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Abstract

The algorithm producing random variables from the multivariate Farlie-Gumbel-Morgenstern distribution is proposed. The construction of this algorithm based on so-called copulas.

Keywords: Farlie-Gumbel-Morgenstern distribution, copulas.

1 Introduction

The Farlie-Gumbel-Morgenstern (FGM) distribution belongs to the family of multivariate distributions modeling dependencies between random variables. Its application is limited to cases with weak or moderate dependencies. The FGM distribution is widely used in practice.

In this paper we present the algorithm generating random variables from the multivariate FGM distribution. The suitable algorithm in the bivariate case has been given by Johnson (1987). The construction of proposed algorithm bases on so-called copulas.

It is usually hardly to generate dependent multivariate random variables. Copulas appear to be a very useful tool to construct and simulate multivariate distributions. They describe a dependence structure between random variables in the specific way, so, in particular, they are the convenient tool to generate dependent random variables. A general algorithm producing random variables from copulas has been proposed by Romano (2002). We take advantage of it to construct the algorithm generating random variables from the FGM distribution.

2 Farlie-Gumbel-Morgenstern distribution

The FGM bivariate distribution (Morgenstern, 1956) has joint cumulative distribution function given by

$$F(x_1, x_2) = F_1(x_1)F_2(x_2)\{1 + \alpha[1 - F_1(x_1)][1 - F_2(x_2)]\} \tag{1}$$

where F_1 and F_2 are the marginal cumulative distributions functions of X_1 and X_2, respectively, and α is a parameter. The joint density function corresponding to (1) is

$$f(x_1, x_2) = f_1(x_1)f_2(x_2)\{1 + \alpha[1 - 2F_1(x_1)][1 - 2F_2(x_2)]\}, \tag{2}$$

where f_1 and f_2 are the marginal density functions of X_1 and X_2, respectively. The coefficient α is a real number, suitably limited so that $f(x_1, x_2)$ is a nonnegative function for all x_1, x_2.

The parameter α describes the association between random variables. The variables X_1 and X_2 are independent if and only if $\alpha = 0$. They are positively associated, if $\alpha > 0$ and negatively associated if $\alpha < 0$.

This family of bivariate distribution was later discussed by Gumbel (1958) and Farlie (1960). However, it seems that the earliest publication with the special form (1), i.e. with uniform marginals F_1 and F_2, is Eyraud (1938). Now this family is usually called the Farlie-Gumbel-Morgenstern distributions or sometimes also the generalized Eyraud distributions. Cambanis (1977) considered constraints on parameter α for different marginals. If variables X_1 and X_2 are absolutely continuous, then $|\alpha| \leq 1$.

It can be shown that the FGM distributions are restricted to describing weak dependencies between random variables. Schucany (1978) proved that for continuous bivariate FGM distributions the correlation coefficient ρ is limited irrespective of the marginal distributions F_1 and F_2, i.e.

$$|\rho(X_1, X_2)| \leq \frac{1}{3}. \tag{3}$$

In particular, when both X_1 and X_2 have a normal distribution, then we even have $|\rho(X_1, X_2)| \leq 1/\pi$. Whereas if X_1 and X_2 are exponentially distributed, then $|\rho(X_1, X_2)| \leq 1/4$. For other restricts on ρ see Schucany (1978).

Example 1. ***The bivariate FGM distributions with the uniform marginals.*** *Figure 1 gives a graphical representation of the bivariate FGM distribution (2) with the uniform marginals* $\mathbb{U}(0, 1)$. *The point* $(0.5, 0.5)$ *is a saddle point of the FGM density function.* $\alpha = 1$ *and* $\alpha = -1$ *correspond to maximum positive and negative dependence between two random variables. The suitable cases with positive and negative values of* α *represent the same surface rotated 90 degrees about the point* $(0.5, 0.5)$. *For* $\alpha = 0$ *the density function corresponds to independent components.*

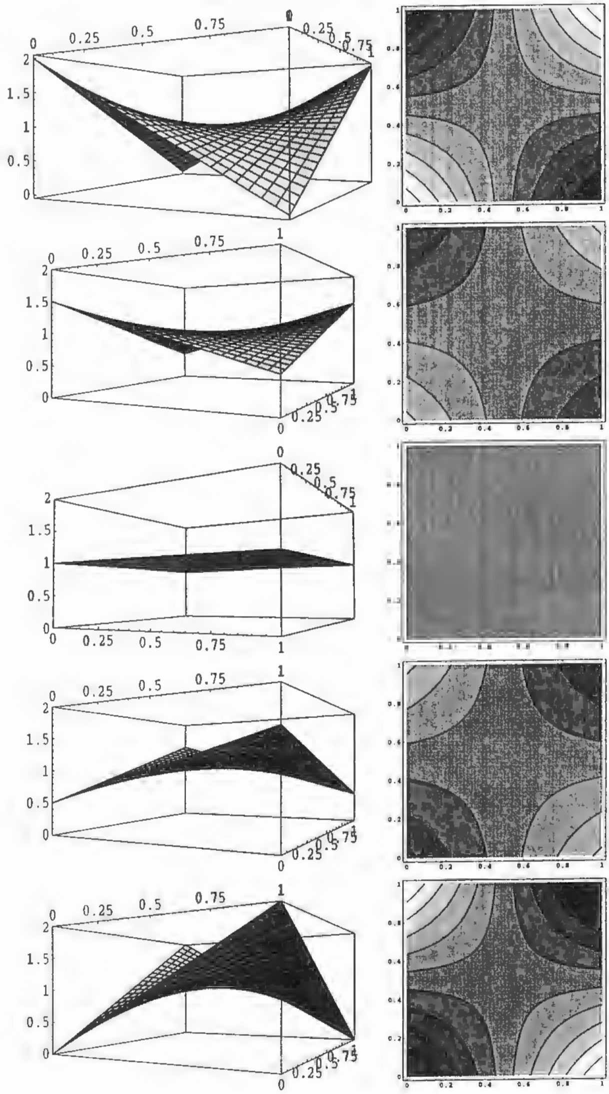

Figure 1: The bivariate FGM distributions with the uniform marginals.

Example 2. ***The bivariate FGM distributions with the exponential marginals.*** *Figure 2 represents the bivariate FGM distribution (2) with the exponential marginal distribution functions* $\mathbb{E}(0,1)$. *Here, the graphs for the same absolute values of* α *represent different surfaces.*

Johnson and Kotz (1975) introduced a general system of multivariate FGM distributions defined by

$$F(x_1,\ldots,x_n) = \prod_{i=1}^{n} F_i(x_i)\left[1+\sum_{j<k}\alpha_{jk}(1-F_j(x_j))(1-F_k(x_k)) + \ldots + \alpha_{12\ldots n}\prod_{k=1}^{n}(1-F_k(x_k))\right], \tag{4}$$

where F_i is the marginal cumulative distribution function of X_i. The coefficients α_{12}, α_{13}, $\alpha_{23},\ldots,\alpha_{12\ldots n}$ are real numbers. Constraints on the values of this coefficients are needed to ensure that $F(x_1,\ldots,x_n)$ is a multivariate cumulative distribution function. If the univariate marginal distributions are absolutely continuous, the joint density function is

$$f(x_1,\ldots,x_n) = \prod_{i=1}^{n} f_i(x_i)\left[1+\sum_{j<k}\alpha_{jk}(1-2F_j(x_j))(1-2F_k(x_k)) + \ldots + \alpha_{12\ldots n}\prod_{k=1}^{n}(1-2F_k(x_k))\right]. \tag{5}$$

Every bivariate pair (X_j, X_k) has (marginal) distribution (2) with $\alpha = \alpha_{jk}$. The function (5) is nonnegative, so coefficients in formulas (4) and (5) must satisfy the following condition

$$1+\sum_{i<j} c_i c_j \alpha_{ij} + \sum_{i<j<k} c_i c_j c_k \alpha_{ijk} + \ldots + \left(\prod_{i=1}^{n} c_i\right)\alpha_{12\ldots n} \geq 0, \tag{6}$$

where $c_i = \pm 1$ for all i. It is worth noting, that for $n = 2$ we get $1 \pm \alpha_{12} \geq 0$, i.e. $|\alpha_{12}| \leq 1$. For $n = 3$, the conditions can be summarized as follows

$$\begin{aligned}
|\,\alpha_{123}\,| &\leq 1+\alpha_{12}+\alpha_{13}+\alpha_{23},\\
|\,\alpha_{13}+\alpha_{23}\pm\alpha_{123}\,| &\leq 1+\alpha_{12},\\
|\,\alpha_{12}+\alpha_{13}\pm\alpha_{123}\,| &\leq 1+\alpha_{23},\\
|\,\alpha_{12}+\alpha_{23}\pm\alpha_{123}\,| &\leq 1+\alpha_{13}.
\end{aligned} \tag{7}$$

If $\alpha_{12} = \alpha_{13} = \alpha_{23} = 1$ in (7), then $\alpha_{123} = 0$. Similarly if $\alpha_{123} = 1$, then $\alpha_{12} = \alpha_{13} = \alpha_{23} = 0$. Introducing higher ordered correlations can lead to reduction of some correlation coefficients among $\alpha_{12}, \alpha_{13}, \alpha_{23}, \ldots, \alpha_{12\ldots n}$. Moreover higher ordered correlations are seldom available in practical applications and they are usually difficult to calculate.

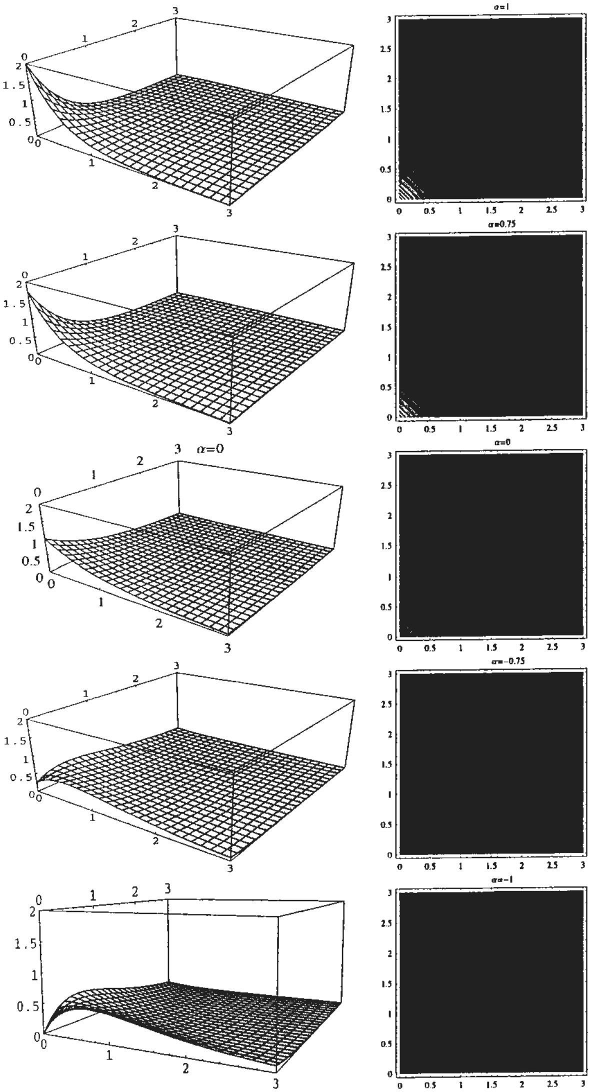

Figure 2: The bivariate FGM distributions with the exponential marginals.

So, further on we will consider a special case of the multivariate FGM distributions, i.e. we suppose that there are only binary correlations between variables, namely

$$F(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_i(x_i) \left[1 + \sum_{j<k} \alpha_{jk}(1 - F_j(x_j))(1 - F_k(x_k)) \right]. \tag{8}$$

Such a truncated model seems to be rich enough. It was successfully applied e.g. for modeling the fatigue crack growth (Sobczyk & Spencer, 1992).

Many of the properties of bivariate FGM distributions generalize. In particular, multivariate FGM distributions characterize weak dependence among the random variables $X_1, X_2, \ldots, X_n$. Additional properties of the FGM family can be found in Johnson & Kotz (1975, 1977). Primarily because of their simple analytical form, FGM distributions have been widely used in modeling, for tests of association and in studying the efficiency of nonparametric procedures. They are also considered (Schucany et al., 1978) as a model for screening variables in a quality control application. Shaked (1975) discussed the analytical appeal of multivariate FGM survival functions in reliability applications and theory of Bayesian survey sampling. For extensive lists of applications and references, see Conway (1983) and Hutchinson & Lai (1990)

3 Copulas

Definition 1. *An n-dimensional copula C is a multivariate distribution function with marginals uniform in* $[0, 1]$ *satisfying the following properties:*

(1) $C : [0,1]^n \to [0,1]$,

(2) $C(u_1, \ldots, u_n) = 0$ *if* $u_i = 0$ *for any* $i = 1, \ldots, n$,

(3) $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ *for each* $i = 1, \ldots, n$ *and all* $u_i \in [0, 1]$,

(4) C is n-increasing.

The following theorem by Sklar is useful in many practical applications.

Theorem 1. *Let F be an n-dimensional distribution function with continuous marginals* $F_1, \ldots, F_n$. *Then it has the following unique copula representation*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{9}$$

For the proof, see Sklar (1996).

It is seen that, copulas are functions that join multivariate distribution functions to their one-dimensional marginals. The basic idea of the copulas is to separate the dependence and the marginal distributions in a multivariate distribution. From Theorem 1 we get the following corollary.

Corollary 1. *Let F be an n-dimensional distribution function with continuous marginals $F_1, \ldots, F_n$ and copula C satisfying (9). Then for any $(u_1, \ldots, u_n) \in [0,1]^n$*

$$C(u_1, \ldots, u_n) = F(F^{-1}(u_1), \ldots, F^{-1}(u_n)), \tag{10}$$

where F_i^{-1} is the inverse function of F_i.

It is easy to give the form of FGM copula. Using Theorem 1 for the distribution function F given by (8) we get

$$\tilde{C}(F_1(x_1), \ldots, F_n(x_n)) = \prod_{i=1}^{n} F_i(x_i) \left[1 + \sum_{j<k} \alpha_{jk}(1 - F_j(x_j))(1 - F_k(x_k))\right]. \tag{11}$$

We will denote the FGM copula by $\tilde{C}$.
For uniformly distributed marginals u_1, u_2 in $[0,1]$ we will rewrite (11) as

$$\tilde{C}(u_1, u_2) = u_1 u_2[1 + \alpha(1 - u_1)(1 - u_2)], \ -1 \leq \alpha \leq 1 \tag{12}$$

in the bivariate case and

$$\tilde{C}(u_1, \ldots, u_n) = \prod_{i=1}^{n} u_i \left[1 + \sum_{j<k} \alpha_{jk}(1 - u_j)(1 - u_k)\right], \ , -1 \leq \sum_{j<k} \alpha_{jk} \leq 1 \tag{13}$$

in the multivariate case.

4 A general algorithm for random variable generation from copulas

From (9) we know that if we have a collection of n-copulas then we automatically have a collection of n-dimensional distributions with whatever one-dimensional marginal distributions we desire. This fact is useful in modeling and simulation using copulas.

A general algorithm for random variable generation from copulas (Romano, 2002) makes use of conditional distribution method. Below it is represented for the bivariate (Algorithm 1) and multivariate (Algorithm 2) cases, separately. Let $\mathbf{U} = (U_1, \ldots, U_n)$ be a random vector. The first algorithm has the following form:

Algorithm 1 (Genest & MacKay, 1986)

(1) generate two independent random variables v_1 and v_2 from the uniform distribution $\mathbb{U}(0,1)$,

(2) set $u_1 = v_1$,

(3) let $C(u_2; u_1) = C_{2|1}(u_1, u_2)$, where

$$C_{2|1}(u_1, u_2) = P(U_2 \leq u_2 | U_1 = u_1) = \frac{\partial C(u_1, u_2)}{\partial u_1}.$$

(4) set $u_2 = C^{-1}(v_2; u_1)$,

(5) the vector (u_1, u_2) is generated from the copula C.

The Algorithm 1 generalizes to the multivariate case as follows:

Algorithm 2 (Romano, 2002)

(1) generate n independent random variables $v_1, v_2, \ldots, v_n$ from the uniform distribution $\mathbb{U}(0, 1)$,

(2) set $u_1 = v_1$,

(3) let $C(u_m; u_1, \ldots, u_{m-1}) = C_{m|1,\ldots,m-1}(u_1, \ldots, u_m)$, $m = 2, \ldots, n$, where

$$\begin{aligned} C_{m|1,\ldots,m-1}(u_1, \ldots, u_m) &= P(U_m \leq u_m | U_1 = u_1, \ldots, U_{m-1} = u_{m-1}) \\ &= \frac{\partial^{m-1}_{u_1 \ldots u_{m-1}} C(u_1, \ldots, u_m, 1, \ldots, 1)}{\partial^{m-1}_{u_1 \ldots u_{m-1}} C(u_1, \ldots, u_{m-1}, 1, \ldots, 1)}, \end{aligned} \tag{14}$$

(4) set $u_m = C^{-1}(v_m; u_1, \ldots, u_{m-1})$, $m = 2, \ldots, n$,

(5) the vector $(u_1, \ldots, u_n)$ is generated from the copula C.

In order to generate random variables $(x_1, \ldots, x_n)$ from a multivariate distribution F with given marginals F_i and copula C we have to transform each u_i using the marginal distributions as follows $x_i = F_i^{-1}(u_i)$, $i = 1, \ldots, n$.

Algorithm 2 is computationally intensive for high values of n. In fact, it is a difficult task to compute the conditional distribution (14).

5 Generation of the FGM distribution

To generate bivariate random variables from the FGM copula $\tilde{C}$ given by (12) we can apply Algorithm 1 which reduces to the following procedure:

Algorithm A (Johnson, 1987)

(1) generate independent random variables v_1 and v_2 from the uniform distribution $\mathbb{U}(0, 1)$,

(2) set $u_1 = v_1$,

(3) calculate

$$A = \alpha(2u_1 - 1) - 1$$

and

$$B = [1 - \alpha(2u_1 - 1)]^2 + 4\alpha v_2(2u_1 - 1),$$

(4) set $u_2 = 2\frac{v_2}{\sqrt{B} - A}$,

(5) the vector (u_1, u_2) is generated from the FGM copula.

Random variables (x_1, x_2) from the FGM distribution with different marginals F_1 and F_2 we obtain by first generating (u_1, u_2) from (12) using Algorithm A and applying $x_1 = F_1^{-1}(u_1)$ and $x_2 = F_2^{-1}(u_2)$. The above algorithm can be found in Romano (2002) and Johnson (1987). The last author introduced it without using copula notation.

Algorithm 2 in Section 4 presents the general way how to generate the random variables from arbitrary multivariate copula C. Here we will use it to generate the multivariate FGM copula $\tilde{C}$ given by (13). The first two steps of Algorithm 2 are easy to make. Let us concentrate on the next one.

Let $C(u_m; u_1, \dots, u_{m-1}) = C_{m|1,\dots,m-1}(u_1, \dots, u_m)$ for $m = 2, \dots, n$. This value we calculate as follows

$$\begin{aligned}
\tilde{C}_{m|1,\dots,m-1}(u_1, \dots, u_m) &= \frac{\partial^{m-1}_{u_1 \dots u_{m-1}} \tilde{C}(u_1, \dots, u_m)}{\partial^{m-1}_{u_1 \dots u_{m-1}} \tilde{C}(u_1, \dots, u_{m-1})} \\
&= \frac{\partial^{m-1}_{u_1 \dots u_{m-1}} u_m \tilde{C}(u_1, \dots, u_{m-1})}{\partial^{m-1}_{u_1 \dots u_{m-1}} \tilde{C}(u_1, \dots, u_{m-1})} \\
&\quad + \frac{\partial^{m-1}_{u_1 \dots u_{m-1}} \prod_{i=1}^m u_i \left[\sum_{j<m} \alpha_{jm}(1 - u_j)(1 - u_m)\right]}{\partial^{m-1}_{u_1 \dots u_{m-1}} \tilde{C}(u_1, \dots, u_{m-1})} \\
&= u_m + \frac{\partial^{m-1}_{u_1 \dots u_{m-1}} \prod_{i=1}^m u_i \left[\sum_{j<m} \alpha_{jm}(1 - u_j)(1 - u_m)\right]}{\partial^{m-1}_{u_1 \dots u_{m-1}} \tilde{C}(u_1, \dots, u_{m-1})} \qquad (15)
\end{aligned}$$

The two following propositions will be useful in computing (15). Their proofs are given in Appendix.

Proposition 1.

$$\partial^{m-1}_{u_1 \dots u_{m-1}} \prod_{i=1}^m u_i \left[\sum_{j<m} (1 - u_j)(1 - u_m)\right] = u_m(u_m - 1) \sum_{j<m} \alpha_{jm}(2u_j - 1) \qquad (16)$$

Proposition 2.

$$\partial^{m-1}_{u_1 \dots u_{m-1}} \tilde{C}(u_1, \dots, u_{m-1}) = 1 + \sum_{j<k} \alpha_{jk}(2u_j - 1)(2u_k - 1). \qquad (17)$$

Combining (15) with (16) and (17) we obtain

$$\tilde{C}_{m|1,\dots,m-1}(u_1,\dots,u_m) = u_m + \frac{u_m(u_m-1)\sum_{j<m}\alpha_{jm}(2u_j-1)}{1+\sum_{j<k}\alpha_{jk}(2u_j-1)(2u_k-1)} \tag{18}$$

for $j < k \leq m$, where $j,k = 1,\dots,m$ and $m = 2,\dots,n$.

Now let us go to step (4) of Algorithm 2. We will find all u_m, $m = 2,\dots,n$ such that $u_m = \tilde{C}^{-1}(v_m; u_1,\dots,u_{m-1})$. So, we have to solve the following equation

$$\tilde{C}_{m|1,\dots,m-1}(u_1,\dots,u_m) = v_m. \tag{19}$$

Let

$$E = 1 + \sum_{j<k}\alpha_{jk}(2u_j-1)(2u_k-1) \tag{20}$$

and

$$F = \sum_{j<m}\alpha_{jm}(1-2u_j). \tag{21}$$

Then equation (19) has the following form

$$-Fu_m^2 + (E+F)u_m - Ev_m = 0. \tag{22}$$

For $F = 0$ we have $u_m = v_m$. Otherwise the solution of (22) is

$$u_m = \frac{E+F\pm\sqrt{\Delta}}{2F}, \tag{23}$$

where

$$\Delta = (E+F)^2 - 4EFv_m. \tag{24}$$

In (23) we choose the sign so that $u_m \in [0,1]$.

It can be shown that

$$(E-F)^2 \leq \Delta \leq (E+F)^2$$

for $v_m \in [0,1]$, $m = 2,\dots,n$.

Now we can write the algorithm producing the random variables from the multivariate FGM distribution. Algorithm 2 applied to the FGM copula (13) reduces to the following procedure:

Algorithm B

(1) generate n independent random variables $v_1, v_2, \ldots, v_n$ from the uniform distribution $\mathbb{U}(0,1)$,

(2) set $u_1 = v_1$,

(3) calculate E and F given by (20) and (21), respectively, for $m = 2, \ldots, n$,

(4) set $u_m = \dfrac{E + F \pm \sqrt{\Delta}}{2F}$ for $m = 2, \ldots, n$, so that $u_m \in [0, 1]$,

(5) the vector $(u_1, \ldots, u_n)$ is generated from the FGM copula.

The vector $(x_1, x_2, \ldots, x_n)$, where $x_1 = F_1^{-1}(u_1), x_2 = F_2^{-1}(u_2), \ldots, x_n = F_n^{-1}(u_n)$ has the FGM distribution given by (8).

Appendix

Proof 1. *Let us compute the first three derivatives of* $\prod_{i=1}^m u_i \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m)$ *in succession with respect to* u_1*,* u_2*,* u_3*,* $3 < l$*. We get as follows:*

$$\partial_{u_1} \prod_{i=1}^m u_i \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m) = \prod_{i=2}^m u_i \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m) + \prod_{i=1}^m u_i \alpha_{1m}(u_m - 1),$$

$$\partial^2_{u_1 u_2} \prod_{i=1}^m u_i \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m) = \prod_{i=3}^m u_i \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m) + \prod_{i=2}^m u_i \alpha_{2m}(u_m - 1) + u_1 \prod_{i=3}^m u_i \alpha_{1m}(u_m - 1),$$

$$\partial^3_{u_1 u_2 u_3} \prod_{i=1}^m u_i \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m) = \prod_{i=4}^m u_i \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m) + \prod_{i=3}^m u_i \alpha_{3m}(u_m - 1) + u_2 \prod_{i=4}^m u_i \alpha_{2m}(u_m - 1) + u_1 \prod_{i=4}^m u_i \alpha_{1m}(u_m - 1).$$

It is easy to notice a regularity. So, we can write the form of k*-th derivative. For* $k < m$ *we have:*

$$\partial^k_{u_1 \ldots u_k} \prod_{i=1}^m u_i \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m) = u_1 \prod_{i=k+1}^m u_i \alpha_{1m}(u_m - 1) + \ldots + u_{k-1} \prod_{i=k+1}^m u_i \alpha_{k-1\,m}(u_m - 1) + \prod_{i=k}^m u_i \alpha_{km}(u_m - 1) + \prod_{i=k+1}^m u_i \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m)$$

Finally we get:

$$\partial^{m-1}_{u_1\dots u_{m-1}} \prod_{i=1}^{m} u_i \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m) = u_1 u_m \alpha_{1m}(u_m - 1) + u_2 u_m \alpha_{2m}(u_m - 1)$$
$$+ \ldots + u_{m-1} u_m \alpha_{m-1\,m}(u_m - 1) + u_m \sum_{j<m} \alpha_{jm}(1-u_j)(1-u_m)$$
$$= u_m \left[u_1 \alpha_{1m}(u_m - 1) + \ldots + u_{m-1}\alpha_{m-1\,m}(u_m - 1) + \sum_{j<m} \alpha_{jm}(u_j - 1)(u_m - 1) \right]$$
$$= u_m(u_m - 1) \sum_{j<m} \alpha_{jm}(2u_j - 1) \quad \blacksquare$$

Proof 2. *Let us start with calculating the first four derivatives of $\tilde{C}(u_1, \ldots, u_l)$, $4 < l$. We have:*

$$\partial_{u_1} \tilde{C}(u_1, \ldots, u_l) = \prod_{i=2}^{l} u_i \left[1 + \sum_{j<k} \alpha_{jk}(1-u_j)(1-u_k) \right] + \prod_{i=1}^{l} u_i \sum_{1<k} \alpha_{1k}(u_k - 1),$$

$$\partial^2_{u_1 u_2} \tilde{C}(u_1, \ldots, u_l) = \prod_{i=3}^{l} u_i \left[1 + \sum_{j<k} \alpha_{jk}(1-u_j)(1-u_k) \right]$$
$$+ \prod_{i=2}^{l} u_i \left[\alpha_{12}(u_1 - 1) + \sum_{2<k} \alpha_{2k}(u_k - 1) \right] + u_1 \prod_{i=3}^{l} u_i \sum_{1<k} \alpha_{1k}(u_k - 1) + \prod_{i=1}^{l} u_i \alpha_{12},$$

$$\partial^3_{u_1 u_2 u_3} \tilde{C}(u_1, \ldots, u_l) = \prod_{i=4}^{l} u_i \left[1 + \sum_{j<k} \alpha_{jk}(1-u_j)(1-u_k) \right]$$
$$+ \prod_{i=3}^{l} u_i \left[\alpha_{13}(u_1 - 1) + \alpha_{23}(u_2 - 1) + \sum_{3<k} \alpha_{3k}(u_k - 1) \right]$$
$$+ u_2 \prod_{i=4}^{l} u_i \left[\alpha_{12}(u_1 - 1) + \sum_{2<k} \alpha_{2k}(u_k - 1) \right] + \prod_{i=2}^{l} u_i \alpha_{23} + u_1 \prod_{i=4}^{l} u_i \sum_{1<k} \alpha_{1k}(u_k - 1)$$
$$+ u_1 \prod_{i=3}^{l} u_i \alpha_{13} + u_1 u_2 \prod_{i=4}^{l} u_i \alpha_{12}$$

$$
\begin{aligned}
\partial^4_{u_1\dots u_4}\tilde{C}(u_1,\dots,u_l) &= \prod_{i=5}^{l} u_i \left[1+\sum_{j<k}\alpha_{jk}(1-u_j)(1-u_k)\right] \\
&+ \prod_{i=4}^{l} u_i \left[\alpha_{14}(u_1-1)+\alpha_{24}(u_2-1)+\alpha_{34}(u_3-1)+\sum_{4<k}\alpha_{4k}(u_k-1)\right] \\
&+ u_3\prod_{i=5}^{l} u_i \left[\alpha_{13}(u_1-1)+\alpha_{23}(u_2-1)+\sum_{3<k}\alpha_{3k}(u_k-1)\right] + \prod_{i=3}^{l}\alpha_{34} \\
&+ u_2\prod_{i=5}^{l} u_i \left[\alpha_{12}(u_1-1)+\sum_{2<k}\alpha_{2k}(u_k-1)\right] + u_2\prod_{i=4}^{l}u_i\alpha_{24} + u_2u_3\prod_{i=5}^{l}u_i\alpha_{23} \\
&+ u_1\prod_{i=5}^{l}u_i\sum_{1<k}\alpha_{1k}(u_k-1) + u_1\prod_{i=4}^{l}u_i\alpha_{14} + u_1u_3\prod_{i=5}^{l}u_i\alpha_{13} + u_1u_2\prod_{i=5}^{l}u_i\alpha_{12}.
\end{aligned}
$$

By the above, we can give the form of s-th derivative. For $s < l$ we obtain:

$$
\begin{aligned}
\partial^s_{u_1\dots u_s}\tilde{C}(u_1,\dots,u_l) &= \sum_{j=2}^{s-1}\left(u_1u_j\prod_{i=s+1}^{l}u_i\alpha_{1j}\right) + u_1\prod_{i=s}^{l}u_i\alpha_{1s} + u_1\prod_{i=s+1}^{l}u_i\sum_{1<k}\alpha_{1k}(u_k-1) \\
&+ \sum_{j=3}^{s-1}\left(u_2u_j\prod_{i=s+1}^{l}u_i\alpha_{2j}\right) + u_2\prod_{i=s}^{l}u_i\alpha_{2s} + u_2\prod_{i=s+1}^{l}u_i\left[\alpha_{12}(u_1-1)+\sum_{2<k}\alpha_{2k}(u_k-1)\right] \\
&+ \dots + \sum_{j=t}^{s-1}\left(u_{t-1}u_j\prod_{i=s+1}^{l}u_i\alpha_{t-1\,j}\right) + u_{t-1}\prod_{i=s}^{l}u_i\alpha_{t-1\,s} \\
&+ u_{t-1}\prod_{i=s+1}^{l}u_i\left[\sum_{w=1}^{t-2}\alpha_{w\,t-1}(u_w-1)+\sum_{t-1<k}\alpha_{t-1\,k}(u_k-1)\right] + \dots + \prod_{i=s-1}^{l}u_i\alpha_{s-1\,s} \\
&+ \prod_{i=s}^{l}u_i\left[\sum_{w=1}^{s-1}\alpha_{ws}(u_w-1)+\sum_{s<k}\alpha_{sk}(u_k-1)\right] + \prod_{i=s+1}^{l}u_i\left[1+\sum_{j<k}(1-u_j)(1-u_k)\right]
\end{aligned}
$$

Hence l-th derivative has the following form:

$$\begin{aligned}
\partial^l_{u_1\ldots u_l}\tilde{C}(u_1,\ldots,u_l) &= 1+\sum_{j<k}(1-u_j)(1-u_k)+u_l\sum_{w=1}^{l-1}\alpha_{wl}(u_w-1)+u_{l-1}{}_l\alpha_{l-1\,l}\\
&+\ldots+u_{t-1}u_l\alpha_{t-1\,l}+\sum_{j=t}^{l-1}u_{t-1}u_j\alpha_{t-1\,j}+\ldots+u_2\sum_{2<k}[\alpha_{2k}(u_k-1)+\alpha_{12}(u_1-1)]\\
&+u_2u_l\alpha_{2l}+\sum_{j=3}^{l-1}u_2u_j\alpha_{2j}+u_1\sum_{1<k}\alpha_{1k}(u_k-1)+u_1u_l\alpha_{1l}+\sum_{j=2}^{l-1}u_1u_j\alpha_{1j}\\
&=1+\alpha_{12}[(1-u_1)(1-u_2)+u_2(u_1-1)+u_1(u_2-1)+u_1u_2]\\
&+\alpha_{13}[(1-u_1)(1-u_3)+u_3(u_1-1)+u_1(u_3-1)+u_1u_3]\\
&+\ldots+\alpha_{1l}[(1-u_1)(1-u_l)+u_l(u_1-1)+u_1(u_l-1)+u_1u_l]\\
&+\alpha_{23}[(1-u_2)(1-u_3)+u_3(u_2-1)+u_2(u_3-1)+u_2u_3]\\
&+\ldots+\alpha_{2l}[(1-u_2)(1-u_l)+u_l(u_2-1)+u_2(u_l-1)+u_2u_l]\\
&+\alpha_{l-1\,l}\big[(1-u_{l-1})(1-u_l)+u_l(u_{l-1}-1)+u_{l-1}(u_l-1)+u_{l-1}u_l\big]\\
&=1+\sum_{j<k}\alpha_{jk}\,[(1-u_j)(1-u_k)+u_k(u_j-1)+u_j(u_k-1)+u_ju_k]\\
&=1+\sum_{j<k}\alpha_{jk}(2u_j-1)(2u_k-1) \quad \blacksquare
\end{aligned}$$

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