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BRIEF NOTES

Governing equations for simple continuum feathers

M. SHAHINPOOR (SHIRAZ)

WE ADOPT a special Cosserat material curve as a continuum model for simple feather structures. We then present the governing equations without presenting the details of derivation. Finally, a set of linear equations for a simple feather structure is presented.

1. Introductions

RECENT technological trends towards mechanisms duplicating the flight of birds and insects [1] demand dynamical analyses of feather structures. This is required for deter-

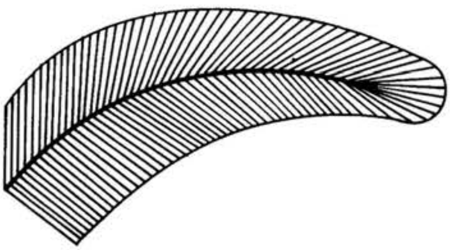

FIG. 1. A simple feather structure.

mining the configurational stability of such structures in flight. For some expositions regarding such analyses the reader is referred to [2]–[10].

The aim of the present work is to adopt a special class of Cosserat material curve as a continuum model for a simple feather structure (Fig. 1). We shall not present the derivation of the governing equations in detail since this can be easily found in [12]–[15].

2. Kinematics of a simple feather structure

We consider a simple feather structure a portion of which is shown in Fig. 1. We adopt a special class of Cosserat material curve for our model such that the generalized coordinate $\mathbf{P} \equiv (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2)$ of any material point X on the curve is an ordered set subject to the following kinematical constraints:

$$\mathbf{d} = \hat{R}(X, t)\mathbf{d}_2, \tag{2.1}$$

$$\mathbf{d}_2 = \hat{R}^T(X,t)\mathbf{d}_1, \tag{2.2}$$

$$\hat{R}^{-1} = \hat{R}^T, \quad \det \hat{R} \neq 0, \tag{2.3}$$

where $\mathbf{r}$ (X, t) is the position vector of each material point X at time t, $d_i(X, t)$, $i = 1, 2$ are two deformable vectors (directors) attached to each point, X_0 playing the role of pairs of feather flags, $\hat{R}$ is an orthogonal and nonsingular rotation matrix, and the superscript T denotes the transpose. Note that the family of vectors $\mathbf{P} \equiv (\mathbf{r}, \mathbf{d}, \mathbf{d}_2)$ generate a nine-dimensional vector space $\mathscr{E}$ whose vector algebra is discussed by SHAHINPOOR [14, 15]. If $\mathscr{K}$ is an operator in $\mathscr{E}$ defined in such a manner that

$$\mathscr{K}: \mathscr{E} \to \mathscr{E}, \tag{2.4}$$

$$\mathbf{Q} \equiv \mathscr{K}\mathbf{P} \equiv (\varrho_0\mathbf{r}+\varrho_{01}\mathbf{d}_1+\varrho_{02}\mathbf{d}_2,\ \varrho_{01}\mathbf{r}_1+\varrho_{12}\mathbf{d}_2+\varrho_1\mathbf{d}_1,\ \varrho_2\mathbf{d}_2+\varrho_{12}\mathbf{d}_1+\varrho_{02}\mathbf{r}), \tag{2.5}$$

where $\varrho_0, \varrho_{01}, \varrho_{02}, \varrho_{1,2}, \varrho_1, \varrho_2$ are all material line measures and functions of X alone. We can now easily derive the governing thermodynamical equations as

$$\frac{\partial \mathscr{K}}{\partial t} + \nabla \cdot (\dot{\mathbf{r}}\mathscr{K}) = 0, \tag{2.6}$$

$$\mathscr{K}\ddot{\mathbf{P}} = \mathbf{T}_{,X} + \mathscr{K}\mathbf{F} - \mathbf{M}, \tag{2.7}$$

$$\mathbf{P}_X \hat{\times} \mathbf{F} + \mathbf{P} \hat{\times} \mathbf{M} = 0, \tag{2.8}$$

$$\dot{e} = \mathbf{T} \circ \dot{\mathbf{P}}_{,X} + \dot{\mathbf{P}} \circ \mathbf{M} + r + q_{,X}, \tag{2.9}$$

$$\dot{\theta}\eta - r - q_{,X} - \theta^{-1}q\theta_{,X} \geqslant 0. \tag{2.10}$$

If one introduces a Helmholtz free energy function $\psi = e - \theta\eta$, then Eqs. (2.9) and (2.10) can be combined to yield the Clausius-Duhem inequality:

$$-\dot{\psi} - \dot{\theta}\eta + \mathbf{T} \circ \dot{\mathbf{P}}_{,X} + \dot{\mathbf{P}} \circ \mathbf{M} + \theta^{-1}q\theta_{,X} \geqslant 0. \tag{2.11}$$

Constitutive relations

We let ψ, η, $\mathbf{T}$, $\mathbf{M}$, $\mathbf{q}$ be the dependent constitutive functions. Taking Truesdell's "equipresence principle" [16] we can write

$$\psi = \hat{\psi}(\mathbf{P}, \mathbf{P}_{,X}, \theta, \dot{\theta}, \theta_{,X}; X), \tag{2.12}$$

$$\eta = \hat{\eta}(\mathbf{P}, \mathbf{P}_{,X}, \theta, \dot{\theta}, \theta_{,X}; X), \tag{2.13}$$

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{P}, \mathbf{P}_{,X}, \theta, \dot{\theta}, \theta_{,X}; X), \tag{2.14}$$

$$\mathbf{M} = \hat{\mathbf{M}}(\mathbf{P}, \mathbf{P}_{,X}, \theta, \dot{\theta}, \theta_{,X}; X), \tag{2.15}$$

$$q = \hat{q}(\mathbf{P}, \mathbf{P}_{,X}, \theta, \hat{\theta}, \theta_{,X}; X). \tag{2.16}$$

Note that no dependence on $\dot{\mathbf{P}}$ is assumed because the axiom of objectivity eliminates such constitutive dependences. With the above descriptions the inequality (2.11) expands to

$$\left(\mathbf{T} - \frac{\partial\psi}{\partial\mathbf{P}_{,X}}\right) \circ \dot{\mathbf{P}} + \left(\mathbf{M} - \frac{\partial\psi}{\partial\mathbf{P}}\right) \circ \dot{\mathbf{P}} - \left(\eta + \frac{\partial\psi}{\partial\theta}\right)\dot{\theta} - \left(\frac{\partial\psi}{\partial\dot{\theta}}\right)\ddot{\theta} - \left(\frac{\partial\psi}{\partial\theta_{,X}}\right)\dot{\theta}_{,X} + (\theta q^{-1})\theta_{,X} \geqslant 0. \tag{2.17}$$

We require that the inequality (2.17) hold for all arbitrary variations in $\dot{\mathbf{P}}_{,X}, \dot{\mathbf{P}}, \hat{\mathbf{P}}, \hat{\theta}, \dot{\theta}_{,X}$. We then find that

$$\mathbf{T} = \frac{\partial\psi}{\partial\mathbf{P}_{,X}}, \quad \frac{\partial\psi}{\partial\dot{\mathbf{P}}} = \frac{\partial\psi}{\partial\dot{\theta}} = \frac{\partial\psi}{\partial\theta_{,X}} = 0, \quad M = \frac{\partial\psi}{\partial\mathbf{P}}, \tag{2.18}$$

$$-\left(\eta + \frac{\partial\psi}{\partial\theta}\right)\dot{\theta} + \theta^{-1} q\theta_{,X} \geqslant 0. \tag{2.19}$$

If we assume that thermodynamic equilibrium is achieved where the temperature is uniform throughout the body and is constant in time, i.e. $\dot{\theta} = \theta_{,X} = 0$, for all X and t, then it immediately follows from Eqs. $(2.18)_2$ and the inequality (2.19) that

$$\eta|_E = -\left.\frac{\partial\psi}{\partial\theta}\right|_E, \quad q|_E = 0, \tag{2.20}$$

$$-2\left.\frac{\partial\eta}{\partial\dot{\theta}}\right|_E \geqslant 0, \tag{2.21}$$

$$-\left.\frac{\partial\eta}{\partial\theta_{,X}}\right|_E + \theta^{-1}\left.\frac{\partial q}{\partial\dot{\theta}}\right|_E \geqslant 0, \tag{2.22}$$

$$\theta^{-1}\left.\frac{\partial q}{\partial\theta_{,X}}\right|_E \geqslant 0, \tag{2.23}$$

where $|_E$ implies at equilibrium.

Similarly, we can define a thermodynamical steady state situation such that $\dot{\theta} = 0$ for all X and t. Thus we are led to the inequality

$$\theta^{-1} q|_{SS}\theta_{,X} \geqslant 0. \tag{2.24}$$

One may use more sophisticated entropy principles (MÜLLER [17], GREEN and LAWS [18]) and arrive at slightly different constitutive relations.

The constitutive functions finally take the form

$$\psi = \hat{\psi}(\mathbf{P}, \mathbf{P}_{,X}, \theta; X), \tag{2.25}$$

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{P}, \mathbf{P}_{,X}, \theta; X), \tag{2.26}$$

$$\mathbf{M} = \hat{\mathbf{M}}(\mathbf{P}, \mathbf{P}_{,X}, \theta; X), \tag{2.27}$$

$$\eta = \hat{\eta}(\mathbf{P}, \mathbf{P}_{,X}, \dot{\theta}, \theta_{,X}, \theta; X), \tag{2.28}$$

$$q = \hat{q}(\mathbf{P}, \mathbf{P}_{,X}, \dot{\theta}, \theta_{,X}, \theta; X). \tag{2.29}$$

Note that if

$$f = \hat{f}(\mathbf{P}, \mathbf{P}_{,X}, \theta, \theta_{,X}, \dot{\theta}; X), \tag{2.30}$$

$$g = \hat{g}(P, {}_{,X}, \theta, P\,\theta_{,X}, \dot{\theta}; X), \tag{2.31}$$

are arbitrary twice differentiable functions satisfying

$$-f\dot{\theta} + g\theta_{,X} \geqslant 0, \tag{2.32}$$

then we can write from the inequalities (2.19) and (2.32)

$$\eta = \frac{-\partial\psi}{\partial\theta} + f - a_1\dot{\theta} + a_2\theta_{,x}, \tag{2.33}$$

$$\theta^{-1}q = g + b_1\dot{\theta} + b_2\theta_{,x}, \tag{2.34}$$

$$a_1 \geqslant 0, \quad b_2 \geqslant 0, \quad a_2 + b_1 \geqslant 0. \tag{2.35}$$

If there exists a differential constraint of the form

$$\mathbf{\Phi}_1 \circ \mathbf{P}_{,x} + \mathbf{\Phi}_2 \circ \mathbf{P} + \mathbf{\Phi}_3\dot{\theta} + \mathbf{\Phi}_4\hat{\theta} + \mathbf{\Phi}_5\dot{\theta}_{,x} = 0, \tag{2.36}$$

then

$$\mathbf{T} = \frac{\partial\psi}{\partial\mathbf{P}_{,x}} + \lambda\mathbf{\Phi}, \tag{2.37}$$

$$\mathbf{M} = \frac{\partial\psi}{\partial\mathbf{P}} + \lambda\mathbf{\Phi}_2, \tag{2.38}$$

$$-\left(\eta + \frac{\partial\psi}{\partial\theta} + \lambda\Phi_3\right)\dot{\theta} + (\theta^{-1}q\theta_{,x}) \geqslant 0, \tag{2.39}$$

$$\frac{\partial\psi}{\partial\mathbf{P}} = 0, \quad \frac{\partial\psi}{\partial\dot{\theta}} - \lambda\Phi_4 = 0, \quad \frac{\partial\psi}{\partial\theta_{,x}} - \lambda\Phi_5 = 0. \tag{2.40}$$

Finally, we present here equations corresponding to a linearized model:

$$\begin{aligned}
\psi &= {}^1/_2 C_1 P_{,x}^2 + C_2\tilde{P}^2 + C_3\tilde{\mathbf{P}} \circ \mathbf{P}_{,x},\\
\mathbf{T} &= C_1\mathbf{P}_{,x} + C_3\tilde{\mathbf{P}},\\
\mathbf{M} &= C_3\mathbf{P}_{,x} + C_2\mathbf{P},\\
\eta &= f - a_1\dot{\theta} + a_2\theta_{,x},\\
\theta^{-1}q &= g + b_1\dot{\theta} + b_2\theta_{,x},\\
\mathscr{K}\ddot{\mathbf{P}} &= C_1\mathbf{P}_{,xx} + C_3\tilde{\mathbf{P}}_{,x} + \mathscr{K}\mathbf{F} - C_3\mathbf{P}_{,x} + C_2\tilde{\mathbf{P}},
\end{aligned}$$

where $C_1, C_2, C_3, a_1, a_2, b_1, b_2$ are material constants.

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COLLEGE OF ENGINEERING
PAHLAVI UNIVERSITY, SHIRAZ, IRAN.

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