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Disturbance of SH-type due to shearing stress discontinuity at the interface of two layers overlying a semi-infinite medium

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THE DISPLACEMENT at the plane free surface, due to sudden introduction of a shearing stress discontinuity which moves after creation, at the interface of two layers of finite thickness, the lower layer overlying a semi-infinite medium of different elastic constants, has been obtained in exact form by the method due to Cagniard and modified by Garvin. Displacements at different points on the free surface have been calculated numerically and results are shown graphically for one particular case.

Rozważany jest dwuwarstwowy ośrodek składający się z górnej warstwy o stałej grubości i dolnej pokrywającej półprzestrzeń i charakteryzujący się różnymi stałymi sprężystości w poszczególnych warstwach. Na płaszczyźnie podziału tych warstw przyłożono skokowo naprężenie ścinające, które następnie rozprzestrzenia się w głąb warstw. Przemieszczenie na płaskiej powierzchni swobodnej otrzymano w postaci zamkniętej, metodą Cagniarda zmodyfikowaną przez Garvina. Przemieszczenia w różnych punktach na powierzchni swobodnej policzono numerycznie, a wyniki przedstawiono graficznie dla jednego przypadku szczególnego.

Рассматривается двухслоистая среда, состоящая с верхнего слоя постоянной толщины и нижнего слоя, заполняющего полупространство. Среда характеризуется разными постоянными упругости в отдельных слоях. На плоскости раздела этих слоев приложено скачкообразное напряжение сдвига, которое затем распространяется вглубь слоев. Перемещение на плоской свободной поверхности получено в замкнутом виде методом Каньяра модифицированным Гарвиным. Перемещения в разных точках на свободной поверхности расчитаны численно; результаты представлены графически для одного частного случая.

Introduction

THE TRANSIENT displacement of SH-type produced at the surface of a homogeneous elastic half-space, due to the sudden introduction of discontinuity in the shearing stress within a semi-infinite medium, has been obtained by NAG [1]. In another paper NAG [2] has considered the displacement at the free surface due to the sudden application of a stress discontinuity, moving along the interface of a layer overlying a semi-infinite medium of different elastic constants. A similar problem [3] was solved by the authors when the stress discontinuity occurs just at the middle of the upper layer overying the semi-infinite medium.

Shearing stress discontinuity occurs in many cases, e.g., 1) inside the earth between two layers if there is a crack which is being filled up by liquid, then there will be a case of discontinuity of shearing stress in that region whereas the normal stress will be continuous, 2) when a layer tends to slide over another layer inside the earth, then shearing stress discontinuity may occur and 3) shearing stress discontinuity may be associated with propagation of cracks in earthquakes.

In the present paper the problem was solved for the case when shearing stress discontinuity occurs at the interface of two layers of finite thickness, the lower layer overlying a semi-infinite medium of different elastic constants. The method of solution is similar to that of the previous problems. The initial displacements were obtained for different types of discontinuity in the shearing stress. The numerical results for one case at two different points were shown graphically.

Formulation of the problem

We consider here three layers of different elastic constants. The upper layer (I) of thickness h_1 overlies the medium (II) of thickness h_2 and the layer (II) overlies the semi-infinite medium (III). Let us take the origin of the coordinates 0 (Fig. 1) at the interface of the layers (I) and (II) and assume that the discontinuity occurs suddenly at the interface

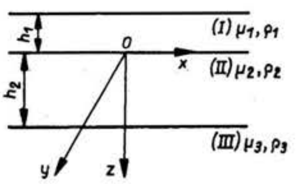

FIG. 1.

$z = 0$ plane and moves with the constant velocity $V(< \beta_1)$. Since only the SH-type of motion is being considered, we can assume that $u = w = 0$ and that v and all quantities are independent of y. The only equation of motion to be satisfied is, with the usual notation,

$$\nabla^2 v = 1/\beta^2 \frac{\partial^2 v}{\partial t^2}, \tag{1}$$

where

$$\beta^2 = \mu/\varrho \quad (\beta_3 > \beta_2 > \beta_1).$$

Now let

$$L[f(t)] = \bar{f}(p) = \int\limits_0^\infty e^{-pt} f(t) dt \tag{2}$$

so that $f(t) = L^{-1}\bar{f}(p)$, p is real and positive. Again, defining

$$\bar{f}(x, p) = \int\limits_{-\infty}^{\infty} \exp(i\xi x) \Phi(\xi, p) d\xi, \tag{3}$$

where $\Phi(\xi, p)$ being the same function as considered by NAG [1].

We can readily obtain for the layers (I), (II) and (III)

$$\bar{v}_1 = \int\limits_{-\infty}^{\infty} (B_1 \cosh \eta_{s_1} z + C_1 \sinh \eta_{s_1} z) \exp(i\xi x) d\xi, \tag{4}$$

$$\bar{v}_2 = \int\limits_{-\infty}^{\infty} (B_2 \cosh \eta_{s_2} z + C_2 \sinh \eta_{s_2} z) \exp(i\xi x) d\xi, \tag{5}$$

(6) $$\bar{v}_3 = \int_{-\infty}^{\infty} D_3 \exp\{-\eta_{s_3} z + i\xi x\} d\xi,$$

where

$$\eta_{s_i} = \left(\xi^2 + \frac{p^2}{\beta_i^2}\right)^{1/2}, \quad i = 1, 2, 3.$$

The boundary conditions are

(7)
$$\begin{aligned}
&\text{(i)} && (\widehat{yz})_1 = 0 && \text{at} \quad z = -h_1,\\
&\text{(ii)} && v_1 = v_2 && \text{at} \quad z = 0,\\
&\text{(iii)} && v_2 = v_3 && \text{at} \quad z = h_2,\\
&\text{(iv)} && (\widehat{yz})_2 = (\widehat{yz})_3 && \text{at} \quad z = h_2,\\
&\text{(v)} && (\widehat{yz})_1 - (\widehat{yz})_2 = S(x, t) H(t) && \text{at} \quad z = 0,
\end{aligned}$$

where $S(x, t)$ is a function of x and t (to be chosen later).

The Laplace transform of the above set of equations (7) will necessarily hold. The boundary condition (i) is satisfied if

(8) $$C_1 \cosh \eta_{s_1} h_1 = B_1 \sinh \eta_{s_1} h_1.$$

The boundary conditions (ii), (iii) and (iv) are satisfied if

(9) $$B_1 = B_2,$$

(10) $$B_2 \cosh \eta_{s_2} h_2 + C_2 \sinh \eta_{s_2} h_2 = D_3 e^{-\eta_{s_3} h_2}$$

and

(11) $$\mu_2 \eta_{s_2} (B_2 \sinh \eta_{s_2} h_2 + C_2 \cosh \eta_{s_2} h_2) = -\mu_3 \eta_{s_3} D_3 e^{-\eta_{s_3} h_2}.$$

We now consider two different forms of $S(x, t)$.

Case 1.

Let

(12) $$S(x, t) = P, \; a \leqslant x \leqslant a + Vt, \quad = 0 \text{ elsewhere},$$

where P is a constant.

From the boundary condition (v) using Eqs. (12) and, as solved by NAG in [1], we have

$$\mu_1 \eta_{s_1} C_1 - \mu_2 \eta_{s_2} C_2 = \frac{P}{2\pi p} \frac{e^{-i\xi a}}{(p/V + i\xi)}.$$

By solving B_1 C_1, B_2, C_2 etc., we have at $z = -h_1$,

(13) $$\bar{v}_1(x_1, -h_1, p) = I = \frac{2P}{\pi p} \int_0^{\infty} \Bigg[\frac{(\mu_2 \eta_{s_2} + \mu_3 \eta_{s_3}) e^{i\xi x_1}}{(p/V + i\xi)[\mu_2 \eta_{s_2}(\mu_1 \eta_{s_1} + \mu_3 \eta_{s_3}) + (\mu_1 \eta_{s_1} \mu_3 \eta_{s_3} + \mu_2^2 \eta_{s_2}^2)]} \times \{e^{-\eta_{s_1} h_1} - K_5 e^{-3\eta_{s_1} h_1} - K_3 e^{-2(\eta_{s_1} h_1 + \eta_{s_2} h_2)} - K_6 e^{-(2\eta_{s_2} h_2 + \eta_{s_1} h_1)} - K_4 e^{-(3\eta_{s_1} h_1 + 2\eta_{s_2} h_2)} - K_3 K_5 e^{-2(2\eta_{s_1} h_1 + \eta_{s_2} h_2)} - K_3 K_6 e^{-2(\eta_{s_1} h_1 + 2\eta_{s_2} h_2)} - K_3 K_4 e^{-4(\eta_{s_1} h_1 + \eta_{s_2} h_2)}\} \Bigg] d\xi,$$

where

$$x_1 = x-a, \quad K_1 = \frac{\mu_3\eta_{s_3}-\mu_1\eta_{s_1}}{\mu_3\eta_{s_3}+\mu_1\eta_{s_1}} < 1,$$

$$K_2 = \frac{\mu_2^2\eta_{s_2}^2-\mu_1\eta_{s_1}\mu_3\eta_{s_3}}{\mu_2^2\eta_{s_2}^2+\mu_1\eta_{s_1}\mu_3\eta_{s_3}} < 1, \quad K_3 = \frac{\mu_2\eta_{s_2}-\mu_3\eta_{s_3}}{\mu_2\eta_{s_2}+\mu_3\eta_{s_3}} < 1,$$

$$K_4 = \frac{K_1(\mu_1\eta_{s_1}+\mu_3\eta_{s_3})\mu_2\eta_{s_2}-K_2(\mu_1\eta_{s_1}\mu_3\eta_{s_3}+\mu_2^2\eta_{s_2}^2)}{(\mu_1\eta_{s_1}+\mu_3\eta_{s_3})\mu_2\eta_{s_2}+(\mu_1\eta_{s_1}\mu_3\eta_{s_3}+\mu_2^2\eta_{s_2}^2} < 1,$$

$$K_5 = \frac{K_1(\mu_1\eta_{s_1}+\mu_3\eta_{s_3})\mu_2\eta_{s_2}+K_2(\mu_1\eta_{s_1}\mu_3\eta_{s_3}+\mu_2^2\eta_{s_2}^2)}{(\mu_1\eta_{s_1}+\mu_3\eta_{s_3})\mu_2\eta_{s_2}+(\mu_1\eta_{s_1}\mu_3\eta_{s_3}+\mu_2^2\eta_{s_2}^2)} < 1,$$

$$K_6 = \frac{(\mu_1\eta_{s_1}+\mu_3\eta_{s_3})\mu_2\eta_{s_2}-(\mu_1\eta_{s_1}\mu_3\eta_{s_3}+\mu_2^2\eta_{s_2}^2)}{(\mu_1\eta_{s_1}+\mu_3\eta_{s_3})\mu_2\eta_{s_2}+(\mu_1\eta_{s_1}\mu_3\eta_{s_3}+\mu_2^2\eta_{s_2}^2)} < 1.$$

After inserting

$$\xi = \frac{\zeta_1 p}{\beta_1}, \quad \frac{\beta_1}{v} = \nu', \quad \frac{\beta_2}{\beta_1} = \nu_1, \quad \frac{\beta_3}{\beta_2} = \nu_2,$$

$$\eta_{s_1} = p(1+\zeta_1^2)^{1/2}/\beta_1, \quad \eta_{s_2} = p(1+\nu_1^2\zeta_1^2)^{1/2}/\beta_2, \quad \eta_{s_3} = p(1+\nu_2^2\zeta_1^2)^{1/2}/\beta_3,$$

we have the first term in Eq. (13) which is given by

$$(14)\quad I_1 = \frac{2P}{\pi p^2}\int_0^\infty \frac{\{\varrho_2(1+\nu_1^2\zeta_1^2)^{1/2}+\varrho_3(1+\nu_2^2\zeta_1^2)^{1/2}\}\exp[-p\{-i\zeta_1 x_1+h_1(1+\zeta_1^2)^{1/2}\}/\beta_1]}{D(\nu'+i\zeta_1)}d\zeta_1,$$

where

$$D = [\varrho_2(1+\nu_1^2\zeta_1^2)^{1/2}\{\varrho_1(1+\zeta_1^2)^{1/2}+\varrho_3(1+\nu_2^2\zeta_1^2)^{1/2}\} + \varrho_1\varrho_3(1+\zeta_1^2)^{1/2}(1+\nu_2^2\zeta_1^2)^{1/2}+\varrho_2^2(1+\nu_1^2\zeta_1^2)].$$

Let

$$t = \{-i\zeta_1 x_1+h_1(1+\zeta_1^2)^{1/2}\}\beta_1^{-1}$$

then, by inversion,

$$\zeta_1(t) = \frac{\beta_1}{x_1^2+h_1^2}[itx_1+h_1\{t^2-(x_1^2+h_1^2)\beta_1^{-2}\}^{1/2}].$$

Now, as obtained by NAG [2], we have

$$(15)\qquad L^{-1}I_1 = \frac{2P}{\pi}\int_0^t (t-\lambda)G_1[\zeta_1(\lambda)]d\lambda \quad \text{since} \quad L[tH(t)] = \frac{1}{p^2}\cdots,$$

where

$$(16)\quad G_1[\zeta_1(t)] = \mathrm{Re}\,\frac{\{\varrho_2(1+\nu_1^2\zeta_1^2)^{1/2}+\varrho_3(1+\nu_2^2\zeta_1^2)^{1/2}\}\dfrac{d\zeta_1}{dt}H\{t-(x_1^2+h_1^2)^{1/2}\beta_1^{-1}\}}{D(\nu'+i\zeta_1)}.$$

By a similar procedure, we have

$$L^{-1}I_m = \frac{2P}{\pi}\int\limits_0^t (t-\lambda)G_m[\zeta_m(\lambda)]d\lambda,$$

where $m = 2, 3, \dots$ *up* to 8,

$$(17)\quad G_2[\zeta_2(t)] = \mathrm{Re}\Big\langle\{\varrho_2(1+\nu_1^2\zeta_2^2)^{1/2}+\varrho_3(1+\nu_2^2\zeta_2^2)^{1/2}\}\,[\varrho_2(1+\nu_1^2\zeta_2^2)^{1/2}\{\varrho_3(1+\nu_2^2\zeta_2^2)^{1/2}+\varrho(1+\zeta_2^2)^{1/2}\} + \{\varrho_2^2(1+\nu_1^2\zeta_2^2)-\varrho_1\varrho_3(1+\zeta_2^2)^{1/2}(1+\nu_2^2\zeta_2^2)^{1/2}\}]\frac{d\zeta_2}{dt}H\{t - (x_1^2+9h_1^2)^{1/2}\beta_1^{-1}\}\Big/D^2(\nu'+i\zeta_2)\Big\rangle,$$

$$(18)\quad G_3[\zeta_3(t)] = \mathrm{Re}\Big\langle\{\varrho_2(1+\nu_1^2\zeta_3^2)^{1/2}+\varrho_3(1+\nu_2^2\zeta_3^2)^{1/2}\}\,[\{\varrho_1(1+\zeta_3^2)+\varrho_3(1+\nu_2^2\zeta_3^2)^{1/2}\}\varrho_2(1+\nu_1^2\zeta_3^2)^{1/2} - \varrho_1\varrho_3(1+\zeta_3^2)^{1/2}(1+\nu_2^2\zeta_3^2)^{1/2}]\frac{d\zeta_3}{dt}H[t-\{x_1^2 + (h_1+2\nu_1h_2)^2\}^{1/2}\beta_1^{-1}]\Big/D^2(\nu'+i\zeta_3)\Big\rangle,$$

$$(19)\quad G_4[\zeta_4(t)] = \mathrm{Re}\,\frac{\{\varrho_2(1+\nu_1^2\zeta_4^2)^{1/2}-\varrho_3(1+\nu_2^2\zeta_4^2)^{1/2}\}\dfrac{d\zeta_4}{dt}H[t-\{x_1^2+4(h_1+h_2\nu_1)^2\}^{1/2}\beta_1^{-1}]}{D(\nu'+i\zeta_4)},$$

$$(20)\quad G_5[\zeta_5(t)] = \mathrm{Re}\Big\langle\{\varrho_2(1+\nu_1^2\zeta_5^2)^{1/2}+\varrho_3(1+\nu_2^2\zeta_5^2)^{1/2}\}\,[\varrho_2(1+\nu_1^2\zeta_5^2)^{1/2}\{\varrho_3(1+\nu_2^2\zeta_5^2)^{1/2}-\varrho_1(1+\zeta_5^2)^{1/2}\} - \{\varrho_2^2(1+\nu_1^2\zeta_5^2)-\varrho_1\varrho_3(1+\zeta_5^2)^{1/2}(1+\nu_2^2\zeta_5^2)^{1/2}\}]\frac{d\zeta_5}{dt}H[t-\{x_1^2+(3h_1+2h_2\nu_1)^2\}^{1/2} \times \beta_1^{-1}]\Big/D^2(\nu'+i\zeta_5)\Big\rangle,$$

$$(21)\quad G_6[\zeta_6(t)] = \mathrm{Re}\Big\langle\{\varrho_2(1+\nu_1^2\zeta_6^2)^{1/2}+\varrho_3(1+\nu_2^2\zeta_6^2)^{1/2}\}[\varrho_2(1+\nu_1^2\zeta_6^2)^{1/2}\{\varrho_3(1+\nu_2^2\zeta_6^2)^{1/2}-\varrho_1(1+\zeta_6^2)^{1/2}\} + \{\varrho_2^2(1+\nu_1^2\zeta_6^2)-\varrho_1\varrho_3(1+\zeta_6^2)^{1/2}(1+\nu_2^2\zeta_6^2)^{1/2}\}]\frac{d\zeta_6}{dt}H[t-\{x_1^2+4(2h_1+h_2\nu_1)^2\}^{2/1} \times \beta_1^{-1}]\Big/D^2(\nu'+i\zeta_6)\Big\rangle,$$

$$(22)\quad G_7[\zeta_7(t)] = \mathrm{Re}\Big\langle\{\varrho_2(1+\nu_1^2\zeta_7^2)^{1/2}-\varrho_3(1+\nu_2^2\zeta_7^2)^{1/2}\}[\{\varrho_1(1+\zeta_7^2)^{1/2}+\varrho_3(1+\nu_2^2\zeta_7^2)^{1/2}\}\varrho_2(1+\nu_1^2\zeta_7^2)^{1/2} - \varrho_1\varrho_3(1+\zeta_7^2)^{1/2}(1+\nu_2^2\zeta_7^2)^{1/2}]\frac{d\zeta_7}{dt}H[t-\{x_1^2+4(h_1 + 2h_2\nu_1)^2\}^{1/2}\beta_1^{-1}]\Big/D^2(\nu'+i\zeta_7)\Big\rangle,$$

(23) $G_8[\zeta_8(t)]$

$$= \mathrm{Re}\Big\langle\{\varrho_2(1+\nu_1^2\zeta_8^2)^{1/2}+\varrho_3(1+\nu_2^2\zeta_8^2)^{1/2}\}[\varrho_2(1+\nu_1^2\zeta_8^2)^{1/2}\{\varrho_3(1+\nu_2^2\zeta_8^2)^{1/2}-\varrho_1(1+\zeta_8^2)^{1/2}\} - \{\varrho_2^2(1+\nu_1^2\zeta_8^2)-\varrho_1\varrho_3(1+\zeta_8^2)^{1/2}(1+\nu_1^2\zeta_8^2)^{1/2}\}]\frac{d\zeta_8}{dt}H[t-\{x_1^2+16(h_1 + h_2\nu_1)^2\}^{1/2}\beta_1^{-1}]\Big/D^2(\nu'+i\zeta_8)\Big\rangle.$$

Hence

$$v_1(x,-h_1,t) = L^{-1}I = L^{-1}\sum_{m=1}^{\infty} I_m \quad \text{(for the initial eight values).}$$

Case 2.

Let

$$S(x,t) = Ph_1\delta(x-Vt). \tag{24}$$

From the boundary condition (v) and Eq. (24), and as obtained by NAG in [2], we have

$$\mu_1\eta_{s_1}C_1-\mu_2\eta_{s_2}C_2 = \frac{Ph_1}{2\pi V(P/V+i\xi)}. \tag{25}$$

Solving for B_1, C_1, B_2, C_2 etc., we have at $z = -h_1$,

$$v_1(x_1-h_1,p) = \frac{2Ph_1}{\pi V}\int_0^{\infty}\Bigg[\frac{(\mu_2\eta_{s_2}+\mu_3\eta_{s_3})e^{i\xi x}}{(p/V+i\xi)\{\mu_2\eta_{s_2}(\mu_1\eta_{s_1}+\mu_3\eta_{s_3})+(\mu_1\eta_{s_1}\mu_3\eta_{s_3}+\mu_2^2\eta_{s_2}^2)\}} \times\{e^{-\eta_{s_1}h_1}-K_5e^{-3\eta_{s_1}h_1}-K_6e^{-(2\eta_{s_2}h_2+\eta_{s_1}h_1)}-K_3e^{-2(\eta_{s_1}h_1+\eta_{s_2}h_2)}-K_4e^{-(3\eta_{s_1}h_1+2\eta_{s_2}h_2)} - K_3K_5e^{-2(2\eta_{s_1}h_1+\eta_{s_2}h_2)}-K_3K_6e^{-2(\eta_{s_1}h_1+2\eta_{s_2}h_2)}-K_3K_4e^{-4(\eta_{s_1}h_1+\eta_{s_2}h_2)}\}\Bigg]d\xi,$$

where K_3, K_4 etc. are defined in Eq. (13).

Hence $v_1(x,-h_1,t) = L^{-1}\sum_{m=1}^{8} I_m$ (for initial eight values) where $L^{-1}I_m$ ($m = 1, 2, \ldots$) are the same as determined in Case 1 except that in this case the integrands do not contain the factor $(t-\lambda)$ and $2P/\pi$ and x_1 are replaced by $2Ph_1/\pi V$ and x respectively.

For example, $L^{-1}I_1$ is given by

$$L^{-1}I_1 = \frac{2Ph_1}{\pi V}\int_0^t G_1[\zeta_1(\lambda)]d\lambda \quad \text{since} \quad L[H(t)] = \frac{1}{p}$$

and $G_1[\zeta_1(t)]$ is the same as defined in Eq. (16).

Numerical calculation and discussion

Numerical values of $Kv_1(x,-h_1,t)$ for $x = 5h_1$ and $x = 10h_1$ were calculated using the same method as that used by the authors in a previous paper [3] and by following GILBERT and KNOPOFF [4], only in the initial stages of the motion, for Case 2. The values of $Kv_1(x,-h_1,t)$ were plotted against the dimensionless quantity τ given by $t = \dfrac{\tau h_1}{\beta_1}$

in Fig. 2. It is being assumed that $K = \frac{\pi V \beta_1^2}{2Ph_1^4}$, $\frac{h_1}{h_2} = \frac{1}{2}$, $\nu_1 = 2$, $\nu_2 = 3$, $\nu' = \sqrt{2}$, $\varrho_1 = \varrho_2 = \varrho_3 = \varrho$. Curve (I) is for $x = 5h_1$ and (II) for $x = 10h_1$. For $x = 5h_1$ and $x = 10h_1$ the times at which the disturbance arrives at the point of observations are 5.06

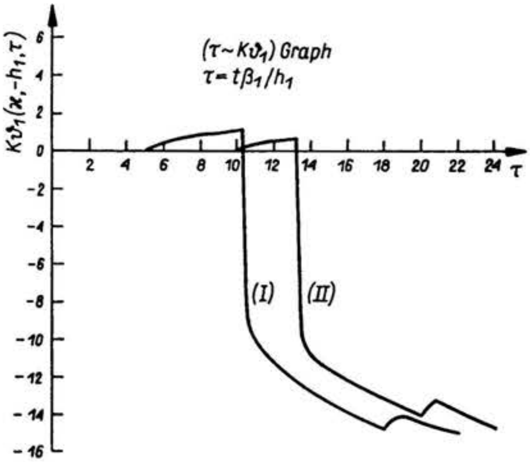

FIG. 2.

and 10.05, respectively. From Fig. 2 it is clear that the curves undergo changes in their slopes as the different pulses arrive after undergoing reflections from the boundaries of the layers. It is also clear that the contributions due to the third pulse for $x = 10h_1$ will arrive in a shorter time than for $x = 5h_1$.

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