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Theory of disclinations in elastic Cosserat media

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THE CONCEPT of disclinations in a micropolar Cosserat medium is exposed in the paper. The starting point is the surface model of Volterra which determines the disclination by means of displacement discontinuities at a certain surface σ. Green's functions are used to derive the general formulae for the displacement **u** and rotation **φ** fields. The displacement discontinuity is then modelled by a suitable distribution of body forces and couples at the surface σ. Transition from distortions to disclinations is shown and the distortional model of a disclination is constructed. In the concluding section, effective solutions for the displacement **u** and rotation fields **φ** are derived in the cases of the fundamental types of disclinations.

W pracy przedstawiono koncepcję dysklinacji w ciele mikropolarnym Cosseratów. Punktem wyjścia jest powierzchniowy model Volterry, określający dysklinację poprzez nieciągłość przemieszczenia na pewnej powierzchni σ. Przy pomocy funkcji Greena znaleziono ogólne wzory na pole przemieszczeń **u** i pole obrotów **φ**. Następnie wymodelowano nieciągłość przemieszczenia na powierzchni σ poprzez rozkład sił i momentów masowych na tej powierzchni. Dalszy ciąg pracy poświęcony jest przejściu od dystorsji do dysklinacji; utworzono dystorsyjny model dysklinacji. W ostatniej części wyprowadzono efektywne rozwiązania na pole przemieszczeń **u** i pole obrotów **φ** dla podstawowych typów dysklinacji.

В работе представлена концепция дисклинаций в микрополярном теле Коссера. Исходной точкой является поверхностная модель Вольтерра, определяющая дисклинацию через разрыв перемещения на некоторой поверхности σ. При помощи функции Грина найдены общие формулы для поля перемещений **u** и поля вращений **φ**. Затем разрыв перемещения на поверхности σ моделирован через распределение сил и массовых моментов на этой поверхности. Продолжение работы посвящено переходу от дисторсии к дисклинации; образована дисторсная модель дисклинации. В последней части работы выведены эффективные решения для поля перемещений **u** и поля вращений **φ** для основных типов дисклинаций.

Introduction

IN THE classical theory of elasticity it is assumed that interactions of individual portions of the medium, the contact interactions, may completely be described by means of the force-stress vector. This assumption results in describing the deformation of the body in terms of the symmetric strain and stress tensors.

The model does not comply, however, with experimental results concerning such cases in which large stress concentrations occur, e.g. in the vicinity of notches or cavities and holes. Considerable differences may also be observed in high frequency wave propagation problems and in granular bodies and polymers; this seems to be the result of disregarding the microstructure of the materials considered.

In order to explain those differences, W. VOIGT [19] introduced in 1887 the notion of couple-stress vector, in addition to the usual force-stress vector. In 1909 the Cosserat brothers [20, 21] outlined the complete theory of asymmetric elasticity. A rigid trihedron ascribed to any point of the deforming body moves and rotates in the process of deformation of the medium. The polar medium obtained consists of material elements, each of which is characterized by six degrees of freedom, and is then used to model the deformation of the body in terms of asymmetric stress and deformation tensors. The Cosserat theory has been developed in recent years by R. A. TOUPIN [22], A. C. ERINGEN and A. S. SUHUBI [23], W. GÜNTHER [24] and W. NOWACKI [25].

The dislocations and disclinations in Cosserat media were considered in several papers. The first author who suggested the necessity of applying the Cosserat theory to those problems was probably E. KRÖNER [1]. In considering the deformations produced by plastic torsion he proposed to introduce the couple-stresses into the theory of dislocations.

Further papers in which the Cosserat theory was used were based on a special version of that theory, and namely on the so-called model with constrained rotations. C. TEODOSIU [2] tackled the problem of determining the stresses produced by dislocations. The starting point of his considerations was the set of non-homogeneous compatibility equations involving the dislocation density tensor. Following the same line of reasoning, M. MISICU [3] constructed the fundamental conditions of compatibility expressed in terms of stresses (the generalized Beltrami equations) for the Cosserat media. A particular problem of determining the stresses produced by dislocations has been solved for the case of a model with constrained rotations.

Another particular problem was dealt with in the paper by Z. KNÉSL and F. SEMEL [4], and namely the calculation of couple-stresses produced by edge dislocations. Here also the starting point was the set of compatibility conditions of the plane state of strain. The equations were reduced to simple (biharmonic and Helmholtz-type) forms by means of the Airy and Mindlin stress functions.

The fundamental theoretical concept of dislocations in a Cosserat medium was outlined by C. A. ERINGEN and W. D. CLAUSS [5]. It was based on the application of non-homogeneous compatibility conditions, their right-hand sides containing the dislocation and disclination density tensors. The concept was then generalized to meromorphic media by the same authors.

K. H. ANTHONY [7] considered the problem of disclinations in Cosserat media by following the concept of A. C. Eringen and W. D. Clauss, and discussed the problem of a screw dislocation. Two-dimensional problems were analyzed by W. NOWACKI [8] who also started from the non-homogeneous compatibility conditions.

The two types of defects discussed here may also be considered using another method. In the classical theory of elasticity the Volterra model of defect description is used; it is expressed as a set of forced deformations on certain surfaces [9–11].

The present paper is aimed at describing the disclinations by means of a Volterra-type model defined for the Cosserat continuum. The derivation of fundamental equations will be followed by a number of simple examples involving disclinations of various types.

The paper combines and generalizes the results obtained in two other papers by the author [12, 13] on the problem of dislocations in micropolar media. Static problems will be discussed in this paper and hence the time-dependent terms of all equations will be disregarded.

1. The Volterra distortion

Let us consider the doubly-connected body subject to self-stresses. Once the deformations of that body are known, the displacements and rotations may be derived by the method given by E. CESARO [28]. Let $\mathring{P}(\mathbf{x})$ be a point at which the displacements $\mathring{u}_i(\mathbf{x})$ and rotations $\mathring{\varphi}_i(\mathring{\mathbf{x}})$ are known. The values of $u_i(\mathbf{x})$ and $\varphi_i(\mathbf{x})$ at another point $P(\mathbf{x})$ are determined by means of a line integral taken along a continuous and rectifiable curve connecting the points $\mathring{P}$ and P

$$(1.1)\qquad u_i(\mathbf{x}) = \mathring{u}_i(\mathring{\mathbf{x}}) + \int_{\mathring{P}}^{P} du_i + nb_i$$

$$= \mathring{u}_i(\mathring{\mathbf{x}}) + \varepsilon_{kji}(x_j - \mathring{x}_j)\mathring{\varphi}_k(\mathring{\mathbf{x}}) + \int_{\mathring{P}}^{P} [\gamma_{li} + \varepsilon_{kji}(x_j - \zeta_j)\varkappa_{lk}]\, d\zeta_l + nb_i,$$

$$(1.2)\qquad \varphi_i(\mathbf{x}) = \mathring{\varphi}_i(\mathring{\mathbf{x}}) + \int_{\mathring{P}}^{P} d\varphi_i + n\omega_i = \mathring{\varphi}_i(\mathring{\mathbf{x}}) + \int_{\mathring{P}}^{P} \varkappa_{li}\, d\zeta_l + n\omega_i,$$

where

$$(1.3)\qquad b_i = \oint_C (\gamma_{li} - \varepsilon_{kji}\zeta_j\varkappa_{lk})\, d\zeta_l,$$

$$(1.4)\qquad \omega_i = \oint_C \varkappa_{li}\, d\zeta_l.$$

Here n is the number of revolutions taken along the contour of integration C.

The integrals (1.3), (1.4) have constant values, independent of the position of the closed contour C which lies within the body and can not be contracted to a single point [14].

Let the displacement and rotation discontinuities occur at the surface σ. This surface is selected in such a way that a cut performed along σ renders the body simply-connected. Let σ^+ and σ^- denote the surfaces neighbouring upon σ, and $P^+(\mathbf{x}^+)$, $P(\mathbf{x})$, $P^-(\mathbf{x}^-)$—the points lying close to each other and located on the respective surfaces σ^+, σ, σ^-.

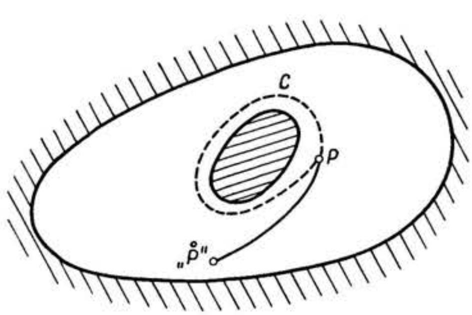

FIG. 1.

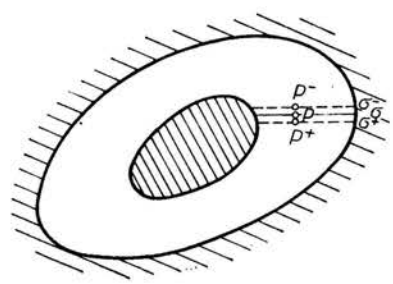

FIG. 2.

After performing the cut along σ let us write down the Cesaro integrals for the simply-connected body, $P^-(\mathbf{x}^-)$ and $P^+(\mathbf{x}^+)$ constituting the lower and upper limits of integration, respectively.

$$(1.5)\qquad u_i^+(\mathbf{x}^+) = u_i^-(\mathbf{x}^-)+\varepsilon_{kji}(x_j^+-x_j^-)\varphi_k^-(\mathbf{x}^-)+\int_{P^-}^{P^+}(\gamma_{li}-\varepsilon_{kji}\zeta_j\varkappa_{lk})d\zeta_l+x_j\int_{P^-}^{P^+}\varepsilon_{kji}\varkappa_{lk}d\zeta_l,$$

$$(1.6)\qquad \varphi_i^+(\mathbf{x}^+) = \varphi_i^-(\mathbf{x}^-)+\int_{P^-}^{P^+}\varkappa_{li}d\zeta_l.$$

Passing with surfaces σ^+, σ^- to the surface σ we obtain

$$x_j^+\to x_j,\qquad x_j^-\to x_j,\qquad x_j^+-x_j^-\to 0$$

and the integrals in Eqs. (1.5) and (1.6) are transformed to the integrals taken over closed contours

$$(1.7)\qquad u_i^+(\mathbf{x}^+)-u_i^-(\mathbf{x}^-) = \oint_C(\gamma_{li}-\varepsilon_{kji}\zeta_l\varkappa_{lk})d\zeta_l+x_j\oint_C\varepsilon_{kji}\varkappa_{lk}d\zeta_l,$$

$$(1.8)\qquad \varphi_i^+(\mathbf{x}^+)-\varphi_i^-(\mathbf{x}^-) = \oint_C\varkappa_{li}d\zeta_l.$$

Using the formulae (1.3) and (1.4) and denoting by $[u_i(\mathbf{x})]_\sigma$ and $[\varphi_i(\mathbf{x})]_\sigma$ the respective displacement and rotation discontinuities calculated at points $x\in\sigma$, we obtain

$$(1.9)\qquad [u_i(\mathbf{x})]_\sigma = b_i+\varepsilon_{kji}x_j\omega_k,$$

$$(1.10)\qquad [\varphi_i(\mathbf{x})]_\sigma = \omega_i.$$

It is seen that if the surface discontinuities of displacements and rotations are introduced into a doubly-connected body in the manner shown above, the discontinuities must be of the forms given by Eqs. (1.9) and (1.10).

In the case of Hooke's bodies, such an operation of introducing the stresses into a simply-connected body was performed by V. Volterra who called the resulting defect a distortion.

2. Surface model of dislocation and disclination

Let us consider an elastic micropolar body which is homogeneous, centrosymmetric and isotropic. A disclination in such a body will be described by discontinuities of the fields of displacements and rotations on a bounded open surface σ (Fig. 3). The discontinuities are given by the following relations:

$$(2.1)\qquad [u_l(\mathbf{x})]_\sigma = B_l(\mathbf{x}) = b_l+\varepsilon_{lqr}\Omega_q(x_r-\mathring{x}_r),$$

$$(2.2)\qquad [\varphi_l(\mathbf{x})]_\sigma = \Omega_l.$$

Here $\mathbf{b}$—Burgers vector, $\mathbf{\Omega}$—Frank vector, $\mathring{\mathbf{x}}$—a point on the axis of rotation.

From the conditions of geometric compatibility on the surface of discontinuity [15, 16]

$$(2.3)\qquad [\nabla_j u_i]_\sigma = \nabla_j[u_i]_\sigma$$

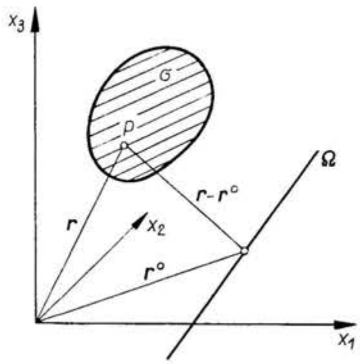

FIG. 3.

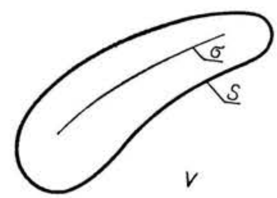

FIG. 4.

we obtain the following relations for the derivatives of displacements and rotations:

$$[\nabla_j u_i]_\sigma = \varepsilon_{pji}\Omega_p, \tag{2.4}$$

$$[\nabla_j \varphi_i]_\sigma = 0, \tag{2.5}$$

what yields the conditions of continuity of the force-stress and couple-stress tensors at the surface of discontinuity

$$[\sigma_{ji}]_\sigma = 0, \quad [\mu_{ji}]_\sigma = 0.$$

Let us surround the surface σ by a closed surface S and denote the region lying outside S by V. The introduction of the displacement and rotation discontinuities, Eqs. (2.1) and (2.2), produces in the body the state of deformation characterized by the displacements u_i, rotations φ_i and stresses σ_{ji}, μ_{ji}. The displacements and rotations should satisfy in the region V the generalized Lamé system of equations

$$L_{ji} u_i + R_{ji}\varphi_i = 0, \tag{2.6}$$

$$D_{ji}\varphi_i + R_{ji} u_i = 0. \tag{2.7}$$

Here the following notations have been introduced:

$$L_{ji} = A_{jpis}\nabla_p\nabla_s = [\lambda\delta_{jp}\delta_{is} + (\mu+\alpha)\delta_{ps}\delta_{ij} + (\mu-\alpha)\delta_{js}\delta_{pi}]\nabla_p\nabla_s,$$

$$D_{ji} = B_{jpis}\nabla_p\nabla_s - 4\alpha\delta_{ij} = [\beta\delta_{jp}\delta_{is} + (\gamma+\varepsilon)\delta_{ps}\delta_{ij} + (\gamma-\varepsilon)\delta_{js}\delta_{pi}]\nabla_p\nabla_s - 4\alpha\delta_{ij},$$

$$R_{ji} = 2\alpha\varepsilon_{jpi}\nabla_p;$$

λ, μ, α, β, γ, ε are material constants of the Cosserat medium. The discontinuities (2.1)–(2.5) play here a role analogous to that of the boundary conditions in a bounded body. It is moreover required that $\mathbf{u}\to 0$ and $\boldsymbol{\varphi}\to 0$ at $\mathbf{x}\to\infty$.

In order to solve the above system of equations let us now introduce the Green functions G_{in}, Φ_{in} and $\hat{G}_{in}$, $\hat{\Phi}_{in}$ for an infinite region. The functions satisfy the corresponding systems of equations:

$$L_{ji}G_{in} + R_{ji}\Phi_{in} + \delta_{jn}\delta_3(\mathbf{x}-\mathbf{x}') = 0, \tag{2.8}$$

$$D_{ji}\Phi_{in} + R_{ji}G_{in} = 0 \tag{2.9}$$

and

$$L_{ji}\hat{G}_{in}+R_{ji}\Phi_{in} = 0, \tag{2.10}$$

$$D_{ji}\hat{\Phi}_{in}+R_{ji}\hat{G}_{in}+\delta_{jn}\delta_3(\mathbf{x}-\mathbf{x}') = 0. \tag{2.11}$$

Since the fields $\mathbf{u}$ and $\boldsymbol{\varphi}$ are of the class $C^{(2)}$ in the entire region V, the following identities must hold true:

$$u_n(\mathbf{x}) = \int_V u_n(\mathbf{x}')\delta_3(\mathbf{x}-\mathbf{x}')dV(\mathbf{x}'), \tag{2.12}$$

$$\varphi_n(\mathbf{x}) = \int_V \varphi_n(\mathbf{x}')\delta_3(\mathbf{x}-\mathbf{x}')dV(\mathbf{x}'). \tag{2.13}$$

Consequently, elimination of the Dirac function (2.12), (2.13) by means of Eqs. (2.8)–(2.11) and application of the formulae (2.6) and (2.7) yields

$$u_n(\mathbf{x}) = -\int_S [A_{ijkl}(u_i G_{kn,l}dS_j - u_{i,j}G_{kn}dS_l)] - \int_S 2\alpha\varepsilon_{ijk}(u_i\Phi_{kn}+\varphi_i G_{kn})dS_j - \int_S [B_{ijkl}(\varphi_i\Phi_{kn,l}dS_j - \varphi_{i,j}\Phi_{kn}dS_l], \tag{2.14}$$

$$\varphi_n(\mathbf{x}) = -\int_S [A_{ijkl}(u_i\hat{G}_{kn,l}dS_j - u_{i,j}\hat{G}_{kn}dS_l] - \int_S 2\alpha\varepsilon_{ijk}(u_i\hat{\Phi}_{kn}+\varphi_i\hat{G}_{kn})dS_j - \int_S [B_{ijkl}(\varphi_i\hat{\Phi}_{kn,l}dS_j - \varphi_{i,j}\hat{\Phi}_{kn}dS_l)]. \tag{2.15}$$

Passing with the surface S to σ and using the conditions (2.1)–(2.5), we obtain

$$u_n(\mathbf{x}) = \int_V dV(\mathbf{x}')[A_{ijkl}G_{kn}(\mathbf{x},\mathbf{x}')\overline{\eta}_{ji,l}(\mathbf{x}') - 2\alpha\varepsilon_{ijk}\Phi_{kn}(\mathbf{x},\mathbf{x}')\overline{\eta}_{ji}(\mathbf{x}') + B_{ijkl}\Phi_{kn}(\mathbf{x},\mathbf{x}')\Theta_{ji,l}(\mathbf{x}')], \tag{2.16}$$

$$\varphi_n(\mathbf{x}) = \int_V dV(\mathbf{x}')[A_{ijkl}\hat{G}_{kn}(\mathbf{x},\mathbf{x}')\overline{\eta}_{ji,l}(\mathbf{x}') - 2\alpha\varepsilon_{ijk}\hat{\Phi}_{kn}(\mathbf{x},\mathbf{x}')\overline{\eta}_{ji}(\mathbf{x}') + B_{ijkl}\hat{\Phi}_{kn}(\mathbf{x},\mathbf{x}')\Theta_{ji,l}(\mathbf{x}')]. \tag{2.17}$$

Here

$$\overline{\eta}_{ji}(\mathbf{x}') = \int_\sigma B_i(\mathbf{x}')\delta_3(\mathbf{x}'-\boldsymbol{\zeta})d\sigma_j(\boldsymbol{\zeta}), \tag{2.18}$$

$$\bar{\Theta}_{ji}(\mathbf{x}') = \int_\sigma \Omega_i\delta_3(\mathbf{x}'-\boldsymbol{\zeta})d\sigma_j(\boldsymbol{\zeta}). \tag{2.19}$$

The displacement and rotation fields given by Eqs. (2.16) and (2.17) are the fields sought for. They satisfy the generalized Lamé system of equations within the entire region outside the surface σ, and the conditions (2.1)–(2.5) on the surface σ. The formulae for displacements and rotations due to a dislocation is obtained by substituting $\boldsymbol{\Omega} = 0$ into Eqs. (2.16)–(2.19):

$$u_n(\mathbf{x}) = \int_V dV(\mathbf{x}')[A_{ijkl}G_{kn}(\mathbf{x},\mathbf{x}')\eta_{ji,l}(\mathbf{x}') - 2\alpha\varepsilon_{ijk}\Phi_{kn}(\mathbf{x},\mathbf{x}')\eta_{ji}(\mathbf{x}')], \tag{2.20}$$

$$\varphi_n(\mathbf{x}) = \int_V dV(\mathbf{x}')[A_{ijkl}\hat{G}_{kn}(\mathbf{x},\mathbf{x}')\eta_{ji,l}(\mathbf{x}') - 2\alpha\varepsilon_{ijk}\hat{\Phi}_{kn}(\mathbf{x},\mathbf{x}')\eta_{ji}(\mathbf{x}')] \tag{2.21}$$

where

$$\eta_{ji}(\mathbf{x}') = b_i \int_\sigma \delta_3(\mathbf{x}'-\boldsymbol{\zeta})\, d\sigma_j(\boldsymbol{\zeta}). \tag{2.22}$$

3. Introduction of fictitious body forces and couples

In the preceding section we considered a homogeneous system of equations in terms of displacements and rotations with the boundary (discontinuity) conditions on the surface σ. Now we shall try to obtain the results previously derived by solving the non-homogeneous system of equations written in terms of displacements and rotations; to that end, certain fictitious body forces and body couples will be used. The forces and couples will be determined by equating the results previously obtained to the displacement and rotation fields produced by those forces and expressed by means of the Green functions. Let us solve the following non-homogeneous system of equations:

$$L_{ji} u_i + R_{ji} \varphi_i + X_j = 0, \tag{3.1}$$

$$D_{ji} \varphi_i + R_{ji} u_i + Y_j = 0. \tag{3.2}$$

Apply now the reciprocity theorem for an infinite region,

$$\int_V (X_i u_i' + Y_i \varphi_i')\, dV = \int_V (X_i' u_i + Y_i' \varphi_i)\, dV. \tag{3.3}$$

Under the assumption that

$$\{u_i', \varphi_i', X_i', Y_i'\} = \{G_{in}, \Phi_{in}, \delta_{in}\delta_3(\mathbf{x}-\mathbf{x}'), 0\}$$

we obtain

$$u_n(\mathbf{x}) = \int_V (X_i G_{in} + Y_i \Phi_{in})\, dV(\mathbf{x}') \tag{3.4}$$

and assuming

$$\{u_i', \varphi_i', X_i', Y_i'\} = \{\hat{G}_{in}, \hat{\Phi}_{in}, 0, \delta_{in}\delta_3(\mathbf{x}-\mathbf{x}')\}$$

we have

$$\varphi_n(\mathbf{x}) = \int_V (X_i \hat{G}_{in} + Y_i \hat{\Phi}_{in})\, dV(\mathbf{x}'). \tag{3.5}$$

On comparing the formulae (3.4) and (3.5) with Eqs. (2.16) and (2.17), we obtain

$$X_k = A_{ijkl}\, \overline{\eta}_{ji,l} = A_{ijkl} \frac{\partial}{\partial x_l'} \int_\sigma B_i(\mathbf{x}')\, \delta_3(\mathbf{x}'-\boldsymbol{\zeta})\, d\sigma_j(\boldsymbol{\zeta}), \tag{3.6}$$

$$\begin{aligned} Y_k &= B_{ijkl}\, \overline{\Theta}_{ji,l} - 2\alpha \varepsilon_{ijk} \overline{\eta}_{ji} \\ &= B_{ijkl} \frac{\partial}{\partial x_l'} \int_\sigma \Omega_i\, \delta_3(\mathbf{x}'-\boldsymbol{\zeta})\, d\sigma_j(\boldsymbol{\zeta}) - 2\alpha\varepsilon_{ijk} \int_\sigma B_i(\mathbf{x}')\, \delta_3(\mathbf{x}'-\boldsymbol{\zeta})\, d\sigma_j(\boldsymbol{\zeta}). \end{aligned} \tag{3.7}$$

The fields of displacements $\mathbf{u}$ and rotations $\boldsymbol{\varphi}$ are now derived by solving the non-homogeneous system of Eqs. (3.1) and (3.2) in which the body forces $\mathbf{X}$ and body couples $\mathbf{Y}$ are given by Eqs. (3.6) and (3.7).

The system of Eqs. (3.1) and (3.2) may also be written as

(3.8) $$A_{ijkl}\gamma_{lk,j}+A_{ijkl}\bar{\eta}_{lk,j}=0,$$

(3.9) $$B_{ijkl}\varkappa_{lk,j}+2\alpha\varepsilon_{ijk}\gamma_{jk}+B_{ijkl}\bar{\Theta}_{lk,j}+2\alpha\varepsilon_{ijk}\bar{\eta}_{jk}=0.$$

Here

$$\gamma_{lk}=u_{k,l}-\varepsilon_{jlk}\varphi_j,\qquad \varkappa_{lk}=\varphi_{k,l}$$

and use has been made of the identities $A_{ijkl}=A_{klij}$, $B_{ijkl}=B_{klij}$.

In the entire region the equilibrium conditions must be satisfied,

(3.10) $$\sigma_{ji,j}=0,$$

(3.11) $$\varepsilon_{ijk}\sigma_{jk}+\mu_{ji,j}=0$$

and hence we obtain the following relations between the fields of force-stresses σ_{ij} and couple-stresses μ_{ij}, and the fields of strains γ_{ij} and torsion-flexure deformations $\varkappa_{ij}$:

(3.12) $$\sigma_{ji}=A_{ijkl}\gamma_{lk}+A_{ijkl}\bar{\eta}_{lk},$$

(3.13) $$\mu_{ji}=B_{ijkl}\varkappa_{lk}+B_{ijkl}\bar{\Theta}_{lk}.$$

4. Distortional model of dislocation and disclination

Let us consider an infinite elastic body containing the initial distortions $\mathring{\gamma}_{ji}$, $\mathring{\varkappa}_{ji}$. If the elastic deformations produced by $\mathring{\gamma}_{ji}$, $\mathring{\varkappa}_{ji}$ are denoted by $\overset{S}{\gamma}_{ji}$, $\overset{S}{\varkappa}_{ji}$, then the complete deformations may be written in the form

(4.1) $$\gamma_{ji}=\mathring{\gamma}_{ji}+\overset{S}{\gamma}_{ji},\qquad \varkappa_{ji}=\mathring{\varkappa}_{ji}+\overset{S}{\varkappa}_{ji}.$$

Stresses σ_{ji}, μ_{ji} are given by the formulae

(4.2) $$\sigma_{ji}=A_{ijkl}\gamma_{lk}-A_{ijkl}\mathring{\gamma}_{lk},$$

(4.3) $$\mu_{ji}=B_{ijkl}\varkappa_{lk}-B_{ijkl}\mathring{\varkappa}_{lk}.$$

The deformations γ_{ji}, $\varkappa_{ji}$ may be expressed in terms of the displacements u_i and rotations φ_i,

(4.4) $$\gamma_{ji}=u_{i,j}-\varepsilon_{kji}\varphi_k,\qquad \varkappa_{ji}=\varphi_{i,j}.$$

Substituting the expressions (4.4) into Eqs. (4.2) and (4.3) and then into the equations of equilibrium

(4.5) $$\sigma_{ji,j}=0,$$

(4.6) $$\varepsilon_{ijk}\sigma_{jk}+\mu_{ji,j}=0$$

the non-homogeneous system of equations is obtained

(4.7) $$L_{ji}u_j+R_{ji}\varphi_j=A_{ijkl}\mathring{\gamma}_{lk,j},$$

(4.8) $$D_{ji}\varphi_j+R_{ji}u_j=B_{ijkl}\mathring{\varkappa}_{lk,j}+2\alpha\varepsilon_{ilk}\mathring{\gamma}_{lk}.$$

Let us introduce the following notations for the right-hand terms of Eqs. (4.7) and (4.8):

(4.9) $$-A_{ijkl}\mathring{\gamma}_{lk,j}=X_i^*,$$

(4.10) $$-B_{ijkl}\mathring{\varkappa}_{lk,j}-2\alpha\varepsilon_{ilk}\mathring{\gamma}_{lk}=Y_i^*.$$

Using the reciprocity theorem for distortions and following the way of reasoning of the preceding section we obtain

$$u_n = \int_V (X_i^* G_{in} + Y_i^* \Phi_{in}) dV, \tag{4.11}$$

$$\varphi_n = \int_V (X_i^* \hat{G}_{in} + Y_i^* \hat{\Phi}_{in}) dV. \tag{4.12}$$

Let us now compare the formulae for displacements and rotations, Eqs. (4.11) and (4.12), and for the force- and couple-stresses, Eqs. (4.2) and (4.3), in the distortional model, with the corresponding formulae (3.4), (3.5) and (3.12), (3.13) obtained by introducing the fictitious body forces and body couples. It is readily seen that they are identical (i.e. all of them represent the solutions of the problem of dislocations and disclinations) provided

$$\mathring{\gamma}_{ji}(\mathbf{x}') = -\overline{\eta}_{ji}(\mathbf{x}') = -\int_\sigma B_i(\mathbf{x}')\delta_3(\mathbf{x}'-\boldsymbol{\zeta})d\sigma_j(\boldsymbol{\zeta}), \tag{4.13}$$

$$\mathring{\varkappa}_{ji}(\mathbf{x}') = -\overline{\Theta}_{ji}(\mathbf{x}') = -\int_\sigma \Omega_i \delta_3(\mathbf{x}'-\boldsymbol{\zeta})d\sigma_j(\boldsymbol{\zeta}). \tag{4.14}$$

Thus the dislocations and disclinations have been modelled by means of the distortions described by Eqs. (4.13) and (4.14). Inserting Eqs. (4.9) and (4.10) into the formulae (4.11) and (4.12), we obtain

$$\begin{aligned} u_n(\mathbf{x}) = -\int_V [&A_{ijkl}\mathring{\gamma}_{ji,l}(\mathbf{x}')G_{kn}(\mathbf{x},\mathbf{x}') \\ &-2\alpha\varepsilon_{ijk}\mathring{\gamma}_{ji}(\mathbf{x}')\Phi_{kn}(\mathbf{x},\mathbf{x}') + B_{ijkl}\mathring{\varkappa}_{ji,l}(\mathbf{x}')\Phi_{kn}(\mathbf{x},\mathbf{x}')]dV(\mathbf{x}'), \end{aligned} \tag{4.15}$$

$$\begin{aligned} \varphi_n(\mathbf{x}) = -\int_V [&A_{ijkl}\mathring{\gamma}_{ji,l}(\mathbf{x}')\hat{G}_{kn}(\mathbf{x},\mathbf{x}') \\ &-2\alpha\varepsilon_{ijk}\mathring{\gamma}_{ji}(\mathbf{x}')\hat{\Phi}_{kn}(\mathbf{x},\mathbf{x}') + B_{ijkl}\mathring{\varkappa}_{ji,l}(\mathbf{x}')\hat{\Phi}_{kn}(\mathbf{x},\mathbf{x}')]dV(\mathbf{x}'). \end{aligned} \tag{4.16}$$

Integration by parts and substitution of the following Green functions [17]

$$\begin{aligned} G_{jn} = \frac{1}{8\pi\mu}\left[\delta_{jn}\nabla^2 R - \frac{\lambda+\mu}{\lambda+2\mu}\nabla_j\nabla_n R\right] \\ + \frac{\alpha}{4\pi\mu(\alpha+\mu)}\left[l^2\nabla_j\nabla_n\left(\frac{e^{-R/l}-1}{R}\right) - \delta_{jn}\frac{e^{-R/l}}{R}\right], \end{aligned}$$

$$\Phi_{jn} = -\frac{1}{8\pi\mu}\varepsilon_{njp}\nabla_p\left(\frac{e^{-R/l}-1}{R}\right),$$

$$\hat{G}_{jn} = -\frac{1}{8\pi\mu}\varepsilon_{njp}\nabla_p\left(\frac{e^{-R/l}-1}{R}\right),$$

$$\hat{\Phi}_{jn} = -\frac{1}{16\pi\mu}\nabla_j\nabla_n\left(\frac{e^{-R/l}-1}{R}\right) + \frac{1}{16\pi\alpha}\nabla_j\nabla_n\left(\frac{e^{-R/h}-e^{-R/l}}{R}\right) + \frac{\mu+\alpha}{16\pi\alpha\mu l^2}\frac{e^{-R/l}}{R}\delta_{jn}$$

yields the formulae

$$(4.17)\qquad u_n(\mathbf{x}) = \frac{1}{8\pi}\int_V \Big[\mathring{\gamma}_{jn}R_{,jpp}+\mathring{\gamma}_{nj}R_{,ppj}-\frac{1}{1-\nu}R_{,npj}\mathring{\gamma}_{jp} + \frac{\nu}{1-\nu}R_{,npp}\mathring{\gamma}_{jj}\Big]dV(\mathbf{x}')+\frac{\alpha}{2\pi(\mu+\alpha)}\int_V \mathring{\gamma}_{ji}[l^2\Gamma_{,ijn}-\Omega_{,i}\delta_{jn}]dV - \frac{1}{8\pi\mu}\int_V \mathring{\varkappa}_{ji}[(\gamma+\varepsilon)\varepsilon_{nip}\Gamma_{,pj}+(\gamma-\varepsilon)\varepsilon_{njp}\Gamma_{,pi}]dV,$$

$$(4.18)\qquad \varphi_n(\mathbf{x}) = -\frac{1}{8\pi\mu}\int_V \mathring{\gamma}_{ji}\Big[(\mu+\alpha)\varepsilon_{nip}\Gamma_{,pj}+(\mu-\alpha)\varepsilon_{njp}\Gamma_{,pi} + \varepsilon_{pji}(\mu\Psi-\alpha\Gamma)_{,pn}+\varepsilon_{nji}\frac{\mu+\alpha}{l^2}\Omega\Big]dV+\frac{1}{8\pi\mu\alpha}\int_V\Big\{\mathring{\varkappa}_{ji}\gamma(\mu\Psi-\alpha\Gamma)_{,nij}+\frac{\mu\beta}{2h^2}\mathring{\varkappa}_{jj}E_{,n} + \frac{\mu+\alpha}{2l^2}[(\gamma+\varepsilon)\mathring{\varkappa}_{jn}\Omega_{,j}+(\gamma-\varepsilon)\mathring{\varkappa}_{nj}\Omega_{,j}]\Big\}dV.$$

Here

$$\Gamma = \frac{e^{-R/l}-1}{R},\qquad \Omega = \frac{e^{-R/l}}{R},\qquad E = \frac{e^{-R/h}-1}{R},$$

$$\Psi = E-\Gamma,\qquad \nu = \frac{\lambda}{2(\lambda+\mu)},\qquad l^2 = \frac{(\mu+\alpha)(\gamma+\varepsilon)}{4\mu\alpha},$$

$$h^2 = \frac{2\gamma+\beta}{4\alpha},\qquad R = [(x_i-x_i')(x_i-x_i')]^{1/2}.$$

In the case of Hooke's body ($\alpha = 0$) we obtain the well-known result

$$(4.19)\qquad u_n(\mathbf{x}) = \frac{1}{8\pi}\int_V\Big[\mathring{\gamma}_{jn}R_{,jpp}+\mathring{\gamma}_{nj}R_{,jpp}-\frac{1}{1-\nu}\mathring{\gamma}_{jp}R_{,jpn}+\frac{\nu}{1-\nu}\mathring{\gamma}_{jj}R_{,npp}\Big]dV,$$

$$(4.20)\qquad \varphi_n(\mathbf{x}) = \frac{1}{2}\varepsilon_{njk}\nabla_j u_k.$$

Equations (4.13), (4.14) and (4.17), (4.18) are useful in considering certain particular cases of dislocations and disclinations in a micropolar medium.

5. Examples

In this section the displacement and rotation fields will be derived for several simple cases of disclinations; let us rewrite Eqs. (4.13) and (4.14) in the following form:

$$(5.1)\qquad \mathring{\gamma}_{kl} = -\delta_k(\sigma)\{b_l+\varepsilon_{lqr}\Omega_q(x_r'-\mathring{x}_r)\},$$

$$(5.2)\qquad \mathring{\varkappa}_{kl} = -\delta_k(\sigma)\Omega_l$$

with the notation

$$\delta_k(\sigma) = \int_\sigma \delta_3(\mathbf{x}'-\boldsymbol{\zeta})\,d\sigma_k(\boldsymbol{\zeta}).$$

Let us assume for the surface of discontinuity the plane $x_1'x_3'$ at negative values of x_1' ($x_2' = 0$, $x_1' < 0$) characterized by the normal vector directed towards negative x_2', $\mathbf{n} = [0, -1,$

0]. The axis of rotation passes through the origin of the coordinate system. Then $d\sigma_2 = -d\zeta_1 d\zeta_3$, while

$$\delta_2(\sigma) = \int_\sigma \delta_3(\mathbf{x}'-\boldsymbol{\zeta})\, d\sigma_2(\boldsymbol{\zeta}) = -\int_{-\infty}^{0} \delta(x_1'-\zeta_1)\, d\zeta_1\, \delta(x_2') \int_{-\infty}^{\infty} \delta(x_3'-\zeta_3)\, d\zeta_3 = = -H(-x_1')\,\delta(x_2'),$$

$H(-x_2')$ denoting the Heaviside function is defined as follows:

$$H(-x_1') = \begin{cases} 0 & \text{for} \quad x_1' > 0, \\ 1 & \text{for} \quad x_1' < 0. \end{cases}$$

Let us observe that the only non-vanishing components of the distortion tensors $\overset{\circ}{\gamma}_{jl}$ and $\overset{\circ}{\varkappa}_{jl}$ in Eqs. (5.1) and (5.2) are

$$\overset{\circ}{\gamma}_{2l} = (b_l + \varepsilon_{lqr}\,\Omega_q x_r)\, H(-x_1')\delta(x_2'), \tag{5.3}$$

$$\overset{\circ}{\varkappa}_{2l} = \Omega_l H(-x_1')\delta(x_2'). \tag{5.4}$$

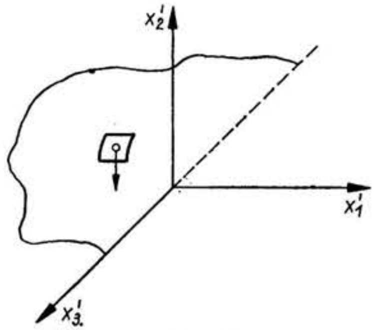

FIG. 5.

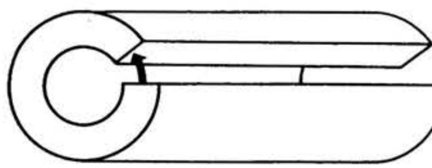

FIG. 6.

It is seen that the above expressions may be composed of two parts, one of them corresponding to the assumption $\mathbf{\Omega} = 0$ and the other to $\mathbf{b} = 0$. The first part is due to a dislocation; it was discussed in the paper [12] and it is not necessary to repeat the derivations here. Let us rather pass to the second part, that is to the assumption $\mathbf{b} = 0$. Consider the case shown in Fig. 6.

The assumption $\Omega_1 = \Omega_2 = 0$, $\Omega_3 \neq 0$ yields the following equations:

$$\overset{\circ}{\gamma}_{21} = \overset{\circ}{\gamma}_{23} = 0, \quad \gamma_{22} = \Omega_3 x_1' H(-x_1')\delta(x_2'), \tag{5.5}$$

$$\overset{\circ}{\varkappa}_{21} = \overset{\circ}{\varkappa}_{22} = 0, \quad \varkappa_{23} = \Omega_3 H(-x_1')\delta(x_2'). \tag{5.6}$$

After substituting the results in Eqs. (4.17) and (4.18) we obtain

$$\begin{aligned} u_1 &= \bar{u}_1 - \frac{\Omega_3(\gamma+\varepsilon)}{4\pi\mu}\left\{\frac{\partial^2}{\partial x_1 \partial x_2}[x_2(I_2-I_1)] - \frac{\partial^2}{\partial x_2^2}(I_5-I_4)\right\}, \\ u_2 &= \bar{u}_2 - \frac{\Omega_3(\gamma+\varepsilon)}{4\pi\mu}\left\{\frac{\partial^2}{\partial x_2^2}[x_2(I_2-I_1)] - \frac{x_2}{l^2}I_2 - \frac{\partial}{\partial x_2}(I_2-I_1)\right\}, \\ u_3 &= 0, \\ \varphi_1 &= 0, \\ \varphi_2 &= 0, \\ \varphi_3 &= -\frac{\Omega_3}{2\pi}\left\{\frac{\partial}{\partial x_2}I_5 - x_2\frac{\partial}{\partial x_1}(I_2-I_1)\right\}. \end{aligned} \tag{5.7}$$

Here

$$I_1 = -\ln r, \quad I_2 = K_0(r/l), \quad I_3 = K_0(r/h),$$

$$I_4 = -x_1(\ln r - 1) - x_2 \operatorname{arctg}\frac{x_2}{x_1}, \quad I_5 = \int_{-\infty}^{0} K_0(\bar{r}/l)\, d\zeta_1,$$

$$r = [(x_1^2 + x_2^2)]^{1/2}, \quad \bar{r} = [(x_1 - \zeta_1)^2 + x_2^2]^{1/2}.$$

$K_0(z)$ is the modified Bessel function of the third kind (McDonald function). Displacements $\bar{u}_1, \bar{u}_2$ have the form

$$\bar{u}_1 = -\frac{\Omega_3}{2\pi}\left[\frac{\nu - 1/2}{1-\nu} x_1(\ln r - 1) + x_2 \operatorname{arctg}\frac{x_2}{x_1}\right],$$

(5.8)

$$\bar{u}_2 = -\frac{\Omega_3}{2\pi}\left[\frac{\nu - 1/2}{1-\nu} x_2(\ln r - 1) - x_1 \operatorname{arctg}\frac{x_2}{x_1}\right]$$

and they represent the solution of the problem of disclination in a Hooke's body.

Let us now consider the case shown in Fig. 7. Here $\Omega_1 \neq 0$, $\Omega_2 = \Omega_3 = 0$ and hence the only non-vanishing components are

(5.9)

$$\overset{\circ}{\gamma}_{22} = -x_3' \Omega_1 H(-x_1')\delta(x_2'),$$

$$\overset{\circ}{\varkappa}_{21} = \Omega_1 H(-x_1')\delta(x_2').$$

Substitution of Eqs. (4.17) and (4.18) yields

$$u_1 = \bar{u}_1 - \frac{\Omega_1 x_3(\gamma+\varepsilon)}{4\pi\mu}\frac{\partial^2}{\partial x_2^2}(I_2 - I_1),$$

$$u_2 = \bar{u}_2 + \frac{\Omega_1 x_3(\gamma+\varepsilon)}{4\pi\mu}\left[\frac{\partial^3}{\partial x_2^3}(I_5 - I_4) - \frac{\partial}{\partial x_2} I_5\right],$$

$$u_3 = \bar{u}_3 + \frac{\Omega_1}{4\pi\mu}\left[(\gamma - \varepsilon)\frac{\partial}{\partial x_1}(I_2 - I_1) - (\gamma+\varepsilon)\frac{\partial^2}{\partial x_2^2}(I_5 - I_4)\right],$$

(5.10)

$$\varphi_1 = \frac{\Omega_1}{2\pi}\left\{\frac{\gamma}{2\mu\alpha}\frac{\partial^2}{\partial x_1 \partial x_2}[\mu(I_3 - I_2) - \alpha(I_2 - I_1)] - \frac{\partial}{\partial x_2} I_5\right\},$$

$$\varphi_2 = -\frac{\Omega_1}{2\pi}\left\{\frac{\gamma}{2\mu\alpha}\frac{\partial^2}{\partial x_2^2}[\mu(I_3 - I_2) - \alpha(I_2 - I_1)] + \frac{\gamma - \varepsilon}{\gamma + \varepsilon} I_2\right\},$$

$$\varphi_3 = -\frac{x_3\Omega_1}{2\pi}\frac{\partial}{\partial x_2}(I_2 - I_1).$$

The displacements $\bar{u}_1, \bar{u}_2, \bar{u}_3$ assume the following values:

$$\bar{u}_1 = \frac{\Omega_1 x_3}{2\pi(1-\nu)}\left[\left(\nu - \frac{1}{2}\right)\ln r + \frac{x_2^2}{2r^2}\right],$$

(5.11)

$$\bar{u}_2 = \frac{\Omega_1 x_3}{2\pi}\left[\frac{x_1 x_2}{2(1-\nu) r^2} - \operatorname{arctg}\frac{x_2}{x_1}\right],$$

$$\bar{u}_3 = \frac{\Omega_1}{2\pi}\left[x_2 \operatorname{arctg}\frac{x_2}{x_1} - \frac{1-2\nu}{2(1-\nu)} x_1(\ln r - 1)\right]$$

and are the solutions for the disclination of that type in Hooke's body. Observe that u_1 for $r = 0$ possesses a logarithmic singularity.

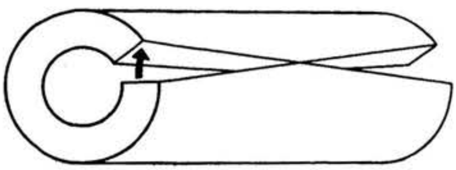

FIG. 7.

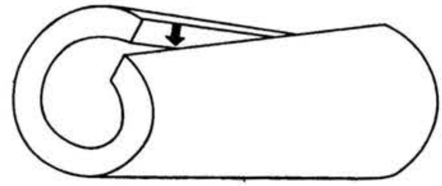

FIG. 8.

Let us now finally consider the case illustrated by Fig. 8. Here $\Omega_1 = \Omega_3 = 0$, $\Omega_2 \neq 0$ and the non-vanishing tensor components are

$$(5.12)\qquad \begin{aligned} \mathring{\gamma}_{21} &= \Omega_2 x_3' H(-x_1')\,\delta(x_2'),\\ \mathring{\gamma}_{23} &= -\Omega_2 x_1' H(-x_1')\,\delta(x_2'),\\ \mathring{\varkappa}_{22} &= \Omega_2 H(-x_1')\,\delta(x_2'). \end{aligned}$$

On substituting these in Eqs. (4.17) and (4.18) we obtain

$$\begin{aligned} u_1 &= \bar{u}_1 - \frac{\Omega_2 x_3(\gamma+\varepsilon)}{4\pi\mu}\,\frac{\partial^2}{\partial x_1\,\partial x_2}(I_2-I_1),\\ u_2 &= \bar{u}_2 - \frac{\Omega_2 x_3}{\pi}\left[\frac{(\gamma+\varepsilon)}{4\mu}\,\frac{\partial^2}{\partial x_2^2}(I_2-I_1) - \frac{\alpha}{(\mu+\alpha)}\,I_2\right],\\ u_3 &= \bar{u}_3 - \frac{\Omega_2\gamma}{2\pi\mu}\,\frac{\partial}{\partial x_2}(I_2-I_1), \end{aligned}$$

$$(5.13)\qquad \begin{aligned} \varphi_1 &= -\frac{\Omega_2}{4\pi}\left\{\frac{\gamma}{\alpha\mu}\,\frac{\partial^2}{\partial x_2^2}[\mu(I_3-I_2)-\alpha(I_2-I_1)] + \frac{\beta}{2\alpha h^2}(I_3-I_1)\right.\\ &\quad\left. - \frac{\partial^2}{\partial x_1^2}\left[(I_9-I_8) - \frac{\alpha}{\mu}(I_8-I_7)\right] + \frac{\alpha+\mu}{\mu}\,\frac{\partial}{\partial x_2}[x_2(I_2-I_1)] - \frac{\mu+\alpha}{\mu l^2}I_8\right\},\\ \varphi_2 &= -\frac{\Omega_2}{4\pi}\left\{\frac{\gamma}{\alpha\mu}\,\frac{\partial^3}{\partial x_2^3}[\mu(I_6-I_5)-\alpha(I_5-I_4)] + \frac{\partial}{\partial x_2}\left[\frac{(\mu+\alpha)\gamma}{\mu\alpha l^2}I_5\right.\right.\\ &\quad\left.\left. + \frac{\beta}{2\alpha h^2}(I_6-I_4)\right] - x_2\frac{\partial}{\partial x_1}(I_3-I_1)\right\},\\ \varphi_3 &= -\frac{\Omega_2 x_3}{4\pi\mu}\left\{(\mu+\alpha)\frac{\partial^2}{\partial x_2^2}(I_5-I_4) + (\mu-\alpha)\frac{\partial}{\partial x_1}(I_2-I_1) - \frac{\alpha+\mu}{l^2}I_5\right\}, \end{aligned}$$

where

$$I_6 = \int_{-\infty}^{0} d\zeta_1 K_0\left(\frac{\bar{r}}{h}\right),\qquad I_7 = -\left(x_1^2+\frac{r^2}{2}\right)\ln r - x_1x_2\,\mathrm{arc\,tg}\frac{x_2}{x_1} + \frac{3}{4}x_1^2,$$

$$I_8 = \int_{-\infty}^{0} d\zeta_1\zeta_1 K_0\left(\frac{\bar{r}}{l}\right),\qquad I_9 = \int_{-\infty}^{0} d\zeta_1\zeta_1 K_0\left(\frac{\bar{r}}{h}\right).$$

The displacements $\bar{u}_1, \bar{u}_2, \bar{u}_3$ have the following form:

$$
\begin{aligned}
\bar{u}_1 &= \frac{\Omega_2 x_3}{2\pi}\left[\operatorname{arc\,tg}\frac{x_2}{x_1}+\frac{x_1 x_2}{2(1-\nu)r^2}\right],\\
\bar{u}_2 &= \frac{\Omega_2 x_3}{4\pi(1-\nu)}\left[(1-2\nu)\ln r+\frac{x_2^2}{r^2}\right],\\
\bar{u}_3 &= \frac{\Omega_2 x_1}{2\pi}\left[\operatorname{arc\,tg}\frac{x_2}{x_1}+\frac{1-2\nu}{2(1-\nu)}(\ln r-1)\right]
\end{aligned}
\tag{5.14}
$$

and represent the solutions for that type of disclination in Hooke's body. Observe that φ_1 and u_2 exhibit logarithmic singularities for $r = 0$.

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