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# On the solution of a certain class of spatial problems in the theory of plastic flow

## I. Foundations

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IN THIS paper, a method is proposed for solution of certain spatial problems of the theory of rigid ideal plastic bodies. The method is based on certain restrictions imposed on the three-dimensional stress and velocity fields. In the first part of the paper, by virtue of the integral principles, the three-dimensional boundary value problem of the theory of plastic flow is reduced to the one-parameter family of two-dimensional boundary value problems. The solutions of these problems must also satisfy certain extra physical criteria of applicability. Applications of the general approach given here will be developed in the second part of the paper.

W pracy zaproponowano metodę rozwiązywania pewnych przestrzennych zagadnień teorii ciał sztywno idealnie plastycznych. Podstawą metody są pewne ograniczenia nałożone na trójwymiarowe pola naprężenia i prędkości. W pierwszej części pracy, korzystając z zasad całkowych, zredukowano trójwymiarowe zagadnienie brzegowe teorii plastycznego płynięcia do jednoparametrowej rodziny dwuwymiarowych zagadnień brzegowych. Ponadto rozwiązania powinny spełniać pewne kryteria fizycznej stosowalności przyjętych założeń. Zastosowania podanej tu ogólnej metody będą przedstawione w drugiej części pracy.

В работе предложен метод решения некоторых пространственных проблем теории жестко-идеально пластических тел. Основой метода являются некоторые ограничения, наложенные на трехмерные поля напряжений и скоростей. В первой части работы, используя интегральные принципы, трехмерная краевая задача теории пластического течения сведена к однопараметрическому семейству двухмерных краевых задач. Кроме этого решения должны удовлетворять некоторым физическим критериям применяемости принятых предположений. Применения, приведенного здесь общего подхода, будут представлены во второй части работы.

### Introduction

THE KNOWN solutions in the theory of plastic flow are usually obtained as solutions of boundary value problems on the plane for the hyperbolic system of partial differential equations. Such two-dimensional boundary value problems are formulated by applying the semi-inverse method (cf. [6] pp. 152, 240) to the limit analysis of plane or axially-symmetric states of plastic flow. An extensive bibliography referring to these problems can be found in [13]. The spatial — i.e. three-dimensional-problems of plastic flow, however, have not so far, apart from certain special cases (cf. [3] pp. 222–224), been investigated.

The aim of the present paper is to perform a limit analysis of a certain class of three-dimensional problems in the theory of plastic flow. To obtain solutions of the problems under consideration, procedures analogous to those used for plane problems can be applied. It is shown that plane strain problems, together with axially-symmetric strain problems of plastic flow, constitute special cases of the more general approach given in the paper.

The approach presented concerns only isotropic rigid-ideal plastic materials in which the generalized Coulomb flow law associated (or not) with the yield condition or the Levy-Mises yield condition must hold. Moreover, this approach can be applied only to certain systems of external loads and certain configurations of the plastic regions inside the body. Suitable criteria of applicability of the proposed limit analysis are given.

In Sect. 1, the postulates constituting the foundations of the analysis are listed. The conception of the approach, formulated in Sect. 3, is based on certain hypothesis concerning state of stress and the distribution of velocities. The governing equations are obtained in Sect. 4, and the criteria under which the limit analysis has physical meaning are introduced in Sect. 5. The plane strain problem and the axially-symmetric problem are in Sect. 6, derived as special cases of the general spatial approach. In Sect. 7 are given examples of application of the proposed analysis. Here we confine ourselves to analytical solutions of simple three-dimensional problems only; more complex spatial problems can be treated by applying known numerical methods to the equations given in the paper.

## Notation

All basic relations of the paper are carried out in a certain orthogonal curvilinear coordinate system $\{z^\alpha\}$ defined in the region $V$ of the physical space(1). The system $\{z^\alpha\}$ is related to the fixed Carthesian orthogonal system $\{x^i\}$ in the physical space:

$$x^i = x^i(z^1, z^2, z^3), \tag{0.1}$$

where the functions on the right-hand sides of the Eq. (0.1) are assumed to be defined and differentiable in $V$ and where $\det[\partial x^i/\partial z^\alpha] \neq 0$. The region $V$ represents the plastic zone of the body under consideration, and an explicit form of Eq. (0.1) will be given in Sect. 3. The indices $i, j$, together with $\alpha, \beta, \gamma, \delta$ run over the sequence 1, 2, 3; the indices $K, L$ run over the sequence 1, 2. The summation convention holds(2). Partial derivatives are denoted by a comma:

$$f_{,\alpha} \equiv \frac{\partial f(z^1, z^2, z^3)}{\partial z^\alpha}, \quad g_{,i} \equiv \frac{\partial g(x^1, x^2, x^3)}{\partial x^i} \quad \text{etc.} \tag{0.2}$$

Components of the metric tensor in the system $\{z^\alpha\}$ are denoted by $g_{\alpha\beta}$ and are given by

$$g_{\alpha\beta} = x^i_{,\alpha} x^j_{,\beta} \delta_{ij}, \tag{0.3}$$

where $\delta_{ij}$ is the Kronecker symbol. The elements of the matrix inverse to $[g_{\alpha\beta}]$ are denoted by $g^{\alpha\beta}$. Covariant derivatives in the system $\{z^\alpha\}$ of vector or tensor fields are denoted by a vertical line:

$$\begin{aligned} v_\alpha|_\beta &= v_{\alpha,\beta} - \begin{Bmatrix}\gamma \\ \alpha\beta\end{Bmatrix} v_\gamma, \\ T^{\alpha\beta}|_\gamma &= T^{\alpha\beta}_{,\gamma} + \begin{Bmatrix}\alpha \\ \delta\gamma\end{Bmatrix} T^{\delta\beta} + \begin{Bmatrix}\beta \\ \delta\gamma\end{Bmatrix} T^{\alpha\delta} \quad \text{etc.,} \end{aligned} \tag{0.4}$$

(1) In some special cases the system $\{z^\alpha\}$ can also be assumed as a Carthesian coordinate system.

(2) No summation is carried out if the same index is repeated more then twice or if it is repeated on the same level.

where the Cristoffel symbols $\begin{Bmatrix} \alpha \\ \delta\gamma \end{Bmatrix}$ can be calculated from (0.1) by means of the known formulas:

$$\begin{Bmatrix} \alpha \\ \delta\gamma \end{Bmatrix} = z^{\alpha}{}_{,i} x^{i}{}_{,\delta\gamma}; \tag{0.5}$$

the matrix $[z^{\alpha}{}_{,i}]$ is an inverse of the matrix $[x^{i}{}_{,\alpha}]$. We also introduce the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the Carthesian components of which are equal to

$$e_{\alpha}{}^{i} = \frac{x^{i}{}_{,\alpha}}{\sqrt{g_{\alpha\alpha}}}, \tag{0.6}$$

where $g_{\alpha\alpha} = x^{i}{}_{,\alpha} x^{j}{}_{,\alpha} \delta_{ij}$ (cf. Eq. (0.3)). We also denote by $e^{\alpha}{}_{i}$ the component of the matrix inverse to the matrix $[e_{\alpha}{}^{i}]$. Since the matrix $[z^{\alpha}{}_{,i}]$ is an inverse of the matrix $[x^{i}{}_{,\alpha}]$, we obtain:

$$e^{\alpha}{}_{i} = z^{\alpha}{}_{,i} \sqrt{g_{\alpha\alpha}}\,. \tag{0.7}$$

Using the Eqs. (0.6), (0.7) we define the physical components of vector and tensor fields by means of the known transformation formulas:

$$\begin{aligned} \sigma^{\alpha\beta} &= e^{\alpha}{}_{i} e^{\beta}{}_{j} x^{i}{}_{,\gamma} x^{j}{}_{,\delta} T^{\gamma\delta} = \sqrt{g_{\alpha\alpha} g_{\beta\beta}}\; T^{\alpha\beta}, \\ d_{\alpha\beta} &= e_{\alpha}{}^{i} e_{\beta}{}^{j} z^{\gamma}{}_{,i} z^{\delta}{}_{,j} \xi_{\gamma\delta} = \frac{\xi_{\alpha\beta}}{\sqrt{g_{\alpha\alpha} g_{\beta\beta}}} \qquad \text{etc.}, \end{aligned} \tag{0.8}$$

where $T^{\alpha\beta}$, $\xi_{\alpha\beta}$ are components of tensors in the coordinate system $\{z^{\alpha}\}$, and $\sigma^{\alpha\sigma}$, $d_{\alpha\beta}$ are physical components of these tensors, respectively, related to this system. Other basic denotations used throughout the paper are listed below:

- $V$ part of the physical space occupied by the material in the limit state (the plastic zone),
- $S$ boundary of the region $V$,
- $\bar{V}$ region $V$ with its boundary,
- $p^{\alpha}$ components of the vector of external surface loads acting on $V$ across $S$,
- $S^{\alpha}$ part of $S$, where the surface load component $p^{\alpha}$ is known,
- $v_{\alpha}$ components of the velocity vector,
- $S_{\alpha}$ part of $S$, on which the velocity component $v_{\alpha}$ is known,
- $f^{\alpha}$ components of density of external loads,
- $\bar{f}^{\alpha}, \bar{p}^{\alpha}$ components of internal body forces and internal surface tractions, respectively,
- $T^{\alpha\beta}$ components of the stress tensor,
- $\sigma^{\alpha\beta}$ physical components of the stress tensor,
- $\sigma_1, \sigma_2, \sigma_3$ principal normal stresses,
- $\xi_{\alpha\beta}$ components of the strain rate tensor,
- $d_{\alpha\beta}$ physical components of the strain rate tensor,
- $d_1, d_2, d_3$ principal values of the matrix $[d_{\alpha\beta}]$.
- $\Sigma_{\mathrm{v}}, \Sigma_{\mathrm{T}}$ surface inside $V$ across which the fields $v_{\alpha}$, $T^{\alpha\beta}$ suffer discontinuities, respectively,
- $D$ power of the plastic deformation (dissipation).

## 1. Fundamentals of the theory

In this Section we shall give those basic relations of the theory of plastic flow which will be used throughout the paper. Let $V$ be a regular region of the physical space, occupied by the material in the limit state, and let $S$ be its boundary. We assume that the material of the body is isotropic and rigid-ideal plastic. This means that at each point of $V$, the yield condition holds in the form:

$$F(\sigma_1, \sigma_2, \sigma_3) = 0, \tag{1.1}$$

where $\sigma_\alpha$ are principal stresses and $F$ is a symmetrical function of all arguments. At the same time, the remaining part of the body is assumed to be rigid. Denoting by $G(\sigma_1, \sigma_2, \sigma_3)$ the plastic potential of an isotropic material, we postulate the equation of plastic flow in the known form:

$$d_\alpha = \lambda \frac{\partial G(\sigma_1, \sigma_2, \sigma_3)}{\partial \sigma_\alpha}, \qquad \xi_{\alpha\beta} = \lambda \frac{\partial G}{\partial T^{\alpha\beta}}, \tag{1.2}$$

where $\lambda \geqslant 0$ is a certain skalar factor which must be determinated in each problem, and $d_\alpha$ are principal values of the strain rate tensor $\xi_{\alpha\beta}$. The special forms of functions $F$ and $G$ with which we have to deal will be specified in Sect. 3.

We denote by $\Sigma_v$ a smooth surface, oriented by the unit normal vector $n_\beta$, across which the flow velocity vector field $v_\alpha$ suffers discontinuity. The jump of the velocity field across $\Sigma_v$ will be denoted by $[v_\alpha] = v_\alpha^+ - v_\alpha^-$, where $v_\alpha^+$, $v_\alpha^-$ are the limit values of the field

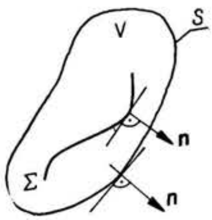


FIG. 1.

$v_\alpha(z^1, z^2, z^3)$ on both sides of $\Sigma_v$ (cf. Fig. 1). Apart from the special situations (see for example [10] pp. 70–71) we assume that $[v_\alpha]n^\alpha = 0$ — i.e. we assume that the projection of the velocity vector in the direction normal to $\Sigma_v$ is continuous across $\Sigma_v$. Analogously, by $\Sigma_T$ we denote the smooth surface across which the stress tensor suffers discontinuities.

We denote by $S_\alpha$ the part of the boundary $S$ on which the flow velocity component $v_\alpha$ is known:

$$v_\alpha = \mathring{v}_\alpha \quad \text{on} \quad S_\alpha. \tag{1.3}$$

Analogously, by $S^\alpha$ we denote the part of $S$ on which the component $p^\alpha$ of the surface traction is prescribed(³):

$$p^\alpha = \mathring{p}^\alpha \quad \text{on} \quad S_\alpha. \tag{1.4}$$

Moreover, for any fixed $\alpha$, the surfaces $S_\alpha$ and $S^\alpha$ have no common points, and $\bar{S}_\alpha \cap \bar{S}^\alpha = S$. We assume that the functions $p^\alpha$ are continuous almost everywhere on $S^\alpha$, $f^\alpha$ are continuous

(³) Usually, it is assumed that $S_1 = S_2 = S_3$ (cf. [6] p. 304) and $S^1 = S^2 = S^3$. In this paper, the surfaces $S_\alpha$ (as also $S^\alpha$, $\alpha = 1, 2, 3$) can be different for different $\alpha$.

almost everywhere in $V$, $T^{\alpha\beta}$ are continuous with their first derivatives in $V-\Sigma_T$, and $[T^{\alpha\beta}]n_\beta = 0$ on $\Sigma_T$. Moreover, we assume that $v_\alpha$ are continuous with their first derivatives in $\bar{V}-\Sigma_v$, and $\xi_{\alpha\beta}$ are continuous in $V-\Sigma_v$.

The approach applied in what follows will be based on certain restrictions given a priori, imposed on the stress tensor field $T^{\alpha\beta}(z^1, z^2, z^3)$ and the flow velocity field $v_\alpha(z^1, z^2, z^3)$ in the region $V$. The exact form of these restrictions will be specified in Sect. 3. If such restrictions are postulated, then the integral principles of continuum mechanics must be used instead of the known differential equilibrium equations $T^{\alpha\beta}|_\beta+f^\alpha = 0$, the boundary conditions $T^{\alpha\beta}n_\beta = p^\alpha$, and the kinematical equations $\xi_{\alpha\beta} = v_{(\alpha|\beta)}$(4). Before we formulate these integral principles, we bear in mind the known concepts of the virtual increments of the fields defined above.

An arbitrary field $\overset{*}{v}_\alpha(z^1, z^2, z^3)$ is said to be the virtual increment of the velocity field $v_\alpha(z^1, z^2, z^3)$ if the field $v_\alpha+\overset{*}{v}_\alpha$ satisfies regularity conditions, boundary conditions and restrictions imposed on the velocity field $v_\alpha$, provided that such conditions and restrictions are linear.

The field $\overset{*}{\xi}_{\alpha\beta}(z^1, z^2, z^3)$ defined by $\overset{*}{\xi}_{\alpha\beta} \equiv \frac{1}{2}(\overset{*}{v}_{\alpha|\beta}+\overset{*}{v}_{\beta|\alpha})$ in $V-\Sigma_v$, will be called the virtual increment of the strain rate tensor field.

An arbitrary field $\overset{*}{T}{}^{\alpha\beta}(z^1, z^2, z^3)$, $\overset{*}{T}{}^{\alpha\beta} = \overset{*}{T}{}^{\beta\alpha}$, is said to be the increment of the stress tensor field $T^{\alpha\beta}(z^1, z^2, z^3)$ if $T^{\alpha\beta}+\overset{*}{T}{}^{\alpha\beta}$ satisfies the regularity conditions, boundary conditions and restrictions imposed on the stress tensor field $T^{\alpha\beta}$, provided that such conditions and restrictions are linear.

The field $\overset{*}{p}{}^\alpha(z^1, z^2, z^3)$, defined almost everywhere on $S$, and the field $\overset{*}{f}{}^\alpha(z^1, z^2, z^3)$, defined in $V$ by means of the formulas $\overset{*}{p}{}^\alpha \equiv \overset{*}{T}{}^{\alpha\beta}n_\beta$, $\overset{*}{f}{}^\alpha \equiv -\overset{*}{T}{}^{\alpha\beta}|_\beta$, will be called respectively the virtual increment of surface tractions and the virtual increment of external loads.

Now, we can formulate two integral principles of continuum mechanics, which constitute the basis for further considerations.

*Principle of the virtual work*. If the fields $p^\alpha$, $f^\alpha$ determine the external loads on $S$ and in $V$ respectively, and the field $T^{\alpha\beta}$ characterizes the state of stress in $V$, then the following relation

$$\oint_S p^\alpha \overset{*}{v}_\alpha dS+ \int_V f^\alpha \overset{*}{v}_\alpha dV = \int_V T^{\alpha\beta}\overset{*}{\xi}_{\alpha\beta} dV+ \int_{\Sigma_v} T^{\alpha\beta}n_\beta[\overset{*}{v}_\alpha]d\Sigma \tag{1.5}$$

must hold for any virtual increments $\overset{*}{v}_\alpha$, $\overset{*}{\xi}_{\alpha\beta}$.

*Principle of the complementary virtual work*. If the fields $v_\alpha$, $\xi_{\alpha\beta}$ determine the velocity of the flow and the strain rate in the body, respectively, then the following relation

$$\oint_S \overset{*}{p}{}^\alpha v_\alpha dS+ \int_V \overset{*}{f}{}^\alpha v_\alpha dV = \int_V \overset{*}{T}{}^{\alpha\beta}\xi_{\alpha\beta} dV+ \int_{\Sigma_v} \overset{*}{T}{}^{\alpha\beta}n_\beta[v_\alpha]d\Sigma \tag{1.6}$$

must hold for any virtual increments $\overset{*}{T}{}^{\alpha\beta}$, $\overset{*}{p}{}^\alpha$, $\overset{*}{f}{}^\alpha$.

(4) It can be proved (see Sects. 5 and 7) that the restrictions imposed on the fields $v_\alpha$, $T^{\alpha\beta}$ are not in general consistent with the differential equilibrium equations, the boundary conditions and the kinematical equations in their classical form given above.

Moreover, the stress field $T^{\alpha\beta}$ and the strain rate field $\xi_{\alpha\beta}$ must satisfy in $V-\Sigma_v-\Sigma_T$ the inequality:

$$D = T^{\alpha\beta}\xi_{\alpha\beta} = \sigma^{\alpha\beta}d_{\alpha\beta} \geqslant 0, \tag{1.7}$$

which represents the known dissipation condition (cf. [10] p. 41). The suitable dissipation condition must also be satisfied on the surface $\Sigma_v$ (cf. [10] p. 66 and Sect. 5 of this paper).

## 2. Geometry of the plastic zone

Hitherto, the form of the plastic zone $V$ and the curvilinear coordinate system $\{z^\alpha\}$ contained in it have been quite arbitrary. In the present paper, we shall deal only with special forms of the plastic zone parametrized by special systems of coordinates in $V$. A typ-

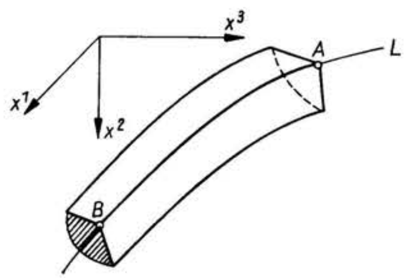


FIG. 2.

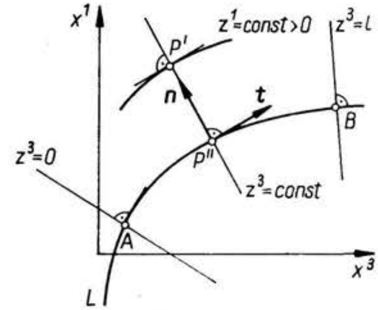


FIG. 3.

ical example of $V$ is given in Fig. 2, where $L$ is a smooth curve situated on the horizontal plane $x^2 = 0$ belonging to the Carthesian orthogonal coordinate system $0x^1x^2x^3$. Let $s$ be the length parameter on the arc $L_{AB}$ of $L$. Moreover, let

$$x^1 = \varphi^1(s), \quad x^3 = \varphi^3(s) \tag{2.1}$$

be a parametric equation of $L_{AB}$. To specify the system of the curvilinear coordinates $z^1, z^2, z^3$ in $V$, we denote by $P'$ an arbitrary point of $V$ situated on the plane $x^2 = 0$. Let $P''$ be an orthogonal projection of $P'$ on $L$. Two unit vectors, one of them being tangent and the other normal to $L$, we denote by $\mathbf{t}$ and $\mathbf{n}$, respectively. Let $z^3 = s =$ const. be a plane normal to $L_{AB}$ and passing by an arbitrary point $P''$, and let $z^1$ be the distance $P'P''$ between the point $P'$ and the curve $L$, provided that $P'$ is situated on the right-hand side on the arc $L_{AB}$ oriented by a unit normal $\mathbf{n}$. It follows that the Carthesian orthogonal coordinates $x^1, x^3$ of the point $P'$ on the plane $x^2 = 0$ are related to the coordinates $z^1, z^3$ on this plane by the relations (cf. Fig. 3)

$$\begin{aligned} x^3 &= \varphi^3(z^3)+n^3(z^3)z^1, \\ x^1 &= \varphi^1(z^3)+n^1(z^3)z^1. \end{aligned} \tag{2.2}$$

At the same time we have $n^3 = t^1$, $n^1 = -t^3$ and $t^3 = d\varphi^3/dz^3$, $t^1 = d\varphi^1/dz^3$. Now let $P$ be an arbitrary point of the region $V$. The curvilinear coordinates $z^1, z^2, z^3$ of the point $P$

will be related to the rectilinear Carthesian coordinates $x^1$, $x^2$, $x^3$ by means of the formulas:

$$
\begin{aligned}
x^1 &= \varphi^1(z^3) - \frac{d\varphi^3(z^3)}{dz^3} z^1,\\
x^2 &= z^2,\\
x^3 &= \varphi^3(z^3) + \frac{d\varphi^1(z^3)}{dz^3} z^1.
\end{aligned}
\tag{2.3}
$$

The Eqs. (2.3) represent a special case of Eqs. $(0.1)_2$. We assume that the region $V$ is uniquely parametrized by the coordinates $z^1$, $z^2$, $z^3$. From the formulas (2.3) and Eqs. (0.5), we obtain the following form of Christoffel symbols:

$$
\begin{aligned}
&\left\{ {1 \atop 3\ 3} \right\} = \varkappa(1 - z^1\varkappa),\\
&\left\{ {3 \atop 1\ 3} \right\} = \left\{ {3 \atop 3\ 1} \right\} = -\frac{\varkappa}{1 - z^1\varkappa},
\end{aligned}
\tag{2.4}
$$

where $\varkappa$ is a curvature of $L$ given by

$$
\varkappa = t^1 \frac{dt^3}{dz^3} - t^3 \frac{dt^1}{dz^3}.
\tag{2.5}
$$

The Christoffel symbols not mentioned in (2.4) are equal to zero. From Eqs. (2.3) and (0.3), we obtain the components $g_{\alpha\beta}$ of the metric tensor of the system $\{z^\alpha\}$:

$$
[g_{\alpha\beta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-z^1\varkappa)^2 \end{bmatrix}, \quad 1 - z^1\varkappa > 0.
\tag{2.6}
$$

Using Eq. (2.6), we obtain from Eqs. (0.8) the physical components $\sigma^{\alpha\beta}$, $d_{\alpha\beta}$ of the stress and strain rate, respectively, in the form:

$$
\begin{aligned}
[\sigma^{\alpha\beta}] &= \begin{bmatrix} T^{11} & T^{12} & (1-z^1\varkappa)T^{13} \\ T^{21} & T^{22} & (1-z^1\varkappa)T^{23} \\ (1-z^1\varkappa)T^{31} & (1-z^1\varkappa)T^{32} & (1-z^1\varkappa)^2 T^{33} \end{bmatrix},\\
[d_{\alpha\beta}] &= \begin{bmatrix} \xi_{11} & \xi_{12} & \dfrac{\xi_{13}}{1-z^1\varkappa} \\ \xi_{21} & \xi_{22} & \dfrac{\xi_{23}}{1-z^1\varkappa} \\ \dfrac{\xi_{31}}{1-z^1\varkappa} & \dfrac{\xi_{32}}{1-z^1\varkappa} & \dfrac{\xi_{33}}{(1-z^1\varkappa)^2} \end{bmatrix}.
\end{aligned}
\tag{2.7}
$$

From now on, all vector and tensor fields used throughout this paper will be related to a coordinate system $\{z^\alpha\}$ given in $V$ by Eqs. (2.3), where Eqs. (2.1) represent an arbitrary smooth curve $L$, situated on a horizontal plane (i.e. the plane $x^2 = 0$ in the Carthesian coordinate system $0x^1x^2x^3$).

## 3. Basic hypothesis and formulation of the problem

An approach to the problem will be based on the limiting analysis of the three-dimensional problems of plastic flow in $V$ (all fields are, in general, dependent on the three variables $z^1$, $z^2$, $z^3$) by means of certain restrictions, given a priori, imposed on the state of stress $T^{\alpha\beta}$ and the velocity of flow $v_\alpha$ in $V$. These restrictions enable us to solve a certain class of three-dimensional problems of the theory of rigid-ideal plastic bodies.

Making use of the curvilinear coordinate system $\{z^\alpha\}$, we postulate that the following two restrictions must hold in $V$:

1. The shear stresses in the planes $z^3 = \text{const.}$ can be disregarded:

$$T^{31} = 0, \quad T^{32} = 0. \tag{3.1}$$

From this assumption, it follows that one of the principal directions of the stress is normal to the planes $z^3 = \text{const.}$

2. The component of the flow velocity vector normal to an arbitrary plane $z^3 = \text{const.}$ can be disregarded:

$$v^3 = 0. \tag{3.2}$$

The Eq. (2.5) states that the flow velocity vectors in $V$ are situated in the planes $z^3 = \text{const.}$ normal to the curve $L$.

The assumptions (2.4), (2.5) are approximatively fulfilled only for a special class of three-dimensional problems of plastic flow. The scope of application of an approach based on the restrictions (3.1), (3.2) will be analysed in Sect. 5. We shall show that the restrictions (3.1), (3.2) enable us to reduce the three-dimensional boundary value problem of plastic flow to a single parameter family (with coordinate $z^3$ as the parameter) of two-dimensional boundary value problems of the hyperbolic type in the variables $z^1$, $z^2$. However, such reduction is possible only for certain kinds of ideal plastic materials.

In the present paper, the plastic potential $G(\sigma_1, \sigma_2, \sigma_3)$ will be assumed in the form $G = \sigma_i - \sigma_j - (\sigma_i + \sigma_j)\sin\overset{*}{\varrho} - c\cos\overset{*}{\varrho}$; the relation $G = \text{const.}$ is defined for any $i, j = 1, 2, 3$; $i \neq j$, and represents a certain convex surface in the space of the principal stresses $\sigma_1, \sigma_2, \sigma_3$. The form of the plastic potential given above was introduced by D. Radencovic (cf. [8] and [10] p. 43) in problems of solid mechanics, where $c$ is a coefficient of cohesion and $\overset{*}{\varrho}$ is a known constant such that $0 < \overset{*}{\varrho} < \varrho < \dfrac{\pi}{2}$, where $\varrho$ is the angle of internal friction. The surfaces $G(\sigma_1, \sigma_2, \sigma_3) = \text{const.}$ are not smooth and are defined by means of six different analytical relations. To make further calculations more concise, we shall extend the domain of the definition of the plastic potential by introducing extra parameters $\mu$, $\nu$ where $0 \leqslant \mu \leqslant 1$, $\nu = 0$ or $\nu = 1$, putting

$$\begin{aligned} \overline{G} = \overline{G}(\sigma_1, \sigma_2, \sigma_3; \mu, \nu) \equiv{} & (1-\sin\overset{*}{\varrho})(1-\nu+\mu\nu)\sigma_1 \\ & + (1+\sin\overset{*}{\varrho})(\mu\nu-\mu-\nu)\sigma_2 + (1-\mu)(2\nu-1-\sin\overset{*}{\varrho})\sigma_3 - 2c\cos\overset{*}{\varrho}. \end{aligned} \tag{3.3}$$

The function $\overline{G}(\sigma_1, \sigma_2, \sigma_3; \mu\,\nu)$ represents a new analytical form of the plastic potential, and has the same physical meaning as the latter if the domain of the function

**Table 1**

| | $\nu = 0$ | $\nu = 1$ | |
|---|---|---|---|
| $\mu = 0$ <br> $\sigma_1 > \sigma_3$ | $G = (1-\sin\overset{*}{\varrho})\sigma_1 - (1+\sin\overset{*}{\varrho})\sigma_3 - 2c\cos\overset{*}{\varrho}$ | $G = -(1+\sin\overset{*}{\varrho})\sigma_2 + (1-\sin\overset{*}{\varrho})\sigma_3 - 2c\cos\overset{*}{\varrho}$ | $\mu = 0$ <br> $\sigma_3 < \sigma_2$ |
| $0 < \mu < 1$ <br> $\sigma_2 = \sigma_3$ | $G = (1-\sin\overset{*}{\varrho})\sigma_1 - (1+\sin\overset{*}{\varrho})\mu\sigma_2 - (1+\sin\overset{*}{\varrho})(1-\mu)\sigma_3 - 2c\cos\overset{*}{\varrho}$ | $G = -(1+\sin\overset{*}{\varrho})\sigma_2 + (1-\sin\overset{*}{\varrho})\mu\sigma_1 - (1-\sin\overset{*}{\varrho})(1-\mu)\sigma_3 - 2c\cos\overset{*}{\varrho}$ | $0 < \mu < 1$ <br> $\sigma_1 = \sigma_3$ |
| $\mu = 1$ <br> $\sigma_1 > \sigma_2$ | $G = [(1-\sin\overset{*}{\varrho})\sigma_1 - (1+\sin\overset{*}{\varrho})\sigma_2 - 2c\cos\overset{*}{\varrho}](2\sigma_3 + 2\zeta)$ | $G = [-(1+\sin\overset{*}{\varrho})\sigma_2 + (1-\sin\overset{*}{\varrho})\sigma_1 - 2c\cos\overset{*}{\varrho}](2\sigma_3 + 2\zeta)$ | $\mu = 1$ <br> $\sigma_1 > \sigma_2$ |
| | $\zeta = 0.5 - \sigma_3$ if Eq. (3.3) holds, | $\mu = 1$, $\zeta = c$, $\overset{*}{\varrho} = 0$, $\sigma_3 = 0.5(\sigma_1+\sigma_2)$ if Eq. (4.16) holds | |

$\bar{G}(\sigma_1, \sigma_2, \sigma_3; \mu, \nu)$ is restricted to the value given in Table 1. Using (1.2) and (3.3), we obtain:

$$
\begin{aligned}
d_1 &= \lambda(1-\nu+\mu\nu)(1-\sin\overset{*}{\varrho}),\\
d_2 &= \lambda(\nu\mu-\mu-\nu)(1+\sin\overset{*}{\varrho}),\\
d_3 &= \lambda(1-\mu)(2\nu-1)[1-(2\nu-1)\sin\overset{*}{\varrho}].
\end{aligned}
\tag{3.4}
$$

The yield condition of the materials under consideration will be assumed in the form which corresponds to that given in [8]:

$$
(1-\sin\varrho)(1-\nu+\mu\nu)\sigma_1 + (1+\sin\varrho)(\mu\nu-\mu-\nu)\sigma_2 + (1+\mu)(2\nu-1-\sin\varrho)\sigma_3 - 2c\cos\varrho = 0,
\tag{3.5}
$$

where the parameters $\mu$, $\nu$ have the same meaning as before.

In what follows, we shall assume that $\sigma_1 \geqslant \sigma_3 \geqslant \sigma_2$. Introducting the function $\eta \equiv \equiv (\sigma_3-\sigma_2)/(\sigma_2-\sigma_1)$, we have:

$$
\sigma_3 = (1-\eta)\sigma_2 + \eta\sigma_1, \quad 0 \leqslant \eta \leqslant 1.
\tag{3.6}
$$

For the time being, $\eta$ is an unknown function.

In Sect. 4 we shall prove that in certain special cases Eqs. (3.3), (3.4) and (3.5) hold also for materials in which the plastic potential has the well known form $G = (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 - 6c^2$, and in which the yield condition is given by $G = 0$.

Now we shall list the basic relations of an approach used in this paper. The problem under consideration will be governed by the following relations which have to be satisfied in $\bar{V}$:

1. The yield condition (3.5).
2. The flow law (3.4).
3. The restrictions (3.2), (1.3) imposed on kinematical fields.
4. The restrictions (3.1), (1.4) imposed on the kinetic fields.

5. The principle of the virtual work (1.5).
6. The principle of the complementary virtual work (1.6).
7. The dissipation conditions (1.7) in $V$ and the dissipation conditions on $\Sigma_v$.

However, the part of the body which is situated outside $\bar{V}$ is assumed to be rigid. The basic unknown fields — namely, the velocity of plastic flow $v_\alpha(z^1, z^2, z^3)$, the strain rate $\xi_{\alpha\beta}(z^1, z^2, z^3)$ and the stress $T^{\alpha\beta}(z^1, z^2, z^3)$ — must be sufficiently smooth, and satisfy all the relations listed above. Due to the hypothesis assumed a priori (3.1), (3.2), the solutions of special problems must be verified. This will be done by means of certain criteria, which will be formulated in Sect. 5.

## 4. Field equations

By the field equations we shall mean here the system of equations which can be deduced from the two integral principles (1.5), (1.6). To obtain these equations, we shall transform the integral relation (1.5), using the divergence theorem. Taking into account that $\overset{*}{\xi}_{\alpha\beta} = \overset{*}{v}_{(\alpha|\beta)}$, $T^{\alpha\beta} = T^{\beta\alpha}$, $[T^{\alpha\beta}]n_\beta = 0$ on $\Sigma_T$, we obtain from (1.5) the following relation (5):

$$\int_V T^{\alpha\beta}\overset{*}{v}_{(\alpha|\beta)}dV = \int_V T^{\alpha\beta}\overset{*}{v}_\alpha|_\beta dV = \int_V (T^{\alpha\beta}\overset{*}{v}_\alpha)|_\beta dV - \int_V T^{\alpha\beta}|_\beta \overset{*}{v}_\alpha dV$$

$$= \oint_S T^{\alpha\beta}\overset{*}{v}_\alpha n_\beta dS - \int_{\Sigma_v} T^{\alpha\beta}[\overset{*}{v}_\alpha]n_\beta d\Sigma - \int_V T^{\alpha\beta}|_\beta \overset{*}{v}_\alpha dV.$$

Combining the Eq. (1.5) and the relation given above, we arrive at:

$$\int_V (f^\alpha + \dot{T}^{\alpha\beta}|^\beta)\overset{*}{v}_\alpha dV + \oint_S (p^\alpha - T^{\alpha\beta}n_\beta)\overset{*}{v}_\alpha dS = 0. \tag{4.1}$$

The foregoing relation must hold for any virtual increments $\overset{*}{v}_\alpha$ of the velocity field $v_\alpha$. According to Eq. (3.2) and the Eqs. (1.3), we have $\overset{*}{v}_3 = 0$ in $V$ and $\overset{*}{v}_K = 0$ on $S_K$. Thus, the relation (4.1) reduces to the form:

$$\int_V (f^K + T^{K\beta}|_\beta)\overset{*}{v}_K dV + \oint_S (p^K - T^{K\beta}n_\beta)\overset{*}{v}_K dS = 0 \tag{4.2}$$

and must be satisfied for any continuous vector field $\overset{*}{v}_K$. In particular Eq. (4.2) must be satisfied for the vector field $\overset{*}{v}_K$ such that $\overset{*}{v}_K|_S = 0$. It follows that Eq. (4.2) is satisfied if and only if the relation

$$\int_V (f^K + T^{K\beta}|_\beta)\overset{*}{v}_K dV = 0, \tag{4.3}$$

(5) The divergence theorem is used here in the form:

$$\int_V w^\alpha|_\alpha dV = \oint_S w^\alpha n_\alpha dS - \int_\Sigma [w^\alpha]n_\alpha d\Sigma,$$

where $V$ is a regular region in $R^3$ with a boundary $S$, $\Sigma$ is a smooth oriented surface across which $w^\alpha$ suffers discontinuity, and $w^\alpha \in C^1(\bar{V} - \Sigma)$.

holds for any continuous functions $\overset{*}{v}_K$ in $V$, equal to zero on $S$ and $\Sigma_T$. Because $\overset{*}{v}_K \in C^1(\bar{V})$, $\overset{*}{v}_K|_S = 0$ and $f^K + T^{K\beta}|_\beta \in C(\bar{V})$, from the du Bois-Reymonde Lemma(6) we obtain:

$$T^{K\beta}|_\beta + f^K = 0 \quad \text{in} \quad V. \tag{4.4}$$

By virtue of Eqs. $(0.4)_2$ and (2.4), assuming that $f^1 = f^3 = 0$, $f^2 = \gamma$, where $\gamma$ is the known volume density of external loads, we rewrite the Eq. (4.4) in the form:

$$\begin{aligned} &T^{11}{}_{,1} + T^{12}{}_{,2} + \varkappa(1-z^1\varkappa)T^{33} - \frac{\varkappa T^{11}}{1-z^1\varkappa} = 0, \\ &T^{21}{}_{,1} + T^{22}{}_{,2} - \frac{\varkappa T^{21}}{1-z^1\varkappa} + \gamma = 0. \end{aligned} \tag{4.5}$$

Because of $\overset{*}{v}_3 = 0$, it does not follow from Eq. (4.1) that $T^{3\beta}|_\beta + f^3$ is equal to zero in $V$. Denoting $\bar{f}^3 \equiv -T^{3\beta}|_\beta$ in $V$, and because of $T^{3\beta}|_\beta = T^{33}{}_{,3}$ (see Eqs. (0.4) and (2.4)) we shall write:

$$T^{33}{}_{,3} + \bar{f}^3 = 0. \tag{4.6}$$

The vector with components $(0, 0, \bar{f}^3)$ will be referred to as a density of internal body forces. The existence of this new force is strictly connected with the condition $\overset{*}{v}_3 = 0$, which follows from the assumption (3.2) and the definition of virtual increments given in Sect. 1.

On the part $S_K$ of the boundary $S$ are prescribed the kinematical boundary conditions (1.3). Since in the Eq. (4.1) the first integral is equal to zero, and since $\overset{*}{v}_K = 0$ on $S_K$ and $\bar{S}^K = S - \bar{S}_K$, we have:

$$\sum_{K=1}^{2} \int_{S^K} (p^K - T^{K\beta} n_\beta) \overset{*}{v}_K dS = 0,$$

for any continuous $\overset{*}{v}_K$. Thus the relation given above must also hold for any $\overset{*}{v}_K$ being equal to zero in $S^K - S'$, where $S'$ can be treated as a region on the plane. It follows that

$$\int_{S'} (p^K - T^{K\beta} n_\beta) \overset{*}{v}_K dS = 0, \tag{4.7}$$

for any $\overset{*}{v}_K$ vanishing on the boundary of the region $S'$. Applying the du Bois-Reymond Lemma to Eq. (4.7) and bearing in mind that $S'$ may be choosen arbitrarily, we obtain finally:

$$T^{K\beta} n_\beta = p^K \quad \text{on} \quad S^K. \tag{4.8}$$

From Eq. (3.2), by virtue of the continuity of the function $v_3$ in $V$, we obtain $\overset{*}{v}_3 = 0$ on $S$. Thus, from Eq. (4.1) it does not follow that $T^{3\beta} n_\beta - p^3$ is equal to zero on $S$.

(6) Let $\varphi \in C(\Omega)$, where $\Omega$ is a regular region in $R^n$. If $\int_\Omega \varphi\psi d\Omega = 0$ holds for any $\psi \in C^1(\bar{\Omega})$ such that $\psi|_{\partial\Omega} = 0$ and $\psi_{\alpha}|_{\partial\Omega} = 0$, then $\varphi \equiv 0$ in $\Omega$, where $\partial\Omega$ is a boundary of $\Omega$ and $\bar{\Omega} = \Omega \cup \partial\Omega$.

Denoting $\bar{p}^3 \equiv T^{3\beta}n_\beta - p^3$, and assuming that the boundary loads are acting exclusively in the parametic planes $z^3 = \text{const.}$ — i.e. that $p^3 = 0$ — we obtain:

$$T^{33}n_3 = \bar{p}^3 \quad \text{on } S^3 \quad \text{and for} \quad 0 < z^3 < l. \tag{4.9}$$

At the same time, we shall write:

$$\bar{p}^K \equiv T^{KL}n_L \quad \text{on} \quad S_K, \tag{4.10}$$

assuming that $p^K = 0$ on $S_K$. The functions $\bar{p}^\alpha$ defined on $S_\alpha$ are unknown boundary reaction forces connected with the conditions (1.3) in which $\mathring{v}_3 = 0$, by virtue of (3.2). Let us transform Eqs. (1.6), using $\overset{*}{p}{}^\alpha = \overset{*}{T}{}^{\alpha\beta}n_\beta$ and $\overset{*}{f}{}^\alpha = -\overset{*}{T}{}^{\alpha\beta}|_\beta$ to obtain:

$$\oint_S \overset{*}{T}{}^{\alpha\beta}n_\beta v_\alpha dS - \int_V \overset{*}{T}{}^{\alpha\beta}|_\beta v_\alpha dV = \int_V \overset{*}{T}{}^{\alpha\beta}\xi_{\alpha\beta}dV + \int_{\Sigma_v} \overset{*}{T}{}^{\alpha\beta}n_\beta[v_\alpha]d\Sigma. \tag{4.11}$$

We assume that $\Sigma_v$ consists of the surface $\bar{\Sigma}_v$ inside the region $V$ and a surface $S_K$ on the boundary of the plastic zone. Let on $S_K$ the jump of the flow velocity vector be equal to $\mathring{v}_K - v_K$, where $v_K$ is a boundary value and $\mathring{v}_K$ is the known function which is assumed to be given on $S_K$. Taking into account the divergence theorem, by virtue of $[\overset{*}{T}{}^{\alpha\beta}]n_\beta = 0$ on $\Sigma_T$, we arrive at:

$$\int_V (\xi_{\alpha\beta} - v_\alpha|_\beta)\overset{*}{T}{}^{\alpha\beta}dV = 0. \tag{4.12}$$

The relation given above must be satisfied by any virtual increment field $\overset{*}{T}{}^{\alpha\beta} = \overset{*}{T}{}^{\beta\alpha}$. Since from (3.1) we have $\overset{*}{T}{}^{K3} = 0$, then

$$\int_V (\xi_{KL} - v_K|_L)\overset{*}{T}{}^{KL}dV + \int_V (\xi_{33} - v_3|_3)\overset{*}{T}{}^{33}dV = 0. \tag{4.13}$$

The Eq. (4.13) must hold for any $\overset{*}{T}{}^{KL} \in C^1(\bar{V})$, $\overset{*}{T}{}^{33} \in C^1(\bar{V})$. Since the integrands in (4.13) are continuous in $V - \Sigma_v$, therefore from du Bois-Reymonde Lemma we obtain:

$$\xi_{KL} = v_{(K|L)}, \quad \xi_{33} = v_3|_3$$

at all points of the region $V - \Sigma_v$. Making use of (0.4) and (2.4), we shall rewrite the foregoing equations in the form:

$$\begin{aligned} \xi_{KL} &= v_{(K|L)}, \\ \xi_{33} &= -\varkappa(1 - z^1\varkappa)v_1. \end{aligned} \tag{4.14}$$

Since $\overset{*}{T}{}^{K3} = 0$, form (4.12) it does not follow that $\xi_{K3} - v_{(K|3)}$ is equal to zero. Defining $\bar{\xi}_{K3} \equiv \xi_{K3} - v_{(K|3)}$, and using Eqs. (0.4), (2.4), we shall write:

$$\begin{aligned} \xi_{13} &= \frac{1}{2}v_{1,3} + \bar{\xi}_{13}, \\ \xi_{23} &= \frac{1}{2}v_{2,3} + \bar{\xi}_{23}. \end{aligned} \tag{4.15}$$

The functions $\bar{\xi}_{13}$, $\bar{\xi}_{23}$ are unknown. Their existence results from the hypothesis (3.1) imposed on the state of stress.

To conclude this section, we shall prove that if $\varkappa = 0$ then Eqs. (3.3), (3.4), (3.5), after substituting $\eta = 0.5$, $\mu = 1$, $\varrho = \overset{*}{\varrho} = 0$, hold for materials with plastic potential

$$G = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 6c^2. \tag{4.16}$$

Because of $(4.14)_2$, for $\varkappa = 0$, we have $\xi_{33} = 0$ — i.e., $d_3 = 0$. Writing Eq. (1.2) in the form $d_i = \bar{\lambda}\partial G/\partial\sigma_i$, we arrive at $d_i = \bar{\lambda}[\sigma_i - 0.5(\sigma_1 + \sigma_2 + \sigma_3)]$. Bearing in mind that $d_3 = 0$, it follows that $\sigma_3 = 0.5\,(\sigma_1 + \sigma_2)$ and Eq. (3.6) holds for $\eta = 0.5$. Thus, the Huber-Mises yield condition leads to $\sigma_1 - \sigma_2 - 2c = 0$, $\sigma_1 > \sigma_2$ which is equivalent to (3.5), when $\mu = 1$, $\varrho = 0$. The flow law associated with the Huber-Mises yield condition gives $d_1 = 0.5\,\bar{\lambda}\,(\sigma_1 - \sigma_2) = \bar{\lambda}c$, $d_2 = 0.5\bar{\lambda}(\sigma_2 - \sigma_1) = -\bar{\lambda}c$, $d_3 = 0$, which is equivalent to Eq. (3.4) for $\mu = 1$, $\overset{*}{\varrho} = 0$, $\lambda = \bar{\lambda}c$ which completes the proof. It follows that all further investigations are valid not only for the media with plastic potential (3.3) but also for plastic potential (4.16); the latter holds only if $\varkappa = 0$ and $\eta = 0.5$, $\mu = 1$, $\overset{*}{\varrho} = \varrho = 0$. The values of the potential $G$ are listed in Table 1, in which both cases are included.

## 5. Formulation of the boundary value problems and their physical meaning

It can easily be observed that in the equilibrium equations (4.5), in the kinematical equation (4.14) and in the boundary conditions (1.3), (1.4), the independent variable $z^3$ plays the role of the parameter — i.e., we do not deal with the derivatives with respect to $z^3$. Since the Eqs. (3.4), (3.5) are algebraic equations, then for any fixed $z^3$ we can formulate the two-dimensional boundary value problem — i.e., the boundary problem in the independent variables $z^1$, $z^2$ only. Thus we conclude that the approach proposed in the paper, based on the hypothesis (3.1), (3.2) reduces the general three-dimensional boundary value problems of the theory of plastic flow, to a system of two-dimensional boundary value problems for $z^3 = \text{const.}$ independently.

The approach developped in the paper, can be modified by introducing a new hypothesis in place of these given by the Eqs. (3.1), (3.2). In particular, we can assume that in each plane normal to an arbitrary spatial curve there exists the same state of stress. In this way, we shall arrive at the another class of spatial problems, which were analysed in [5] p. 28.

Taking into account the equations obtained in the preceding section of the paper, we shall formulate now all basic relations of Sect. 4 in the orthonormal local coordinate systems. Let us define the functions:

$$\omega = \omega(z^1, z^2) \equiv \frac{\varkappa(z^3)}{1 - z^1\varkappa(z^3)}. \tag{5.1}$$

Symbol $\omega$ represents a curvature of the cylindrical surface $z^1 = \text{const.}$ for a fixed $z^3$. The equilibrium equations (4.5) with the denotation (5.1) have the form:

$$\begin{aligned} \sigma^{11}{}_{,1} + \sigma^{12}{}_{,2} - \omega\sigma^{11} + \omega\sigma^{33} &= 0, \\ \sigma^{21}{}_{,1} + \sigma^{22}{}_{,2} - \omega\sigma^{12} + \gamma &= 0. \end{aligned} \tag{5.2}$$

Analogous, from Eq. (4.6) we obtain:

$$\left(\frac{\sigma^{33}}{(1 - z^1\varkappa)^2}\right)_{,3} + \bar{f}^3 = 0. \tag{5.3}$$

In view of $\sigma^{K3} = 0$, the boundary conditions (4.8) can now be written in the form:

$$\sigma^{KL} n_L = p^K \quad \text{on} \quad S^K. \tag{5.4}$$

The Eqs. (5.2) must be satisfied almost everywhere in $V$, and the conditions (5.4) must hold on the smooth parts of the surfaces $S^K$. On the surfaces $S_K$, where the kinematic boundary conditions $v_K = \mathring{v}_K$ are known, we obtain:

$$\sigma^{KL} n_L = \bar{p}^K \quad \text{on} \quad S_K. \tag{5.5}$$

Let us transform Eqs. (5.2) by means of the following formulas, which constitute the modified version of the known Levy formulas:

$$\begin{aligned} \sigma^{11} &= \sigma + (\sigma \sin\varrho + \hat{c})\cos 2\varphi + h, \\ \sigma^{22} &= \sigma - (\sigma \sin\varrho + \hat{c})\cos 2\varphi + h, \\ \sigma^{12} &= (\sigma \sin\varrho + \hat{c})\sin 2\varphi; \end{aligned} \tag{5.6}$$

the functions $h$, $\hat{c}$ are here defined by:

$$\begin{aligned} \varrho > 0: &\quad h \equiv -c\,\mathrm{ctg}\,\varrho, \quad \hat{c} \equiv 0; \\ \varrho = 0: &\quad h \equiv -\gamma z^2, \quad \hat{c} \equiv c. \end{aligned}$$

By virtue of (3.6), and from the well known relations

$$\left.\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix}\right\} = \frac{\sigma^{11} + \sigma^{22}}{2} \pm \sqrt{\left(\frac{\sigma^{11} - \sigma^{22}}{2}\right)^2 + \left(\frac{\sigma^{12} + \sigma^{21}}{2}\right)^2}, \quad \sigma_3 = \sigma^{33}, \tag{5.7}$$

we obtain:

$$\sigma^3 = \sigma + h + (2\eta - 1)(\sigma \sin\varrho + \hat{c}). \tag{5.8}$$

The right-hand sides of (5.7) and (5.8) fullfil the yield condition (3.5) for $\mu = 1$. From Eqs. (5.6), we conclude that $\sigma = \sigma(z^1, z^2, z^3)$, $\varphi = \varphi(z^1, z^3, z^3)$ are equal to

$$\sigma = \frac{\sigma^{11} + \sigma^{22}}{2} - h,$$

$$\varphi = \frac{1}{2} \operatorname{arctg} \frac{2\sigma^{12}}{\sigma^{11} - \sigma^{22}}.$$

Note that the functions $\sigma$, $\varphi$ depend not only on $z^1$, $z^2$ (such a situation occurs in plane and axially-symmetric problems) but also on $z^3$, because the three-dimensional problem of plastic flow is analysed here. Substituting the right-hand sides of Eqs. (5.6) and (5.8) into (5.2), we obtain

$$\begin{aligned} &(1 + \sin\varrho \cos 2\varphi)\sigma_{,1} - 2\sin 2\varphi(\sigma \sin\varrho + \hat{c})\varphi_{,1} + \sin\varrho \sin 2\varphi\, \sigma_{,2} \\ &\qquad + 2\cos 2\varphi(\sigma \sin\varrho + \hat{c})\varphi_{,2} + \omega(\sigma \sin\varrho + \hat{c})(2\eta - 1 - \cos 2\varphi) = 0, \\ &\sin\varrho \sin 2\varphi\, \sigma_{,1} + 2\cos 2\varphi(\sigma \sin\varrho + \hat{c})\varphi_{,1} + (1 - \sin\varrho \cos 2\varphi)\sigma_{,2} \\ &\qquad + 2\sin 2\varphi(\sigma \sin\varrho + \hat{c})\varphi_{,2} - \omega \sin 2\varphi(\sigma \sin\varrho + \hat{c}) + h_{,2} + \gamma = 0. \end{aligned} \tag{5.9}$$

This is a quasi-linear hyperbolic system of partial differential equations. At the same time, the following relations along the characteristic lines hold (cf. [13] p. 186):

$$dz^2 = \mathrm{tg}(\varphi \pm \varepsilon) dz^1, \quad \varepsilon \equiv \frac{\pi}{4} - \frac{\varrho}{2},$$

$$\text{(5.9)} \quad d\sigma \pm 2(\sigma \mathrm{tg}\varrho + \hat{c}) d\varphi = (h_{,2} + \gamma)(dz^2 \pm \mathrm{tg}\varrho\, dz^1) + \omega(\sigma \mathrm{tg}\varrho + \hat{c})\{\cos\varrho\, dz^1 + (2\nu - 1)[1 + (2\nu - 1)\sin\varrho] dz^2\}.$$

Analogously, by substituting (4.8) into (4.3), we arrive at:

$$\text{(5.10)} \qquad \left(\frac{\sigma + h + (2\eta - 1)(\sigma \sin\varrho + \hat{c})}{(1 - z^1 \varkappa)^2}\right)_{,3} + \bar{f}^3 = 0.$$

Let us also substitute the right-hand sides of Eqs. (5.6), (5.8) into Eqs. (5.4), (5.5) and into the condition:

$$\text{(5.11)} \qquad \sigma^{33} n_3 = \bar{p}^3 \quad \text{on} \quad S_3 \quad \text{and for} \quad 0 < z^3 < l.$$

Thus we have arrived at the complete system of static boundary conditions related to the field equations (5.9) and (5.10). As the unknown functions (for fixed $\eta$) we take the functions $\sigma$, $\varphi$, $\bar{f}^3$ defined in $V$, the functions $\bar{p}^K$ defined on $S_K$, and $\bar{p}^3$ defined on $S_3$ (and for $0 < z^3 < l$ when $n_3 \neq 0$). It can be observed that $\bar{f}^3$, $\bar{p}^3$ are uniquely determined by (5.10), (5.11) and by (5.8) for a fixed $\eta$. The occurrence of the unknown functions $\bar{p}^K$ on $S_K$ is due to the kinematic boundary conditions (1.3). In what follows, two special kinds of problems will be considered.

1. *The plastic zone V is situated along the curved line* — i.e., $\varkappa \neq 0$. In this case, the plastic potential will be assumed in the form (3.3) and the yield condition in the form (3.5). To obtain a solution of the boundary value problems for the system (5.9), we may introduce the condition of complete plasticity, putting $\eta = 0$ or $\eta = 1$(7), (cf. [6] p. 278). After obtaining the functions $\sigma$, $\varphi$ (they may depend on $z^3$, because in the general case $\omega$, $c$, $\varrho$, $\gamma$, $p^K$ also depend on $z^3$), we calculate $\bar{f}^3$ from (5.10) and $\bar{p}^3$ from (5.11) and (5.8).

2. *The plastic zone V is situated along the straight line* — i.e., $\varkappa = 0$. In this case, the plastic potential will be assumed in the form (4.16), in which $\eta = 0.5$, $\mu = 1$, $\varrho = \varrho^* = 0$ (taking the plastic potential in the form (3.3) we shall obtain the undetermined value of $\sigma^{33}$, cf. [10] p. 50) .The functions $\sigma$, $\varphi$ are obtained now as solutions of the boundary value problem for the system of Eqs. (5.9). The foregoing function depend on $z^3$, when the boundary conditions and the weight by volume of the subsoil depend on $z^3$. From the Eq. (5.10), we obtain $\bar{f}^3$ and from Eqs. (5.11), (5.8), we can calculate the function $\bar{p}^3$.

Now we pass to kinematical analysis of the problem under consideration. To this end, we shall rewrite Eqs. (1.2) into the following form:

$$\text{(5.12)} \qquad d_{\alpha\beta} = \lambda \sum_i \frac{\partial G}{\partial \sigma_i} \frac{\partial \sigma_i}{\partial \sigma^{\alpha\beta}} = \sum_i d_i \frac{\partial \sigma_i}{\partial \sigma^{\alpha\beta}}.$$

(7) The condition of complete plasticity $\eta = 0$, or $\eta = 1$ cannot be applied if the plastic potential has the form (4.16), because the components of the plastic flow $v_1$, $v_2$, $v_2 = 0$ cannot be determined from kinematic equations (we are dealing with two functions $v_1$, $v_2$ in three equations, cf. [4] p. 320). The Eqs. (4.16) will be used only when $\varkappa = 0$.

Making use of (5.8) and denoting $\Delta \equiv (\sigma^{11}-\sigma^{22})+(\sigma^{12}+\sigma^{21})^2$, we transform Eq. (5.12) to the slightly changed form:

$$
\begin{aligned}
d_{11} &= \frac{d_1+d_2}{2}+\frac{d_1-d_2}{2}\,\frac{\sigma^{11}-\sigma^{22}}{\sqrt{\Delta}}, \quad d_{13}=0,\\
d_{22} &= \frac{d_1+d_2}{2}-\frac{d_1-d_2}{2}\,\frac{\sigma^{11}-\sigma^{22}}{\sqrt{\Delta}}, \quad d_{23}=0,\\
d_{12} &= \frac{d_1-d_2}{2}\,\frac{\sigma^{12}+\sigma^{21}}{\sqrt{\Delta}}, \quad d_{33}=0.
\end{aligned} \tag{5.13}
$$

Let $\varphi_d$ be the angle between the first of the proper vectors of the matrix $[d_{KL}]$ and the plane $z^2 = \text{const}$. Making use of (5.13), we obtain:

$$
\mathrm{tg}\,2\varphi_d = \frac{2d_{12}}{d_{11}-d_{22}} = \frac{2\sigma^{12}}{\sigma^{11}-\sigma^{22}} = \mathrm{tg}\,2\varphi. \tag{5.14}
$$

The foregoing equation represents the coaxiality of the proper vectors of matrixes $[d_{\alpha\beta}]$ and $[\sigma^{\alpha\beta}]$ in isotropic plastic materials. The Eq. (5.14) for $\varphi = \varphi_d$ yields:

$$
(d_{11}-d_{22})\sin\varphi - 2d_{12}\cos\varphi = 0. \tag{5.15}
$$

This is the first kinematic equation. The second equation will be obtained from (3.4) by multiplying Eqs. $(3.4)_1$ by $(1-\sin\mathring{\varrho})$, Eq. $(3.4)_2$ by $(1+\sin\mathring{\varrho})$, Eq. $(3.4)_3$ by $[1+(2\nu-1)]$ $\sin\mathring{\varrho}$. After summing up termwise the relations obtained, we observe that the sum of their right-hand sides is equal to zero, and we arrive at:

$$
d_1+d_2+d_3+[d_1-d_2+(2\nu-1)d_3]\sin\mathring{\varrho} = 0. \tag{5.16}
$$

At the same time, we have $d_1+d_2+d_3 = d_{11}+d_{22}+d_{33}$, and from Eqs. (5.13), (5.14) it follows that

$$
d_{11}-d_{22} = (d_1-d_2)\frac{\sigma^{11}-\sigma^{22}}{\sqrt{(\sigma^{11}-\sigma^{22})^2+(\sigma^{12}+\sigma^{21})^2}} = \frac{d_1-d_2}{\sqrt{1-\mathrm{tg}^2 2\varphi}} = (d_1-d_2)\cos 2\varphi.
$$

Thus we can write the Eq. (5.16) in the form:

$$
(d_{11}+d_{22}+d_{33})\cos 2\varphi+(d_{11}-d_{22})\sin\mathring{\varrho}+(2\nu-1)d_{33}\sin\mathring{\varrho}\cos 2\varphi = 0. \tag{5.17}
$$

The final form of the kinematical equations can be obtained by substituting into Eqs. (5.15), (5.17) the following relations:

$$
\begin{aligned}
d_{KL} &= \frac{1}{2}(v_{K,L}+v_{L,K}),\\
d_{33} &= -\omega v_1,
\end{aligned} \tag{5.18}
$$

which result from Eqs. (4.14). Thus the system of kinematical equations will be represented by:

$$
\begin{aligned}
&(v_{1,1}-v_{2,2})\sin 2\varphi-(v_{1,2}+v_{2,1})\cos 2\varphi = 0,\\
&(v_{1,1}+v_{2,2}-\omega v_1)\cos 2\varphi+\sin\mathring{\varrho}(v_{1,1}-v_{2,2})-\omega(2\nu-1)\sin\mathring{\varrho}\cos 2\varphi v_1 = 0.
\end{aligned} \tag{5.19}
$$

The foregoing system is hyperbolic. Along the characteristics, we have the relations (cf. [10] p. 196):

$$dz^2 = \mathrm{tg}(\varphi+\varepsilon)dz^1, \quad \varepsilon \equiv \frac{\pi}{4}-\frac{\varrho}{2}, \tag{5.20}$$

$$dv_1+\mathrm{tg}(\varphi\pm\varepsilon)dv_2-\omega v_1\frac{1+(2\nu-1)\sin\mathring{\varrho}}{\sin 2(\varphi\pm\varepsilon)}dz^2 = 0.$$

Bearing in mind that $d_{13} = d_{23} = 0$ (cf. (5.13)) and using Eq. (4.15), we conclude that in spatial problems we have also to deal with the further two kinematical equations:

$$\frac{v_{1,3}}{2(1-z^1\varkappa)}+\overline{d}_{13} = 0, \tag{5.21}$$

$$\frac{v_{2,3}}{2(1-z^1\varkappa)}+\overline{d}_{23} = 0.$$

From the equations given above, we can calculate the values of $\overline{d}_{13}$ and $\overline{d}_{23}$, provided that the flow velocity field $v_\alpha$ is known. The boundary conditions for Eqs. (5.19) have the form:

$$v_K = \mathring{v}_K \quad \text{on} \quad S_K. \tag{5.22}$$

Now let us determined the value of the function $\lambda$. We use Eq. (3.4)

$$d_1+d_2+d_3 = -2\lambda\sin\varrho,$$

and then from Eq. (5.16) we obtain:

$$\lambda = \frac{1}{2}[d_1-d_2+(2\nu-1)d_3] \quad \text{for} \quad \sin\mathring{\varrho} \neq 0. \tag{5.23}$$

If the kinematical fields $v_\alpha$, $d_1$, $d_2$ and the function $\lambda$ are known, we have to calculate the values of the dissipation function. Taking into account Eqs. (1.7), (3.4), we obtain:

$$D = \sigma^{\alpha\beta}d_{\alpha\beta} = \sum_i \sigma_i d_i = \lambda[\sigma_1(1-\sin\mathring{\varrho})(1-\nu+\mu\nu)$$
$$+\sigma_2(1+\sin\mathring{\varrho})(\mu\nu-\nu-\mu)+\sigma_3(1-\mu)(2\nu-1-\sin\mathring{\varrho})].$$

By virtue of the yield condition (3.5) we arrive at:

$$D = \lambda[\sigma_1(\sin\varrho-\sin\mathring{\varrho})(1-\nu+\mu\nu)+\sigma_2(\sin\mathring{\varrho}-\sin\varrho)(\mu\nu-\mu-\nu) \tag{5.24}$$
$$+\sigma_3(1-\mu)(\sin\varrho-\sin\mathring{\varrho})+2c\cos\varrho].$$

From the condition $\lambda > 0$ and from Eq. (3.4), we also have:

$$d_1 \geqslant 0, \quad d_2 \leqslant 0, \quad d_3(2\nu-1) \geqslant 0. \tag{5.25}$$

The foregoing inequalities have to be satisfied together with the condition $D \geqslant 0$ at each point of the plastic zone $V$, which is not situated on the singular surfaces $\Sigma_v$ and $\Sigma_T$.

The condition which has to be satisfied on the lines of intersections of the surface $\Sigma_v$ and the parametric planes $z^3 = \text{const.}$ has the form:

$$\frac{1}{2}(\sigma_1-\sigma_2)[\mathbf{v}] = \sigma\sin\varrho[\mathbf{v}] \geqslant 0, \tag{5.26}$$

and can be found in [10] p.66.

The basic equations and the corresponding boundary conditions which have been obtained in this section, reduce the spatial problem of the plastic flow to the single parameter family of two-dimensional problems given by Eqs. (5.9), (5.4), (5.19), (5.22) (for every $z^3 = \text{const.}$ independently) and to Eqs. (5.10), (5.11) (5.5), (5.21). The Eqs. (5.9) and (5.19) have a form analogous to that of the well known equations of plane strain and axially-symmetric problems. However, all unknown functions which occur in the equations obtained depend in general not only on the independent variables $z^1$, $z^2$ but also on the independent variable $z^3$, since the functions $\omega$, $\varrho$, $c$, $\gamma$, $p^k$, $\mathring{v}_K$ can depend also on $z^3$. The solutions of the problems described by Eqs. (5.9) can be obtained for any $z^3 \in (0, l)$ as solutions of the well known initial value problems. Familiar methods for solving hyperbolic systems of equations (cf. [6] pp. 168—176) can be applied here. On this way, we obtain first the functions $\sigma(z^1, z^2, z^3)$, $\varphi(z^1, z^2, z^3)$ and the functions $\sigma^{KL}(z^1, z^2, z^3)$ from the Eqs. (5.6), and $\sigma^{33}(z^1, z^2, z^3)$ from Eq. (5.8). Analogously, the boundary value problems for Eqs. (5.19), (5.22) can be solved for any $z^3 \in (0, l)$; thus, we obtain the functions $v_K(z^1, z^2, z^3)$. Making use of the numerical approach, we cannot obtain solutions of the boundary value problems under consideration for every $z^3 = \text{const.}$ belonging to $(0, l)$, but only for a finite number of values $z^3$ taken from this interval. The final results in $(0, l)$ can be obtained by the interpolation method. Now suppose, that the functions $\sigma^{\alpha\beta}$, $v_\alpha$, are known; then we can calculate $\bar{f}^3$ from (5.10) (putting $\eta = 0$ for $\varkappa = 0$ and $\eta = 0$ or 1 for $\varkappa \neq 0$), and next obtain $\bar{p}^3$ from (5.11), (5.8) and $\bar{d}_{13}$, $\bar{d}_{23}$ from (5.21).

The field of the stress tensor $T^{\alpha\beta}(z^1, z^2, z^3)$ and the flow velocity vector field $v_\alpha(z^1, z^2, z^3)$ so obtained (note, that the form of these fields may not be uniquely determined, [6] pp. 113—114) satisfy all assumptions in Sect. 3. Moreover, if $\bar{f}^3 = 0$, $\bar{p}^3 = 0$, $\bar{d}_{13} = 0$, $\bar{d}_{23} = 0$, then also the relations $T^{\alpha\beta}|_\beta + f^\alpha = 0$, $\xi_{\alpha\beta} = v_{(\alpha|\beta)}$ in $V$ and $T^{\alpha\beta}n_\beta = p^\alpha$ on $S^\alpha$ also hold. In the general case, the fields $\bar{f}^3, \bar{p}^3, \bar{d}_{13}, \bar{d}_{23}$ are not equal to zero. From the foregoing analysis, it follows that the functions $\bar{f}^3$, $\bar{p}^3$ represent the forces which maintain the kinematic restrictions $v_3 = 0$ introduced a priori. Analogously, the fields $\bar{d}_{13}$, $\bar{d}_{23}$ are extra rates of deformations which maintain the stress restrictions $T^{13} = 0$, $T^{23} = 0$ introduced a priori. Thus we can also interpret the value $\alpha \equiv \max(|\bar{f}^3|, |\bar{p}^3|, |\bar{d}_{13}|, |\bar{d}_{23}|)$ as the measure of the influence of the restrictions referred to on applications of the approach given in the paper. This means that if the fields $\bar{f}^3, \bar{p}^3, \bar{d}_{13}, \bar{d}_{23}$ are sufficiently small, then the solutions obtained can be treated as a sufficiently good approximation of the problem under consideration. The term "sufficiently small" has to be intepreted as "of the same order of accuracy as the numerical calculations or grapfical methods or any other approximations" and must be analysed in each particular problem under consideration (cf. Sect. 7). Moreover, the approach proposed in the paper can be used only in problems where assumptions of the rigid plastic body can be applied. However, it is known that the concept of the rigid plastic body introduces an error which is difficult to estimate (cf. [6] p. 148].

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