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# Equations of steady flow through slightly curved multifilament bundles

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A STEADY filtration flow through a multifilament bundle of slightly curved, almost parallel fibres is considered. The small curvature of fibre lines is due to their high inner longitudinal tension. The presented equations allow to find pressure and velocity fields and forces acting on fibres. High, almost constant inner tension, as depending on reological properties of fibres, is not evaluated here. Fibre lines, assumed in first step as straight lines, may be consecutively determined in the next step by means of corresponding differential equation. Interesting case of flow through multifilament bundle, on which particular attention is paid, is the transversal flow. In transversal flow all longitudinal components of gradients being very small, the filtration relative to the bundle occurs across fibre lines. The wringing of plane or axisymmetrical convergent bundles is presented as example. The equations describing flow through straight parallel fibres are also presented as particular case.

Rozpatrywany jest ustalony przepływ filtracyjny przez wiązkę przędzonych włókien, których słabo zakrzywione linie są prawie równoległe. Warunkiem słabego zakrzywienia włókien jest wystarczająco silne ich napięcie rozciągającymi siłami wewnętrznymi. Dla danej geometrii i kinematyki wiązki wyprowadzone równania pozwalają kolejno wyznaczyć rozkład ciśnień, pole prędkości i pole sił działających na włókna. Silne, prawie stałe napięcie wewnętrzne, jako zależne od własności reologicznych włókien, nie jest określane w przedstawionym schemacie obliczeń. Linie włókien, przyjęte w pierwszym kroku jako proste, mogą być wyznaczane w kolejnym przybliżeniu przez zastosowanie określającego je równania różniczkowego. Ważnym przypadkiem przepływu wewnątrz wielowłóknowych wiązek jest przepływ poprzeczny, na który zwrócono szczególną uwagę. W przepływie tym składowe wzdłużne wszystkich gradientów stają się bardzo małe i przepływ filtracyjny względem wiązki odbywa się w kierunku poprzecznym do włókien. Jako przykład rozpatrzono tu wyżymanie płynu z płaskiej i z osiowosymetrycznej, zbieżnej wiązki prostoliniowych włókien. Przedstawiono również równania opisujące przepływ wewnątrz wiązki włókien prostoliniowych i równoległych.

Рассматривается установившееся фильтрационное течение через пучки пряденных волокон, которых слабо искривленные линии почти параллельны. Условием слабого искривления волокон является достаточно сильное их напряжение растягивающими внутренними силами. Для заданной геометрии и кинематики пучка выведены уравнения, позволяющие последовательно определить распределение давлений, поле скоростей и поле сил действующих на волокна. Сильное, почти постоянное внутреннее напряжение, как зависящее от реологических свойств волокон, не определяется в представленной схеме расчетов. Линии волокон, принятые в первом шагу как прямые, могут быть определены в последовательном приближении путем применения определяющего их дифференциального уравнения. Важным случаем течения внутри многоволокновых пучков является поперечное течение, на которое обращено особенное внимание. В этом течении продольные составляющие всех градиентов становятся очень малыми и фильтрационное течение по отношению к пучку происходит в поперечном направлении к волокнам. Как пример рассмотрено здесь выжимание жидкости из плоского и из осесимметричного сходящегося пучка прямолинейных волокон. Представлены тоже уравнения описывающие течение внутри пучка прямолинейных и параллельных волокон.

## 1. Introduction

CHEMICAL fibres, spun in liquid or gaseous environment, undergo hydrodynamic effects which may influence spinning processes. Very often a large number of almost parallel, slightly curved, flexible and extensible fibres is spun together in a multifilament bundle, the interior of which may be considered as an anisotropic porous deformable medium. The filtration of the surrounding fluid through such a porous medium is the object of our interest.

The equations of steady flows of a viscous fluid through arbitrarily curved multifilament bundles were derived previously [1]. However, their application requires tedious calculations of terms containing curvilinear coordinates of tensor quantities. In our practically interesting case of slightly curved bundles, these calculations may be simplified and we will present them below.

Taking into account all assumptions defining the considered model of multifilament bundles [1], we will assume additionally that the deviations of fibre lines from parallel straight lines are very small. Consequently, introducing a small parameter $\varepsilon \ll 1$, we will assume that the longitudinal derivatives of functions characterizing the geometry of fibres lines are small and of higher order with respect to $\varepsilon$. Only the lowest order approximation shall be taken into account.

## 2. General equations

Flows of a viscous incompressible fluid through a multifilament bundle of incompressible but flexible and extensible fibres separated one from another by the surrounding fluid were considered previously [1] and now we will present briefly the obtained general equations. The equations are presented in two coordinate systems: Cartesian $x^1$, $x^2$, $x^3$, or curvilinear $s^0, s^1, s^2$, (Fig. 1), related one to another by the transformation

$$x^i = x^i(s^0, s^1, s^2), \quad i = 1, 2, 3, \quad x^3 \equiv x^0, \tag{2.1}$$

which allows to find its Jacobian $J = \det(\partial x^i/\partial s^k)$ and the coordinates of the metric tensor $g_{ik}$ or $g^{ik}$ in both reference frames. Equations (2.1) describe simultaneously the fibre lines: $s^0$ being the fibre length and $\partial x^i/\partial s^0$ the unit vector tangent to a fibre.

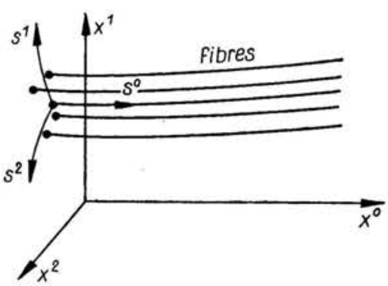


FIG. 1.

To determine the geometry, the kinematics and the permeability of the interior of a bundle, further information is needed. In steady motion the velocity $v^i = v(\partial x^i/\partial s^0)$ of fibres should be tangent to them, so only one scalar quantity characterizes here the velocity field. The porosity $(1-\varphi)$ is characterized by the ratio $\varphi$ of fibre volume to bundle volume. The porosity, or the quantity $\varphi$, is the main parameter determining the nondimensional filtration tensor $F^{il}$, with the main axes directed along and across fibre lines. The inner structure of a bundle is characterized not only by $\varphi$ but also by the area $S$ of the transversal section of the bundle, per one fibre. We will also introduce the tensor

(2.2) $$H_{kl} = g_{kl} + \frac{\varphi}{1-\varphi} g_{km} g_{ln} \frac{\partial x^m}{\partial s^0} \frac{\partial x^n}{\partial s^0}$$

characterizing the geometry of the bundle.

The laws of conservation of the number and volume of fibres

(2.3) $$J/S = F(s^1, s^2), \quad \varphi S v = Q_f(s^1, s^2),$$

give the restrictions imposed on $S$ and $\varphi v$ along fibre lines.

The surrounding fluid with the density $\varrho$ and the viscosity coefficient $\mu$ flows through the bundle with an absolute filtration velocity $u^i$ in the fixed reference frame. The relative filtration velocity $w^i$ in the reference frame moving with fibres is determined by the relation

(2.4) $$w^i = u^i - (1-\varphi)v^i = u^i - (1-\varphi)v(\partial x^i/\partial s^0).$$

In the filtration theory a fundamental role is played by Darcy's law, stating the linear relation between the relative filtration velocity $w^i$ and the gradient of piezometric pressure $\nabla_k(p-\mathring{p})$ where $p$ is the pressure and $\mathring{p}$ is the static distribution of pressure due to the potential body force $f^k$. In our case of a deformable porous medium composed of extensible and flexible fibres not contacting one with another, the linear relation between the relative filtration velocity $w^i$ and the resistance force acting on the medium was found [1] in a more general form:

(2.5) $$w^i = -\frac{S}{\mu} F^{il}\left[\nabla_l(p-\mathring{p}) - (p-\mathring{p}) H_{kl} \frac{\partial}{S\partial s^0}\left(\varphi S \frac{\partial x^k}{\partial s^0}\right)\right], \quad \nabla^k \mathring{p} = \varrho f^k,$$

with an additional term taking into account the deformation of fibres. For parallel, straight, cylindrical fibres this term disappears and the inner filtration is described by a "classical" Darcy's law.

Also in our considerations the filtration law (2.5) plays a fundamental role and introducing it into the continuity equation for the fluid $\nabla_i u^i = 0$, we obtain the main equation

(2.6) $$\nabla_i\left\{\frac{F^{il}}{\mu}\left[S\nabla_l(p-\mathring{p}) - (p-\mathring{p}) H_{kl} \frac{\partial}{\partial s^0}\left(\varphi S \frac{\partial x^k}{\partial s^0}\right)\right]\right\} = \frac{\partial(vS)}{S\partial s^0},$$

which allows to find the pressure field for given characteristics of the bundle. Using then the momentum equations for the fluid and for the fibres

(2.7) $$q^k = -\varrho\varphi S f^k - S\nabla^k(p-\mathring{p}) + \frac{\partial}{\partial s^0}\left(pS\varphi \frac{\partial x^k}{\partial s^0}\right),$$

(2.8) $$\frac{\partial}{\partial s^0}\left(T \frac{\partial x^k}{\partial s^0}\right) = -\varrho_f \varphi S f^k - q^k,$$

we may consecutively determine the force $q^k$ acting on a unity length of the fibre and the inner tension $T$ ($\varrho_f$ is the density of fibre material). It should be emphasized however that the geometry of fibre lines is not arbitrary here, because Eqs. (2.8) should be fulfilled by $x^i(s^0, s^1, s^2)$ (2.1).

The above equations are presented in an invariant form but, to apply them, we must choose a reference frame not always convenient to perform calculations. In the case of slightly curved fibres it is easier to transform these equations to a form convenient for analysis. Also in practical applications this case is the most important.

## 3. Filtration flow through slightly curved fibre lines

Let us choose for our particular case a Cartesian system of coordinates $x^3 \equiv x^0, x^1, x^2$, with the $x^0$-axis almost parallel to all fibres (Fig. 1). In this reference frame the fibre lines equations (2.1) may be presented in the form(1)

$$(3.1)\qquad \begin{aligned} x^0 &= s^0 - \varepsilon\xi(\sigma, s^1, s^2), \quad \sigma = \varepsilon s^0, \quad \varepsilon \ll 1,\\ x^\alpha &= \xi^\alpha(\sigma, s^1, s^2), \quad \alpha = 1, 2, \end{aligned}$$

where $s^0, s^1, s^2$ may be considered as a curvilinear system of coordinates with the transformation matrix

$$(3.2)\qquad \frac{\partial x^i}{\partial s^k} = \left\| \begin{matrix} 1-\varepsilon^2\dfrac{\partial\xi}{\partial\sigma}, & -\varepsilon\dfrac{\partial\xi}{\partial s^1}, & -\varepsilon\dfrac{\partial\xi}{\partial s^2} \\ \varepsilon\dfrac{\partial\xi^1}{\partial\sigma}, & \dfrac{\partial\xi^1}{\partial s^1}, & \dfrac{\partial\xi^1}{\partial s^2} \\ \varepsilon\dfrac{\partial\xi^2}{\partial\sigma}, & \dfrac{\partial\xi^2}{\partial s^1}, & \dfrac{\partial\xi^2}{\partial s^2} \end{matrix} \right\|.$$

In our approximation the coordinate $s^0$ is equal to the fibre length and we will require that it should also be quasi-orthogonal to other coordinate lines $s^\alpha$ ($\alpha = 1, 2$), according to

$$g_{ik}\frac{\partial x^i}{\partial s^0}\frac{\partial x^k}{\partial s^n} = \begin{cases} 1+0(\varepsilon^4), & n = 0,\\ 0(\varepsilon^3), & n = 1, 2. \end{cases}$$

Taking into account these conditions, we obtain

$$(3.3)\qquad \begin{aligned} 2\frac{\partial\xi}{\partial\sigma} &= \left(\frac{\partial\xi^1}{\partial\sigma}\right)^2 + \left(\frac{\partial\xi^2}{\partial\sigma}\right)^2,\\ \frac{\partial\xi}{\partial s^\alpha} &= \frac{\partial\xi^1}{\partial\sigma}\frac{\partial\xi^1}{\partial s^\alpha} + \frac{\partial\xi^2}{\partial\sigma}\frac{\partial\xi^2}{\partial s^\alpha}, \quad \alpha = 1, 2. \end{aligned}$$

The Jacobian $J$ of the three-dimensional transformation of variables (2.1) here becomes almost equal to the Jacobian $j$ of the two-dimensional transformation $(3.1)_2$

$$(3.4)\qquad J = j + 0(\varepsilon^2), \quad j = \det\left(\frac{\partial x^\alpha}{\partial s^\beta}\right), \quad \alpha, \beta = 1, 2.$$

(1) By means of Latin letter indices we will denote coordinates in three-dimentional space $x^i$ ($i = 0, 1, 2$). Also another notation, a Greek letter shall be simultaneously used for two-dimensional space $x^\alpha$ or $s^\alpha$ ($\alpha = 1, 2$), the coordinates in the $x^0$—direction being considered as scalars.

The geometrical interpretation of the parameter $\varepsilon$ is very simple. If we introduce the curvature of fibre line

$$\frac{1}{R} = \varepsilon^2 \sqrt{\left[\frac{\partial^2 \xi^1}{(\partial\sigma)^2}\right]^2 + \left[\frac{\partial^2 \xi^2}{(\partial\sigma)^2}\right]^2},$$

we find that the ratio of the fibre length $s^0 = \sigma/\varepsilon$ to the curvature radius $R$ is of the order $\varepsilon$: $s^0/R = 0(\varepsilon)$. As only the lowest order approximation is the object of our interest, the higher order terms with respect to $\varepsilon$ shall be omitted in further considerations.

The Cartesian coordinates of the metric tensor obviously form a diagonal unit matrix. In the curvilinear system $s^0, s^1, s^2$, we obtain for them in our approximation

(3.5)
$$g_{ik} = \left\| \begin{matrix} 1, & 0 & 0 \\ 0 & \multicolumn{2}{c}{\|g_{\alpha\beta}\|} \\ 0 & & \end{matrix} \right\|, \quad g^{ik} = \left\| \begin{matrix} 1, & 0 & 0 \\ 0 & \multicolumn{2}{c}{\|g^{\alpha\beta}\|} \\ 0 & & \end{matrix} \right\|,$$
$$g_{\alpha\beta} = \frac{\partial\xi^1}{\partial s^\alpha}\frac{\partial\xi^1}{\partial s^\beta} + \frac{\partial\xi^2}{\partial s^\alpha}\frac{\partial\xi^2}{\partial s^\beta}, \qquad g^{\alpha\beta} = \frac{1}{g}\left\| \begin{matrix} g_{22}, & -g_{12} \\ -g_{21}, & g_{11} \end{matrix} \right\|.$$
$$g = j^2.$$

For the unit vector $\partial x^i/\partial s^0$ tangent to fibres and for the curvature vector $\partial^2 x^i/(\partial s^0)^2$ we find in the $x^0, x^1, x^2$ frame:

(3.6x)
$$\frac{\partial x^i}{\partial s^0} = \left\| 1, \ \varepsilon\frac{\partial\xi^1}{\partial\sigma}, \ \varepsilon\frac{\partial\xi^2}{\partial\sigma} \right\|,$$
$$\frac{\partial^2 x}{(\partial s^0)^2} = \left\| -\varepsilon^3\frac{\partial^2\xi}{(\partial\sigma)^2}, \ \varepsilon^2\frac{\partial^2\xi^1}{(\partial\sigma)^2}, \ \varepsilon^2\frac{\partial^2\xi^2}{(\partial\sigma)^2} \right\|$$

and in the $s^0, s^1, s^2$ frame:

$$\frac{\partial x^i}{\partial s^0} = \| 1, \ 0, \ 0 \|,$$

(3.6s)
$$\frac{\partial^2 x^i}{(\partial s^0)^2} = \frac{\varepsilon^2}{j}\left\| 0, \ \left| \begin{matrix} \frac{\partial^2\xi^1}{(\partial\sigma)^2}, & \frac{\partial^2\xi^2}{(\partial\sigma)^2} \\ \frac{\partial\xi^1}{\partial s^2}, & \frac{\partial\xi^2}{\partial s^2} \end{matrix} \right|, \ -\left| \begin{matrix} \frac{\partial^2\xi^1}{(\partial\sigma)^2}, & \frac{\partial^2\xi^2}{(\partial\sigma)^2} \\ \frac{\partial\xi^1}{\partial s^1}, & \frac{\partial\xi^2}{\partial s^1} \end{matrix} \right| \right\|.$$

To describe tensor quantities we may use either the Cartesian $x^0, x^1, x^2$ or the curvilinear $s^0, s^1, s^2$ coordinate system. As the local anisotropy of the bundle is determined by fibre lines geometry, we will use prevailably the curvilinear reference frame $s^0, s^1, s^2$.

In this curvilinear reference frame we find

(3.7)
$$H_{kl} = \left\| \begin{matrix} 1/(1-\varphi), & 0, & 0 \\ 0 & \multicolumn{2}{c}{\| g_{\alpha\beta} \|} \\ 0 & & \end{matrix} \right\|$$

and the filtration tensor $F^{il}$ may be presented in the form

(3.8)
$$F^{il} = \left\| \begin{matrix} F_{\|}, & 0, & 0 \\ 0 & \multicolumn{2}{c}{\| F_{\perp} g^{\alpha\beta} \|} \\ 0 & & \end{matrix} \right\|,$$

where the filtration coefficients along and across fibres are denoted by $F_{\|}$ and $F_{\perp}$.

Applying the known rules of tensor calculus, as for instance

$$\nabla_i\left[F^{il}\frac{S}{\mu}\nabla_l(p-\mathring{p})\right] = \frac{1}{\sqrt{g}}\frac{\partial}{\partial s^i}\left[\sqrt{g}\,\frac{S}{\mu}F^{il}\frac{\partial(p-\mathring{p})}{\partial s^l}\right]$$

$$= \frac{1}{j}\frac{\partial}{\partial s^0}\left[j\frac{S}{\mu}F_{\parallel}\frac{\partial(p-\mathring{p})}{\partial s^0}\right]+\frac{1}{\sqrt{g}}\frac{\partial}{\partial s^\alpha}\left[\sqrt{g}\,\frac{S}{\mu}F_{\perp}g^{\alpha\beta}\frac{\partial(p-\mathring{p})}{\partial s^\beta}\right]$$

$$= \frac{\partial}{\partial s^0}\left[\frac{SF_{\parallel}}{\mu}\frac{\partial(p-\mathring{p})}{\partial s^0}\right]+\nabla_\alpha\left[\frac{SF_{\perp}}{\mu}\nabla^\alpha(p-\mathring{p})\right]+\varepsilon\frac{SF_{\parallel}}{\mu}\frac{\partial j}{j\partial\sigma}\frac{\partial(p-\mathring{p})}{\partial s^0},$$

we may transform Eq. (2.6) to the form

$$\frac{\partial}{\partial s^0}\left\{\frac{SF_{\parallel}}{\mu}\frac{\partial[(1-\varphi)(p-\mathring{p})]}{(1-\varphi)\partial s^0}\right\}+\nabla_\alpha\left[\frac{SF_{\perp}}{\mu}\nabla^\alpha(p-\mathring{p})\right] = \frac{\partial(Sv)}{S\partial s^0}. \tag{3.9}$$

In an analogous way, from Eq. (2.5), we obtain

$$\begin{aligned} u^0-(1-\varphi)v &= w_0 = -\frac{SF_{\parallel}}{\mu}\frac{\partial[(1-\varphi)(p-\mathring{p})]}{(1-\varphi)\partial s^0},\\ u^\alpha &= w^\alpha = -\frac{SF_{\perp}}{\mu}\nabla^\alpha(p-\mathring{p}), \quad \alpha = 1,2. \end{aligned} \tag{3.10}$$

For given geometry, kinematics and permeability of a bundle, we may determine the inner filtration flow in the slightly curved fibres approximation by means of Eq. (3.9) and (3.10). In this approximation all derivatives $\partial/\partial s^0$ of geometrical quantities $\xi$, $\xi^\alpha$, $j$ have been omitted. Other derivatives, as not obligatorily small, are taken into account.

To find the force $q^k$, with which the fluid acts on a unit length of fibres, the momentum equation (2.7) should be used. Neglecting higher order terms, we find its tangent $q^0$ and normal $q^\alpha$ ($\alpha = 1, 2$) components

$$\begin{aligned} q^0 &= S\left\{-\frac{\partial[(1-\varphi)(p-\mathring{p})]}{\partial s^0}+\mathring{p}\frac{\partial\varphi}{\partial s^0}\right\},\\ q^\alpha &= S[-\nabla^\alpha(p-\mathring{p})-\varrho\varphi f^\alpha], \quad \alpha = 1,2. \end{aligned} \tag{3.11}$$

Taking from Eq. (2.8) only the $s^0$-component and eliminating $q^0$, we find for the inner tension in fibres $T$ the following equation:

$$\frac{\partial T}{\partial s^0} = S\left\{\frac{\partial[(1-\varphi)p]}{\partial s^0}-\varrho_f\varphi f^0\right\}. \tag{3.12}$$

The transversal components of Eq. (2.8) give a differential equation of fibre lines

$$T\frac{\partial^2 x^\alpha}{(\partial s^0)^2} = (\varrho-\varrho_f)\varphi S f^\alpha+S\nabla^\alpha(p-\mathring{p}), \quad \alpha = 1,2. \tag{3.13}$$

Since $\partial^2x^\alpha/(\partial s^0)^2 = \varepsilon^2\partial^2\xi^\alpha/(\partial\sigma)^2$ is very small and the terms on the right hand side do not vanish with $\varepsilon\to 0$, the inner tension $T$ should be sufficiently large $T = 0(\varepsilon^{-2})$ so as to maintain the fibres in a slightly curved form. On the other side, from Eq. (3.12), we obtain $\partial T/\partial s^0 = 0(1)$ and, consequently, the large tension $T$ should be almont constant $T = 0\ (\varepsilon^{-2}) = \text{const}\ [1+0(\varepsilon^2)]$. The small curvature $\partial^2x^\alpha/(\partial s^0)^2 = 0(\varepsilon^2)$ of extended fibres is due, according to Eq. (3.13), either to buoyancy force $(\varrho-\varrho_f)f^\alpha\varphi S\Delta s^0$ or to the transversal component of the drag force $S\Delta s^0\nabla^\alpha(p-\mathring{p}) = -\mu w^\alpha/F_\perp$ acting on an element of the fibre length $\Delta s^0$. However, both forces should be small in comparison with the tension $T$. To fulfil this condition for a bundle of the length and thickness $L$ and $b$ undergoing the pressure difference $\Delta p$, two criterial numbers

$$B = (\varrho-\varrho_f)|f^\alpha|\varphi S\frac{L}{T}, \quad P = \frac{SL\Delta p}{bT}, \tag{3.14}$$

should be sufficiently small, at least of the order $\varepsilon$. These numbers characterize the influence of buoyancy force and pressure difference on the transversal deformation of fibre lines.

It should be stressed that the assumed geometrical form of fibre lines is not arbitrary, because Eq. $(3.1)_2$ $x^\alpha = \xi^\alpha(\varepsilon s^0, s^1, s^2)$ should fulfil the differential equation (3.13). However, for sufficiently small $B$ and $P$ the deflection of fibre lines would not be very large and the shape of straight line may be generally taken as the first approximation. Equation (3.13) could then be used in an iterative scheme to correct the previous approximation of fibre lines (Eqs. (3.1)).

Till now there has been assumed nothing about the coefficient of viscosity $\mu$ which, in further considerations, shall always be taken as constant.

## 4. Transversal filtration flow

### 4.1. Simplified equations

Till now, by introducing $\varepsilon \ll 1$, we assumed only that the considered bundles were composed of slightly curved, almost parallel fibres. This assumption concerned the geometrical shape of fibre lines only; no assumption about the rate of stretching or about the pressure field have been introduced. As a result, all longitudinal derivatives of geometrical quantities $\partial/\partial s^0 = \varepsilon\partial/\partial\sigma$ were assumed to be small, but nothing was stated about the order of $\partial\varphi/\partial s^0$ and $\partial(p-\mathring{p})/\partial s^0$.

During spinning processes two cases are possible. In some particular regions, as in the vicinity of the spinerette, the longitudinal derivatives of pressure and, often, also the stretching rate of fibres are very intensive and should not be disregarded. However, other regions exist, as well regions where these longitudinal derivatives $\partial/\partial s^0$ are very small and where further simplifications of equations are admissible. In these regions, according to Eq. (3.10) $w^0 = 0$, the relative filtration flow should be almost perpendicular to the fibre lines. This particular case of transversal filtration through a bundle shall be the object of our interest now.

For this case, from Eqs. (3.9) to (3.13), neglecting higher derivatives in respect of $s^0$, we find

$$\nabla_\alpha\left[\frac{SF_\perp}{\mu}\nabla^\alpha(p-\mathring{p})\right] = \frac{\partial(Sv)}{S\partial s^0}, \tag{4.1}$$

$$u^0 = (1-\varphi)v, \quad u^\alpha = -\frac{SF_\perp}{\mu}\nabla^\alpha(p-\mathring{p}), \quad \alpha = 1, 2, \tag{4.2}$$

$$q^0 = 0, \quad q^\alpha = -S[\nabla^\alpha(p-\mathring{p})+\varrho\varphi f^\alpha], \tag{4.3}$$

$$\varepsilon^2 T = \text{const} = 0(1), \quad \frac{\partial^2 \xi^\alpha}{(\partial\sigma)^2} = \frac{S}{\varepsilon^2 T}[(\varrho-\varrho_f)\varphi f^\alpha + \nabla^\alpha(p-\mathring{p})]. \tag{4.4}$$

These equations describe the transversal filtration flow in slightly curved and slightly stretched, almost parallel fibres.

#### 4.2. Wringing of convergent bundles

As examples, let us consider plane (Fig. 2) or axisymmetrical (Fig. 3) bundles of convergent fibres moving at the constant speed $v$ through the surrounding fluid, which remains in static equilibrium. The tension $T$ in fibres shall be assumed to be so high that $B = P = 0$; straight line approximation of fibre lines is admissible. The fibres are distributed uniformly across the bundle within the angle $2\varepsilon \ll 1$. The region surrounding singularity in the intersection of fibre lines, as physically unjustified, will be excluded from our considerations.

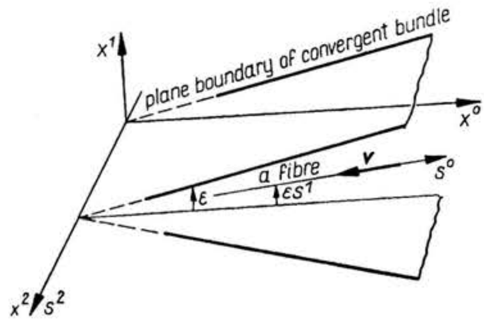


FIG. 2.

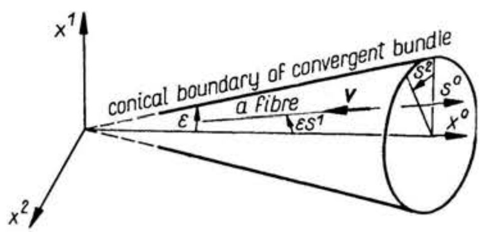


FIG. 3.

Let us introduce the Cartesian coordinates $x^0, x^1, x^2$ and, for plane (Fig. 2) or axisymmetrical (Fig. 3) cases, cylindrical or spherical reference frames $s^0, s^1, s^2$ respectively, where $\varepsilon s^1$ is the angle of convergence and $-1 \leqslant s^1 \leqslant 1$. In these reference frames the functions $\xi(\sigma, s^1, s^2)$ $(3.1)_1$ and $\xi^\alpha(\sigma, s^1, s^2)$ $(3.1)_2$ may be presented in the approximate form

$$\xi = \frac{\sigma}{2}(s^1)^2, \quad \left\{\begin{array}{ll} \xi^1 = \sigma s^1, & \xi^2 = s^2 \\ \xi^1 = \sigma s^1 \cos s^2, & \xi^2 = \sigma s^1 \sin s^2 \end{array}\right\} \text{ for } \left\{\begin{array}{l}\text{plane}\\ \text{or}\\ \text{axi-sym.}\end{array}\right\} \text{ case.} \tag{4.5}$$

Afterwards, from Eq. (3.5) we find

$$g_{12} = g_{21} = 0, \quad g_{11} = \frac{1}{g^{11}} = (\sigma)^2, \quad g_{22} = \frac{1}{g^{22}} = (\sigma s^1)^{2k}, \tag{4.6}$$
$$\sqrt{g} = j = \sigma(\sigma s^1)^k,$$

and from Eq. (2.3) for $v$ = const we obtain

$$S = \frac{1}{N}\left(\frac{\sigma}{\varepsilon}\right)^{k+1} = \frac{(s^0)^{k+1}}{N}, \quad \varphi = \frac{Q_f N}{v}\frac{1}{(s^0)^{k+1}}, \tag{4.7}$$

where

$$k = \begin{cases} 0 \\ 1 \end{cases} \quad \text{for} \quad \begin{Bmatrix} \text{plane} \\ \text{or} \\ \text{axi-sym.} \end{Bmatrix} \text{case,} \tag{4.8}$$

and $N$ is the number of fibres per unit angle and per unit length in the plane case or per unit spherical angle in the axisymmetric case.

Now, Eq. (4.1) and (4.2) $\partial/\partial s^2 = 0$ for the two-dimensional flow may be reduced to the form

$$\begin{gathered} \frac{\partial}{\partial s^1}\left[(s^1)^k \frac{\partial(p-\mathring{p})}{\partial s^1}\right] = (k+1)\frac{\mu N v}{F_\perp(\varphi)}\left(\frac{s^1}{s^0}\right)^k \frac{1}{(s^0)^2}, \\ u^0 = (1-\varphi)v, \quad u^1 = -\frac{F_\perp(\varphi)}{\mu N}(s^0)^{1+k}\frac{\partial(p-\mathring{p})}{\partial s^1}. \end{gathered} \tag{4.9}$$

In static equilibrium, on the boundary $s^1 = \pm 1$ the pressure is equal to the piezometric pressure $p(s^0, \pm 1, s^2) = \mathring{p}(s^0, \pm 1, s^2)$ and the solution of Eq. (4.9) in the reference frame $s^0, s^1, s^2$, is

$$\begin{gathered} p-\mathring{p} = -\frac{\mu N v}{2F_\perp(\varphi)}\frac{1-(s^1)^2}{(s^0)^{2+k}}, \\ u^0 = (1-\varphi)v, \quad u^1 = -v\frac{s^1}{s^0}, \quad u^2 = 0. \end{gathered} \tag{4.10}$$

The physical components of filtration velocity are

$$u^0 = (1-\varphi)v, \quad u_\perp = \frac{u^1}{\sqrt{g^{11}}} = -v\frac{\xi^1}{x^0}. \tag{4.11}$$

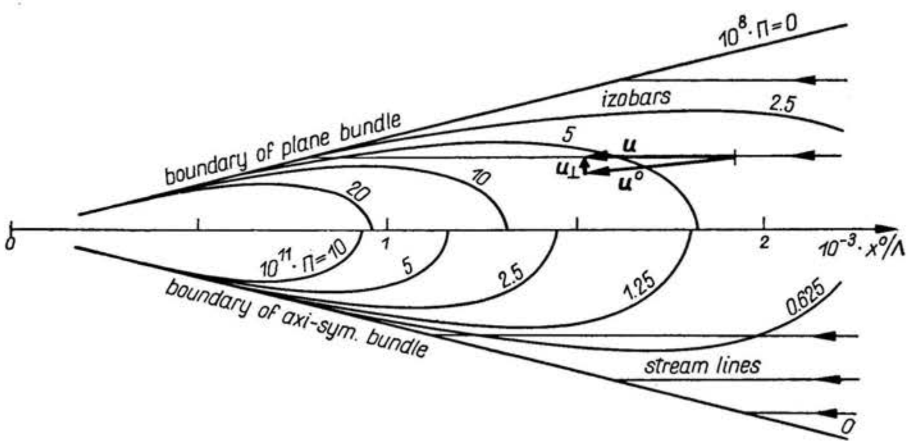


FIG. 4.

To give an illustration of the pressure distribution, we will now restrict ourselves to very rare bundles $\varphi \ll 1$, for which [2, 3]:

$$F_{\perp}(\varphi) \approx \frac{1}{8\pi}\left(\ln\frac{1}{\varphi} - 1.5\right).$$

For this case, (Fig. 4), a uniform velocity field with parallel straight stream lines is obtained and the pressure distribution is given by the formula

$$(4.12) \qquad \Pi = -\frac{\Lambda^{2+k}(p-\mathring{p})}{4\pi\mu N v} = \frac{1-(s^1)^2}{\left(\frac{s^0}{\Lambda}\right)^{2+k}\left[\ln\left(\frac{s^0}{\Lambda}\right)^{1+k} - 1.5\right]}, \quad \Lambda = \left(\frac{Q_f N}{v}\right)^{\frac{1}{1+k}}.$$

Now we may see (Fig. 4) that in the considered case the values of longitudinal components of the pressure gradient are smaller than those of transversal ones and that the approximation of transversal flow may here be satisfactorily applied.

## 5. Uniform bundles composed of straight parallel fibres

### 5.1. Simplified equations

For the limiting case of infinitely large inner tension, the fibre lines become straight lines. Now, we will consider particular cases of bundles composed of straight, parallel fibres distributed uniformly in space. The fibres may eventually be stretching along their axes in dependence on one variable $s^0$ only.

For this case $\varepsilon = 0$, the coordinates $x^0 \equiv s^0$ coincide and the transformation $x^{\infty} \equiv \xi^{\alpha}(s^1, s^2)$, $\alpha = 1, 2$, may be eventually used to introduce the more convenient for calculations cylindrical coordinates $s^0, s^1, s^2$. The area $S$ per one fibre and the flow rate $Q_f$ of fibres should be constant and $\varphi(x^0)$, $v(x^0) = Q_f/[S\varphi(x^0)]$ should depend on $x^0 = s^0$ only.

On the basis of the additional assumption introduced here, Eq. (3.9) may be reduced to the form

$$(5.1) \qquad \frac{\partial}{\partial x^0}\frac{\partial[(1-\varphi)(p-\mathring{p})]}{(1-\varphi)\partial x^0} + \frac{F_{\perp}}{F_{\parallel}}\nabla_{\alpha}\nabla^{\alpha}(p-\mathring{p}) + \frac{\partial F_{\parallel}}{F_{\parallel}\partial x^0}\frac{\partial[(1-\varphi)(p-\mathring{p})]}{(1-\varphi)\partial x^0} = -\frac{\mu Q_f \varphi'}{F_{\parallel}(S\varphi)^2}.$$

The other equations (3.10) to (3.12) remain the same and we will not write them here.

For very rare bundles, $\varphi \ll 1$, the filtration coefficients $F_{\parallel}, F_{\perp}$ may be presented in the form [3]

$$(5.2) \qquad F_{\parallel} = 2F_{\perp} = \frac{1}{4\pi}\left(\ln\frac{1}{\varphi} - 1.5\right),$$

for which Eq. (5.1) may still be reduced to the form

$$(5.3) \qquad \frac{\partial^2(p-\mathring{p})}{(\partial x^0)^2} + \frac{1}{2}\nabla_{\alpha}\nabla^{\alpha}(p-\mathring{p}) = \frac{\varphi'(x^0)}{\varphi(x^0)\left[\ln\frac{1}{\varphi(x^0)} - 1.5\right]}\left[\frac{\partial(p-\mathring{p})}{\partial x^0} - \frac{4\pi\mu Q_f}{S^2\varphi(x^0)}\right].$$

### 5.1. Homogeneous bundles

In the case of homogeneous bundles, $\varphi$ = const the right hand side in Eq. (5.3) disappears and by corresponding change of coordinates, as for instance $x^0 = \sqrt{2}x$, we obtain for pressure the Laplace equation

$$\Delta(p-\mathring{p}) = 0. \tag{5.4}$$

The filtration flows through homogeneous bundles in plane and axisymmetric cases in the vicinity of the spinerette were considered previously [3] to [6] and some of the obtained results were compared with experiments [7, 8]. For bundles with asymmetric section the solution of the Laplace equation (5.4) would require numerical methods.

## 6. Final remarks

The equations presented above with boundary conditions additionally added, allow us to determine the steady filtration flow through bundles of slightly curved fibres moving along their axes with the velocity $v(s^0, s^1, s^2)$. The fibre lines are defined by a small parameter $\varepsilon$ and by the functions $\xi(\sigma, s^1, s^2)$, $\xi^\alpha(\sigma, s^1, s^2)$. The inner structure of the bundle is determined by giving the area per one fibre $S(0, s^1, s^2)$ and the porosity $1-\varphi(0, s^1, s^2)$ in one section $s^0 = 0$, other sections $s^0 \neq 0$ being obtained from Eq. (2.3). The permeability of the bundle is determined by two coefficients $F_{\parallel}(\varphi)$ and $F_{\perp}(\varphi)$ along and across fibres, respectively. To describe anizotropic properties of boundles, a curvilinear system of coordinates $s^0, s^1, s^2$ is introduced, in which all terms should be expressed. Using the curvilinear reference frame connected with deformable fibres, we might expect that some difficulties would appear in such problems, in which the geometry of fibres depends strongly on acting forces. In our cases of slightly curved fibres such interaction between fibres and the surrounding fluid should be rather weak.

After solving the pressure equation (3.9), we may find from Eqs. (3.10) the filtration velocity field $u^0, u^1, u^2$, and the forces $q^0, q^1, q^2$, acting on a unit length of a fibre. It should not be forgotten that the obtained contravariant vector coordinates in the curvilinear reference frame are not equal to physical coordinates which may be easily found afterwards according to known rules of tensor calculus.

The inner tension $T$ in fibres is not determined by the presented model, supplementary information about the rheological properties of fibres would be needed here. The quasi-constant large quantity $T$ exerts a great influence on the geometry of fibre lines described by the differential equation (3.13). This equation states that the functions $\xi(\sigma, s^1, s^2)$ (3.1), $\xi^\alpha(\sigma, s^1, s^2)$ $(3.1)_2$, defining the geometry of fibre lines, may not be chosen arbitrarily. They should fulfil the conditions of orthonormalization, Eq. (3.3), and, additionally, the differential equation (3.13). However, for $B \ll 1$, $P \ll 1$, a straight line shape may be assumed as a good approximation.

The assumption of slightly curved fibres is strongly related to the conditions

$$B \ll 1, \quad P \ll 1, \tag{6.1}$$

imposed on the criterial numbers $B$ and $P$, Eq. (3.14). To fulfil these conditions and to apply the equations presented above, the inner tension $T$ should be sufficiently large in comparison to other forces acting on fibres. For infinitely high tension $T$, we obtain the limiting case $B = P = 0$, for which the approximation of straight line fibres should be exact. This limiting case may be explored further as the first step in an iterative scheme of determining the geometrical shape of fibre lines.

Very often, when all derivatives in respect of $s^0$ are negligibly small, it is not necessary to use the equations of filtration flow in their general form Eqs. (3.9), (3.10), because their simplified form of transversal flow, Eqs. (4.1), (4.2) may here give satisfactory approximation and is much easier to solve. The transversal flow approximation reduces a three-dimensional problem with three independent variables $s^0, s^1, s^2$ to a solution of the differential equation (4.1) with two independent variables $s^1, s^2$, and a given function of $s^0$. Many practically interesting cases may be solved using the transversal flow approximation, but in some particular cases, such as the vicinity of the spinerette, the longitudinal derivatives of pressure should be taken into account.

The obtained equations, either in general or in simplified form for transversal flow, with properly chosen boundary conditions, allow us to determine approximately many practically interesting cases of filtration flows in deformable bundles of spun fibres. It seems that the presented approximation should describe satisfactorily such cases for which the inner tension is sufficiently high. Knowledge of flow phenomena in multifilament bundles may allow to study more thoroughly the physico-chemical processes which accompany spinning technology.

## References

1. A. SZANIAWSKI, *Equations of steady flow through multifilament bundles*, Bull. Acad. Polon. Sci., Série Sci. techn., **24**, 11, 1976.
2. L. SPIELMAN, S. L. GOREN, *Model for pressure drop and filtration efficiency in fibrous media*, Environmental Science and Technology, 2, 1968.
3. A. SZANIAWSKI, A. ZACHARA, *Penetracja kąpieli do strefy formowania wiązki włókien sztucznych*, Polimery, **19**, 3–4, 1974.
4. A. SZANIAWSKI, A. ZACHARA, *Wpływ różnicy ciśnień po obu stronach płaskiej wiązki przędzonych włókien chemicznych na przepływ w jej wnętrzu*, Polimery, **20**, 2, 1975.
5. A. SZANIAWSKI, *O przepływach wewnątrz wiązek włókien*, Polimery, **20**, 7, 1975.
6. A. SZANIAWSKI, A. ZACHARA, *Hydrodynamic problems of multifilament spinning*, Chemicke Vlakna, **26**, č. 1–2, 1976.
7. V. A. MIEDVIEDIEV, V. G. BOVA, A. T. SERKOV, *Raspredelenije davlenia v puchkie formirujushchikhsia volokon v zonie filiery*, Chim. Volokna, 6, 1973.
8. A. ZACHARA, *Rozkład ciśnień wewnątrz wiązki przędzonych włókien chemicznych*, Polimery, **20**, 5, 1975.

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