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Two invariants-dependent models of granular media

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THE THEORY of plasticity is applied to describe the behaviour of granular materials. It is shown that the yield condition of the Granta-Gravel model [5] corresponds to a very simple case of density hardening. When the second invariant of plastic strain deviator is considered in the yield condition, peaks in the stress-strain diagrams may be explained and the volumetric strains may be described in a way which includes the basic features of real behaviour. To illustrate the behaviour, the axially symmetric homogeneous stress is considered on the basis of two simple models.

Do opisu zachowania się materiałów ziarnistych zastosowano teorię plastyczności. Pokazano, że warunek plastyczności typu Granta-Gravela [5] odpowiada bardzo prostemu przypadkowi wzmocnienia gęstościowego. Uzależniając warunek plastyczności od drugiego niezmiennika dewiatora odkształcenia plastycznego można wyjaśnić wierzchołki krzywych naprężenie-odkształcenie oraz opisać odkształcenie objętościowe w sposób zawierający podstawowe cechy rzeczywistego zachowania się ośrodka. Dla ilustracji rozważanego zagadnienia zastosowano dwa proste modele opisujące jednorodny osiowo-symetryczny stan naprężenia.

Для описания поведения зернистых материалов применена теория пластичности. Показано, что условие пластичности типа Гранта-Гравеля [5] отвечает очень простому случаю плотностного упрочнения. Связывая условие пластичности с вторым инвариантом девиатора пластических деформаций, можно выяснить пики на кривых напряжение-деформация и описать объемную деформацию таким образом, чтобы описание содержало основные свойства действительного поведения среды. Для иллюстрации рассматриваемой проблемы применены две простые модели, описывающие однородное, осесимметричное напряженное состояние.

1. Assumptions and general relations

GENERAL remarks on the application of the theory of plasticity to the description of the mechanical behaviour of granular materials may be found in the books written by SZCZEPIŃSKI and MRÓZ [1, 2]. This paper is based on the concepts introduced into soil mechanics by ROSCOE [3, 4] and developed by WROTH and SCHOFIELD [5]. Its aim is to construct a simple model which may be used to discuss structure — subsoil interaction problems. The approach to the problem follows the trend introduced by MRÓZ [6] in which the starting point to soil mechanics is the general theory of plasticity.

In this paper the behaviour of granular media is discussed within the elasto-plastic theory. It is assumed that the strains are small and may be represented as the sum of reversible elastic strains and plastic strains. The material is isotropic and for elastic strains Hooke's law is valid.

It is assumed there exists a flow rule given by the following equation:

$$\dot{\varepsilon}_{ij}^{pl} = \dot{\lambda} \frac{\partial F}{\partial \sigma_{ij}}, \tag{1.1}$$

where $\dot{\varepsilon}^{pl}_{ij}$ are the increments of plastic strains, $\dot{\lambda}$ is a scalar function, F is the plastic potential which, on account of isotropy, depends upon the invariants of stress and plastic strain.

In the paper the associated flow rule is assumed. Thus, the function F represents at the same time the yield condition. The triaxial test is the basic test for granular materials. Thus, until more informations from the true triaxial test are available, it is reasonable to assume that the yield condition depends upon the first invariants of stress and plastic strain tensors and the second invariants of the corresponding deviators. Therefore, the change in the yield condition depends on the plastic strains only.

It is assumed that the yield condition is given by the following equation:

$$F = \sqrt{J_2} + f(I_1, \varepsilon^{pl}_{ii}, N) = 0, \tag{1.2}$$

where I_1 is the first invariant of the stress tensor, J_2 is the second invariant of the stress deviator, ε^{pl}_{ii} is the first invariant of the plastic strain tensor, N is the square root of the second invariant of the plastic strain deviator. Thus,

$$I_1 = \sigma_{ii}, \quad J_2 = \frac{1}{2}\sigma^{\text{dev}}_{ij}\sigma^{\text{dev}}_{ij}, \quad N^2 = \frac{1}{2}\varepsilon^{pl\,\text{dev}}_{ij}\varepsilon^{pl\,\text{dev}}_{ij}, \tag{1.3}$$

where here and in the following summation convention is used for repeated indices and the superscript dev denotes deviator. It must be stressed that Eq. (1.2) introduces simplifications. It is assumed that F is a sum of the square root of the second invariant of the stress deviator and an arbitrary function of the other invariants. To justify the simplification one may say that the Mises-Schleicher yield condition has a similar form.

Substitution of Eq. (1.2) into Eq. (1.1) yields:

$$\dot{\varepsilon}^{pl}_{ij} = \dot{\lambda}\left(\frac{\partial f}{\partial I_1}\delta_{ij} + \frac{1}{2\sqrt{J_2}}\sigma^{\text{dev}}_{ij}\right). \tag{1.4}$$

It should be noted that in the case J_2 is zero, the second expression in the brackets in Eq. (1.4) is an undefined symbol. This fact results in a corner in the yield surface at the $J_2 = 0$ point.

In a standard way $\dot{\lambda}$ may be calculated from the condition that when there is an increase in stress and plastic strain the new stress state is on a new yield surface. Thus, considering the total differential of F expressed by Eq. (1.2) substitution of $\dot{\varepsilon}^{pl}_{ij}$ given by Eq. (1.1) yields the following result:

$$\dot{\lambda} = \frac{1}{A}\left(\frac{\partial f}{\partial I_1}\delta_{rs} + \frac{1}{2\sqrt{J_2}}\sigma^{\text{dev}}_{rs}\right)\dot{\sigma}_{rs}, \tag{1.5}$$

where A is a scalar function given by the following equation:

$$A = -3\frac{\partial f}{\partial \varepsilon^{pl}_{nn}}\frac{\partial f}{\partial I_1} - \frac{1}{2\sqrt{J_2}}\frac{1}{2N}\frac{\partial f}{\partial N}\sigma^{\text{dev}}_{kl}\varepsilon^{pl\,\text{dev}}_{kl}. \tag{1.6}$$

Equation (1.5) may be presented in an alternative form when summations are carried out and the definitions of invariants given by Eqs. (1.3) are taken into account. It follows:

$$\dot{\lambda} = \frac{1}{A}\left(\frac{\partial f}{\partial I_1}\dot{I}_1 + \dot{\sqrt{J_2}}\right). \tag{1.7}$$

Substitution of $\dot{\lambda}$ expressed by Eq. (1.5) into Eq. (1.4) yields the following relation between the increments of stress and plastic strain:

$$(1.8)\qquad \dot{\varepsilon}_{ij}^{pl} = \frac{1}{A}\left(\frac{\partial f}{\partial I_1}\delta_{ij} + \frac{1}{2\sqrt{J_2}}\sigma_{ij}^{\text{dev}}\right)\left(\frac{\partial f}{\partial I_1}\delta_{rs} + \frac{1}{2\sqrt{J_2}}\sigma_{rs}^{\text{dev}}\right)\dot{\sigma}_{rs}.$$

From Eq. (1.8) it follows that the value of the scalar function A plays an important role in the description of plastic behaviour. If A goes to infinity the plastic strains go to zero and if A goes to zero the plastic strains go to infinity. In the second case it means that there is a perfect plastic flow.

The total strains are the sum of elastic and plastic strains. Thus, when the elastic strains are expressed by Hooke's law for isotropic materials it follows:

$$(1.9)\qquad \dot{\varepsilon}_{ij} = A_{ijrs}\dot{\sigma}_{rs},$$

where

$$A_{ijrs} = \frac{1+\nu}{E}\delta_{ir}\delta_{js} - \frac{\nu}{E}\delta_{ij}\delta_{rs} + \frac{1}{A}\left(\frac{\partial f}{\partial I_1}\delta_{ij} + \frac{1}{2\sqrt{J_2}}\sigma_{ij}^{\text{dev}}\right)\left(\frac{\partial f}{\partial I_1}\delta_{rs} + \frac{1}{2\sqrt{J_2}}\sigma_{rs}^{\text{dev}}\right)$$

and E, ν are elastic constants.

By simple but lengthy calculations the inverse relation to Eq. (1.9) may be calculated. It follows:

$$(1.10)\qquad \dot{\sigma}_{ij} = B_{ijrs}\dot{\varepsilon}_{rs},$$

where

$$B_{ijrs} = \frac{E}{1+\nu}\delta_{ir}\delta_{js} + \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{rs} - \frac{\left(\frac{E}{1-2\nu}\frac{\partial f}{\partial I_1}\delta_{ij} + \frac{E}{1+\nu}\frac{1}{2\sqrt{J_2}}\sigma_{ij}^{\text{dev}}\right)\left(\frac{E}{1-2\nu}\frac{\partial f}{\partial I_1}\delta_{rs} + \frac{E}{1+\nu}\frac{1}{2\sqrt{J_2}}\sigma_{rs}^{\text{dev}}\right)}{A + 3\left(\frac{\partial f}{\partial I_1}\right)^2\frac{E}{1-2\nu} + \frac{1}{2}\frac{E}{1+\nu}}.$$

From the expression for B_{ijrs} it can be seen what it means that A is large. If A is large compared with $E/2(1+\nu)$ the elastic behaviour is dominant.

For further discussions it is convenient to obtain expressions for $\dot{\varepsilon}_{ii}^{pl}$, the first invariant of the increments of the plastic strain tensor. From Eq. (1.8), when the definitions of invariants given by Eqs. (1.3) are considered, it follows:

$$(1.11)\qquad \dot{\varepsilon}_{ii}^{pl} = 3\frac{\partial f}{\partial I_1}\frac{1}{A}\left(\frac{\partial f}{\partial I_1}\dot{I}_1 + \dot{\sqrt{J_2}}\right).$$

It may be seen from Eq. (1.11) that an increase in the value of the second invariant of the stress deviator causes a change in volume.

According to the definition given in Eqs. (1.3) it follows:

$$(1.12)\qquad \dot{N} = \frac{1}{2N}\varepsilon_{ij}^{pl\,\text{dev}}\dot{\varepsilon}_{ij}^{pl\,\text{dev}}.$$

Substitution of increments of plastic strains given by Eq. (1.8) after simple manipulations yields

$$(1.13)\qquad \dot{N} = \frac{1}{A}\frac{1}{2N}\frac{1}{2\sqrt{J_2}}\sigma_{ij}^{\text{dev}}\varepsilon_{ij}^{pl\,\text{dev}}\left(\frac{\partial f}{\partial I_1}\dot{I}_1 + \dot{\sqrt{J_2}}\right).$$

2. The homogeneous axisymmetric case

Let us consider in detail the case investigated in the triaxial test. The axes of the Cartesian coordinate system are the principal axes of the stress tensor, σ_x is the vertical principal stress and $\sigma_y = \sigma_z$ are the horizontal principal stresses. In this case the stress deviator is given by the following matrix:

$$(2.1) \qquad [\sigma_{ij}^{\mathrm{dev}}] = (\sigma_x - \sigma_y) \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix},$$

and the stress invariants are given by the following relations:

$$(2.2) \qquad I_1 = \sigma_x + 2\sigma_y, \quad \sqrt{J_2} = \frac{1}{\sqrt{3}} |\sigma_x - \sigma_y|.$$

It is worthwhile to note that in this case the stress deviator depends only upon $(\sigma_x - \sigma_y)$.

The corresponding expressions for the plastic strain are similar. It is necessary only to replace σ_x and σ_y by ε_x^{pl} and ε_y^{pl}.

It is easy to verify that in this case

$$(2.3) \qquad \sigma_{ij}^{\mathrm{dev}} \varepsilon_{ij}^{pl\,\mathrm{dev}} = \pm 2N \sqrt{J_2}.$$

Substitution of Eq. (2.3) into Eq. (1.13) yields

$$(2.4) \qquad \dot{N} = \pm \frac{1}{2A} \left(\frac{\partial f}{\partial I_1} \dot{I}_1 + \dot{\sqrt{J_2}} \right).$$

From the theory of plasticity it is known that $\dot{\lambda}$ is always positive in plastic flow. Thus, a comparison with Eq. (1.6) indicates that in Eq. (2.3) the plus sign must be chosen. It follows immediately that for $\sqrt{J_2} \neq 0$ the following relation is satisfied:

$$(2.5) \qquad \dot{N} = \frac{1}{2} \dot{\lambda}.$$

To obtain analytical solutions, it is convenient to choose N as the independent variable. Let us consider the case $I_1 = \mathrm{const}$. In this case Eq. (2.4) reduces to the following differential equation:

$$(2.6) \qquad \frac{d\sqrt{J_2}}{dN} = 2A.$$

To solve the differential equation it is necessary to specify the function $f(I_1, \varepsilon_{ii}^{pl}, N)$.

The case of hydrostatic pressure needs special consideration. In this case $\sqrt{J_2}$ and N are zero. The first invariant of the plastic deformation may be calculated directly from the yield condition given by Eq. (1.2) which reduces to the form

$$(2.7) \qquad f(I_1, \varepsilon_{ii}^{pl}) = 0.$$

3. A simplified density hardening model

Let us assume that the yield condition does not depend upon the second invariant of the plastic strain deviator. For very small elastic strains, compared to plastic ones, the first invariant of the plastic strain tensor is proportional to the change in volume. Thus, in this case the hardening depends upon density [5, 6].

For density hardening the scalar function A defined in Eq. (1.6) is given by the following equation:

$$A = -3\frac{\partial f}{\partial \varepsilon_{nn}^{pl}} \frac{\partial f}{\partial I_1}. \tag{3.1}$$

Let us assume that there exists a critical straight line in the I_1, $\sqrt{J_2}$ plane on which a perfect plastic flow occurs. The equation of the critical line has the following form:

$$MI_1 - \sqrt{J_2} + c = 0, \tag{3.2}$$

where M and c are constants.

To obtain a simple model let us assume that

$$\begin{aligned} \frac{\partial f}{\partial I_1} &= \alpha \frac{MI_1 + c - \sqrt{J_2}}{MI_1 + c}, \\ \frac{\partial f}{\partial \varepsilon_{ii}^{pl}} &= \varkappa_0 + \varkappa_1 I_1, \end{aligned} \tag{3.3}$$

where α, $\varkappa_0$ and $\varkappa_1$ are constants.

From the first equation of the set (3.3) it may be seen that A is zero on the critical line. The expression in the denominator was chosen in such a way that the expression is dimensionless. The second equation of the (3.3) is a linear function of I_1.

The yield condition which satisfies Eqs. (3.3) takes the following form:

$$F = \sqrt{J_2} + (MI_1 + c)\left[\ln \frac{MI_1 + c}{MI_1^* + c} - \frac{K}{M}\varepsilon_{ii}^{pl}\right] = 0, \tag{3.4}$$

where I_1^* is a constant of integration.

The obtained yield condition has the form of the yield condition used in the Granta Gravel model developed in Cambridge [5]. One may thus say that this yield condition is suitable from the point of view of physical consideration or one may say that it corresponds to a very simple case of density hardening.

When the second deviator of the stress tensor is zero (hydrostatic pressure) the yield condition given by Eq. (3.4) reduces to the following form:

$$F = \ln \frac{MI_1 + c}{MI_1^* + c} - \frac{K}{M}\varepsilon_{ii}^{pl}. \tag{3.5}$$

Thus, there are no plastic strains for $I_1 \leqslant I_1^*$ and this is the physical meaning of the constant I_1^*.

Substitution of Eq. (3.4) into Eq. (3.1) yields

$$A = 3K(MI_1 + c - \sqrt{J_2}). \tag{3.6}$$

The plastic volume change may be calculated directly from the Eq. (1.4). Substitution of f for the model considered above yields

$$\dot{\varepsilon}_{ii}^{pl} = 3\dot{\lambda} M \frac{MI_1 + c - \sqrt{J_2}}{MI_1 + c}. \tag{3.7}$$

Now, $\dot{\lambda}$ is always positive in plastic flow. Thus there is a plastic decrease in volume for points below the critical line and plastic increase in volume for points above the critical line.

To discuss the properties of the model let us solve the case $I_1 = \text{const}$ in the triaxial test. When A expressed by Eq. (3.6) is substituted into Eq. (2.6) the following differential equation results:

$$\frac{d\sqrt{J_2}}{dN} = 6K(MI_1 + c - \sqrt{J_2}). \tag{3.8}$$

The solution with the initial condition that for $N = N_0$ (initial value of N), $\sqrt{J_2} = \sqrt{J_2^*}$ (value at the initial yield surface) takes the following form:

$$\sqrt{J_2} - (MI_1 + c) = [\sqrt{J_2^*} - (MI_1 + c)]e^{-6K(N-N_0)}. \tag{3.9}$$

It is seen immediately from the solution that when N goes to infinity the solution approaches the critical line. When, for the initial yield condition, the point I_1, $\sqrt{J_2^*}$ is above the critical line the expression in the square brackets on the right side is positive and the solution approaches the critical line from above.

4. Simple two invariants-dependent models

In the derivation of relations for the two invariants-dependent models, let us assume that the new model goes to the density hardening model when N is neglected. Thus let us assume that the first equation of the set (3.3) is still valid. In view of the yield condition this equation may be written in the following form:

$$\frac{\partial f}{\partial I_1} = M \frac{MI_1 + c + f(I_1, \varepsilon_{ii}^{pl}, N)}{MI_1 + c}. \tag{4.1}$$

Integration with respect to I_1 yields

$$f(I_1, \varepsilon_{ii}^{pl}, N) = (MI_1 + c)[\ln(MI_1 + c) + G(\varepsilon_{ii}^{pl}, N)], \tag{4.2}$$

where $G(\varepsilon_{ii}^{pl}, N)$ is an arbitrary function of the indicated variables.

As a first approximation, let us consider the case when G is just the sum of a term proportional to ε_{ii}^{pl} and a function of N. It follows:

$$F = \sqrt{J_2} + (MI_1 + c)\left[\ln\frac{MI_1 + c}{MI_1^* + c} - \frac{K}{M}\varepsilon_{ii}^{pl} - g(N)\right] = 0. \tag{4.3}$$

A comparison of Eq. (4.3) to Eq. (3.4) shows that the yield surfaces in the I_1, $\sqrt{J_2}$ are the same, but in the density hardening model the succeeding yield surfaces correspond to unique values of ε_{ii}^{pl}. In the case described by Eq. (4.3) the same yield surface may correspond to different pairs of values of ε_{ii}^{pl} and N.

Let us assume that $g(N)$ is chosen in such a way that:

$$g(0) = 0. \tag{4.4}$$

In the special case, when during the loading the stress deviator is proportional to a constant deviator, for the yield condition described by Eq. (4.3) the scalar function A is given by the following equation:

$$A = 3K\left[\left(1+\frac{g'(N)}{6K}\right)(MI_1+c)-\sqrt{J_2}\right]. \tag{4.5}$$

If the function $g(N)$ is chosen in such a way that:

$$\lim_{N\to\infty} g'(N) = 0 \tag{4.6}$$

for large values of N, the stress points approach the critical line. However, for smaller values of N the stress points may go over the critical line. It is possible that A is equal to zero for a certain value of N. Then, there may be there are local peaks in the stress-strain diagrams.

A suitable function $g(N)$ may be found experimentally. To have an insight into the possibilities of the model two simple examples of functions $g(N)$ are considered.

As the first example let us consider $g(N)$ given by the following equation:

$$g(N) = \begin{cases} g\dfrac{N}{N_k} & \text{for} \quad N \leqslant N_k, \\ g & \text{for} \quad N > N_k, \end{cases} \tag{4.7}$$

where N_k is a constant. The graph of the function is shown in Fig. 1.

Substitution into Eq. (4.5) yields:

$$A = \begin{cases} 3K\left[\left(1+\dfrac{g}{6KN_k}\right)(MI_1+c)-\sqrt{J_2}\right] & \text{for} \quad N \leqslant N_k, \\ 3K\left[(MI_1+c)-\sqrt{J_2}\right] & \text{for} \quad N > N_k. \end{cases} \tag{4.8}$$

From Eq. (4.8) it may be seen that there are two straight lines in the I_1, $\sqrt{J_2}$ plane where A may be zero (Fig. 2). These lines define three regions. If the values of the invariants at the

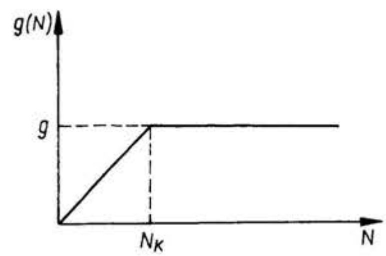

FIG. 1.

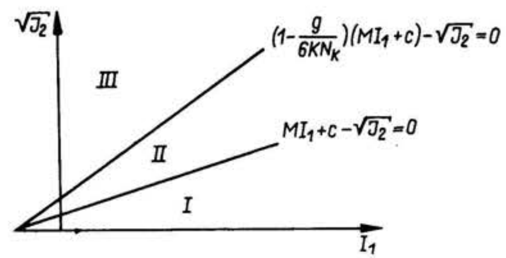

FIG. 2.

moment the initial yield surface is reached are in the I or II region, the stress points may approach the upper line but never go over it. For $N > N_k$ the stress points go to the lower straight line. It is worthwhile to note that in A it is only important whether $N \leqslant N_k$, but there is no direct influence of N on the values of A in both regions. At the value $N = N_k$

there is a discontinuity in A which means that there is a discontinuity in the first derivative of the stress-strain diagram. If, at the initial yield condition, I_1 and $\sqrt{J_2^*}$ are above the upper line, the loads will decrease from the very beginning of the plastic flow.

The plastic volume changes may be found directly from the yield condition given by Eq. (4.3). For N going to infinity it follows:

$$\lim_{N\to\infty} \varepsilon_{ii}^{pl} = \frac{M}{K}\left[\ln\frac{MI_1+c}{MI_1^*+c}+1-g\right]. \tag{4.9}$$

Hence, the choice of g has a direct influence on the position of the upper line in Fig. 2 and on the plastic volume changes.

To illustrate the behaviour let us solve the axially symmetric case in which I_1 is kept constant. Integration of Eq. (2.7) yields

$$\begin{cases} \dfrac{\sqrt{J_2}}{MI_1+c} = 1+\dfrac{g}{6KN_k} \\ \qquad\qquad +\left[\dfrac{\sqrt{J_2^*}}{MI_1+c}-\left(1+\dfrac{g}{6KN_k}\right)\right]e^{-6K(N-N_0)} \quad \text{for} \quad N\leqslant N_k,\, N_0<N_k, \\ \dfrac{\sqrt{J_2}}{MI_1+c} = 1+\dfrac{g}{6KN_k}e^{-6K(N-N_k)} \\ \qquad\qquad +\left[\dfrac{\sqrt{J_2^*}}{MI_1+c}-\left(1+\dfrac{g}{6KN_k}\right)\right]e^{-6K(N-N_0)} \quad \text{for} \quad N> N_k,\, N_0<N_k, \\ \dfrac{\sqrt{J_2}}{MI_1+c} = 1+\left[\dfrac{\sqrt{J_2^*}}{MI_1+c}-1\right]e^{-6K(N-N_0)} \quad \text{for} \quad N_0\geqslant N_k, \end{cases} \tag{4.10}$$

where N_0 is the initial value of N and $\sqrt{J_2^*}$ is the value of the second invariant of the stress deviator at the initial yield surface. It may be easily seen from the solutions given by Eqs. (4.10) that for $N_0 \geqslant N_k$ the model goes over to the density hardening model.

The numerical solutions for $c = 0$ are shown in Figs. 3, 4, 5 and 6. In the case $N_0 = 0$ the choice of the value for N_k fixes the position of the peak. The dimensionless variable $\sqrt{J_2}/MI_1$ approaches 1 when N goes to infinity. In the numerical solution $g/6KN_k$ was chosen equal to 0.4 which corresponds to the assumption that the upper line is given by the equation $\sqrt{J_2}/MI_1 = 1+0.4$.

For different values of $\sqrt{J_2^*}/MI_1$ there are different curves starting from the corresponding point. The choice of the value $6KN_k$ has an influence on the "steepness" of the curve. When N_0/N_k is not equal to zero the peak moves to the left (Fig. 4). In the case $N_0/N_k = 1.0$ the behaviour corresponds to the case of density hardening (Fig. 5). In Fig. 6 the corresponding diagrams for the plastic volume strains are shown. The solutions indicate that in the two invariants-dependent model it is possible to obtain volumetric strains which correspond to the experimental data.

As the second example, let us consider the following function:

$$g(N) = g(1-e^{-\beta N}). \tag{4.11}$$

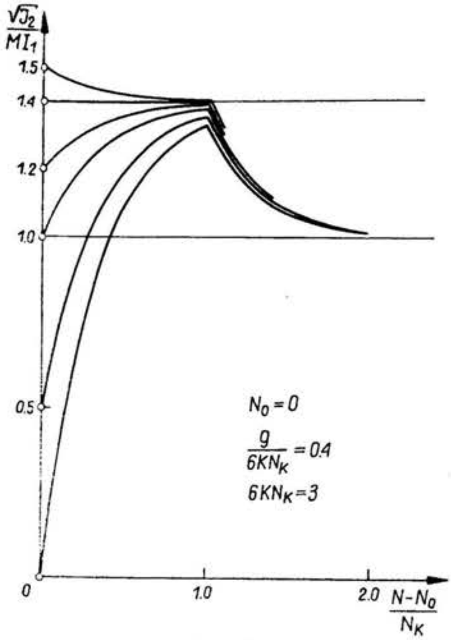

FIG. 3.

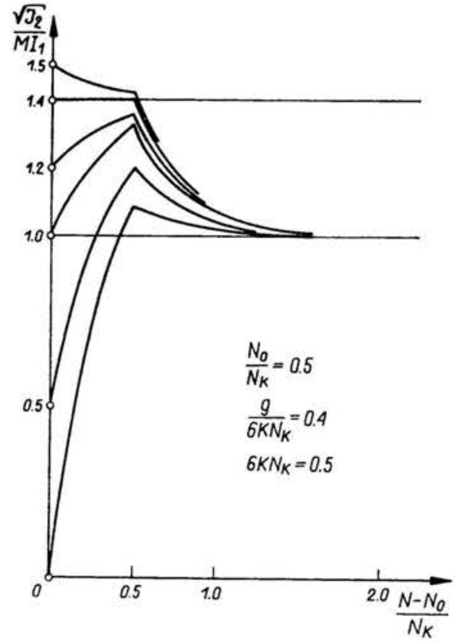

FIG. 4.

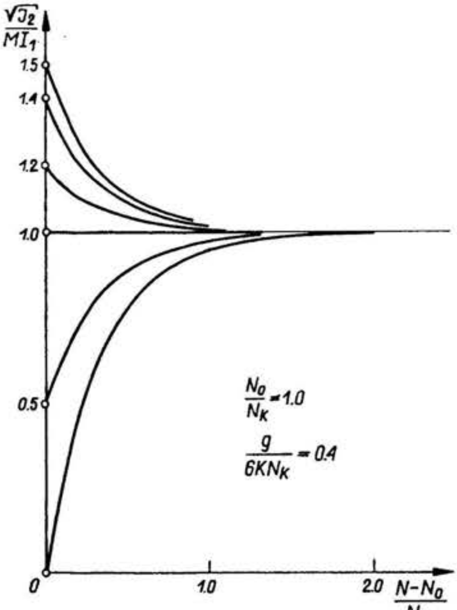

FIG. 5.

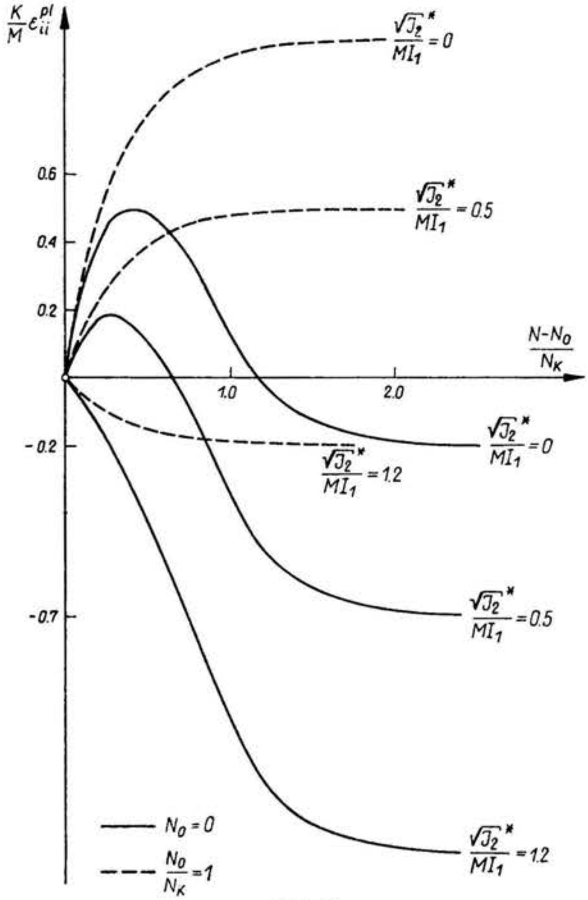

FIG. 6.

The diagram of the function is shown in Fig. 7. Substitution into Eq. (4.5) yields

$$A = 3K\left[\left(1+\frac{g\beta}{6K}e^{-\beta N}\right)(MI_1+c)-\sqrt{J_2}\right]. \tag{4.12}$$

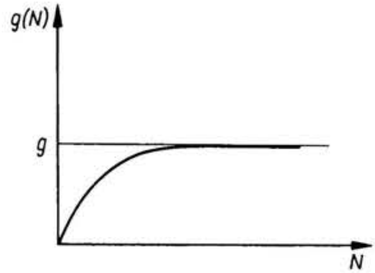

FIG. 7.

For this choice of $g(N)$ the function A is continuous and has continuous derivatives. For the axially symmetric case and constant I_1 the following solutions were obtained:

$$\frac{\sqrt{J_2}}{MI_1+c}-1 = (\varkappa-1)e^{-x}+\frac{g^*\beta}{\beta-6K}(e^{-x}-e^{-\varepsilon x}) \tag{4.13}$$

for $\beta \neq 6K$ and

$$\frac{\sqrt{J_2}}{MI_1+c}-1 = (\varkappa-1)e^{-x}+g^*xe^{-x} \tag{4.14}$$

for $\beta = 6K$ where $x = 6K(N-N_0)$, $\quad g^* = ge^{-\beta N_0}$,

$$\varkappa = \sqrt{J_2^*}/(MI_1+c), \quad \varepsilon = \beta/6K.$$

It should be noted that N_0 has influence on g^* only. For large values of N_0, g^* approaches zero and the behaviour is described by the density hardening model. It is not necessary

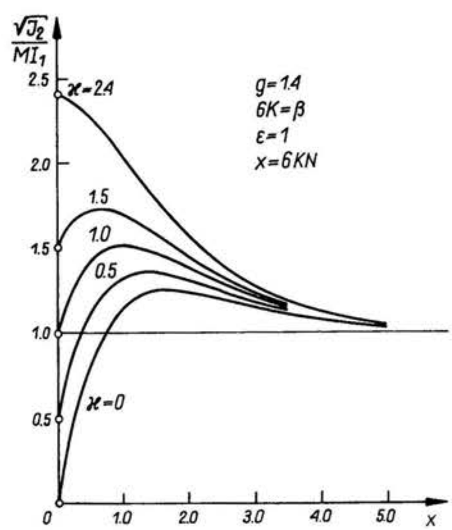

FIG. 8.

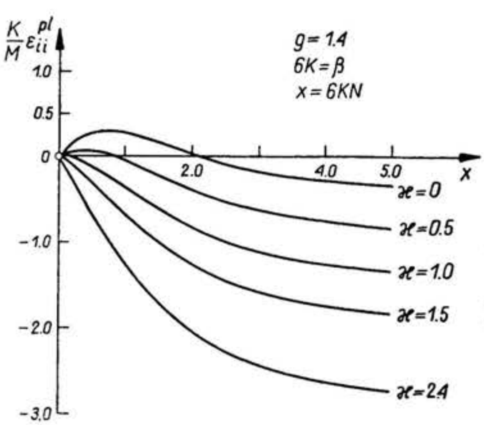

FIG. 9.

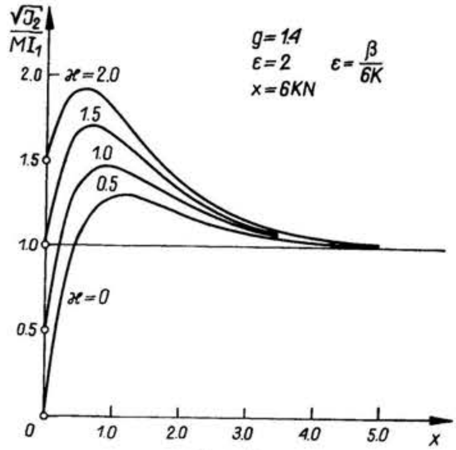

FIG. 10.

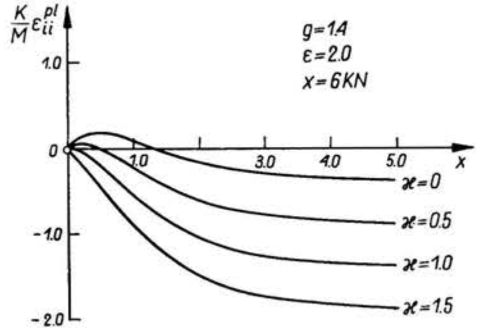

FIG. 11.

to discuss the influence of N_0 separately. We may assume that $N_0 = 0$ and take an appropriate value of g^*.

Diagrams for two cases are shown in Figs. 8, 9, 10 and 11. The results are smooth curves. The disadvantages in this model are that there are difficulties in explaining the influence of the parameteis on the solutions. It may be stated, however, that the diagrams include the essential features of granular material behaviour.

5. Conclusions

In the density hardening model all the solutions approach the critical line, but it is not possible to get an appropriate description of the volumetric changes.

In the case the model depends on two invariants of plastic strain it is possible to consider peaks in the strain-stress diagrams and to take into account that at the beginning of loading the volume decreases and when the stress deviator increases the volume increases. It is thus possible to model the behaviour when hardening is succeeded by a softening of the material.

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