | \varrho(x, y) < r\}$, where $y \in *T$ and $r > 0$. As a basis for the topology in $*T$ can be also taken the set of all open S-balls, $S(x, {}^0r) = \{y | {}^0\varrho(x, y) < {}^0r\}$, where ${}^0r$ is a positive standard real number. In this case, we shall deal with the S-topology in $*T$. It is not difficult to see that Q-topology is finer then S-topology. The real valued function $f(x)$, defined on the set $D \subset *T$, is said to be S-continuous at $y \in D$, if for every standard $\varepsilon > 0$ there exists a standard $\delta > 0$ such that $\varrho\big(f(x), f(y)\big) < \varepsilon$ for any $y \in D$, such that $\varrho(x, y) < \delta$.

Throughout the paper we also use some further notions which are listed below.

The set $D \subset *T$ will be called Q-discrete if for every $x, y \in D$ the following conditions hold:

$$(\forall x, y)(\exists r)[\varrho(x, y) > r], \quad (\exists {}^0a)(\forall x, y)[{}^0\varrho(x, y) < {}^0a], \tag{A.1}$$

where $r \in *R^+$ is a positive real number and ${}^0\alpha \in R^+$ is a positive standard real number. It follows that for each $x \in D$ there exists a Q-ball with a center x, which does not contain any other element of D.

The set $D \subset *T$ will be called S-discrete if for every $x, y \in D$ we may write:

$$(\exists {}^0r)(\forall x, y)[{}^0\varrho(x, y) > {}^0r], \quad (\exists {}^0a)(\forall x, y)[{}^0\varrho(x, y) < {}^0a], \tag{A.2}$$

where ${}^0r$ is a positive standard real number. Thus for each $x \in D$ there exists an S-ball with a centre x not containing any other element of D.

The set $D \subset *T$ will be called S-continuous if

$$(\forall {}^0r)(\forall x)(\exists y)[{}^0\varrho(x, y) < {}^0r], \tag{A.3}$$

i.e. if for each $x \in D$ here does not exist an S-ball with a center not containing any other point belonging to D.

References

1. A. ROBINSON, *Non-standard analysis*, North-Holland Publ. Comp., American Elsevier Publ. Comp., 1974.
2. Cz. WOŹNIAK, *On the relation between discrete and continuum mechanics of certain material systems*, Arch. Mech., **28**, 571–581, 1976.
3. A. E. GREEN, J. E. ADKINS, *Large elastic deformations*, At the Clarendon Press, Oxford 1960.
4. Cz. WOŹNIAK, *Non-standard approach to the theory of elasticity*, Bull. Acad. Polon. Sci., Série Sci. Techn., **5**, 229–240, 1976.

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