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On nonlocal diffusion of gases

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A NONLOCAL continuum approach to nonreacting binary gas mixtures is given in this paper. The balance laws, jump conditions, and the constitutive equation are obtained generalizing the classical inviscid fluids to include the nonlocal effects. Thermodynamic restrictions are used to obtain the specific forms of the constitutive relations and the nonlocal field residuals for the binary mixtures. As one of the consequences of nonlocal effects, it is observed that, by contrast with the classical case, the hydrostatic pressure of the constituents is a space variable quantity. Finally, the diffusion equation for small disturbance is obtained.

W pracy zaproponowane jest nowe przedstawienie nielokalnej kontynualnej teorii mieszaniny dwóch niereagujących ze sobą gazów. Podane są równania zachowania, warunki dla skoków odpowiednich wielkości oraz równania konstytutywne, stanowiące uogólnienie klasycznej teorii nielepkich cieczy na przypadek nielokalnych efektów. W celu wyspecyfikowania formy równań konstytutywnych oraz nielokalnych reziduów dla dwuskładnikowych mieszanin wykorzystano ograniczenia termodynamiczne. Zaobserwowano, że w przeciwieństwie do klasycznej teorii występowanie nielokalnych efektów powoduje, że ciśnienie hydrostatyczne staje się wielkością zmieniającą się w przestrzeni. Wyprowadzone jest również równanie dyfuzji dla małych zaburzeń.

В работе предложен новый подход к нелокальной континуальной теории смешения двух нереагируемых с собой газов. Приведены уравнения сохранения, условия для скачков соответствующих величин, а также определяющие уравнения, составляющие обобщение классической теории невязких жидкостей на случай нелокальных эффектов. С целью специфицирования формы определяющих уравнений и нелокальных вычетов для двухкомпонентных смесей использованы термодинамические ограничения. Наблюдается, что в противовес классической теории выступание нелокальных эффектов вызывает факт, что гидростатическое давление становится величиной меняющейся в пространстве. Выведено тоже уравнение диффузии для малых возмущений.

1. Introduction

THE IMPORTANCE of nonlocal interatomic forces in the frequency spectra of solids has long been recognized (c.f. [8–9] and references there given). However, the incorpotation of these effects into continuum is only of recent origin. In this context, we may mention the works of KUNIN [4], KRÖNER [10] and the recent works of ERINGEN [1] on solids, and of DEMIRAY [2] on elastic dielectrics.

The nonlocal effects in fluid mechanics have been explored by ERINGEN [11, 12]. In these works, Eringen showed that the nonlocal theory includes the surface tension in the constitutive equations.

The main purpose of the present study is to establish a nonlocal theory of nonreacting binary mixtures extending the range of classical inviscid fluids to include nonlocal effects. The balance laws and associated jump conditions of nonlocal mixtures are given in Sect. 2, and the second law of thermodynamics in Sect. 3. In Sect. 4, we develop a set of nonlinear and linear constitutive equations for nonlocal mixtures of inviscid fluids. Invariance requirements and appropriate thermodynamical restrictions are studied. Finally, as some applications of the theory, the hydrostatics of mixtures and the problem of nonlocal diffusion are discussed. It is observed that, by contrast with the classical case, the hydrostatic pressure of the constituents is a space variable quantity rather than a constant. It is quite probable that this fact is the result of surface tension which may exist in the gases.

2. Motion, kinematics and balance equations

We consider a collection of continua called the body B, consisting of n elements capable of reaction in a region $V+S$ of Euclidean E_3 space. A material point $\mathbf{X}_{(\alpha)}$ of αth continuum ($\alpha = 1, 2, \ldots, n$) at time t is carried to a spatial position $\mathbf{x}$ through its appropriate motion:

$$\mathbf{x} = \mathbf{x}_{(\alpha)}(\mathbf{X}_{(\alpha)}, t), \quad \alpha = 1, 2, \ldots, n. \tag{2.1}$$

Throughout this work, Greek indices enclosed within brackets mark the species. These indices are freely placed either in subscript or superscript positions, to avoid crowding with tensorial indices. The summation convention is applied only on the repeated latin indices (not on the Greek indices).

The inverse motion of the constituents occupying the spatial point $\mathbf{x}$ is given by

$$\mathbf{X}_{(\alpha)} = \mathbf{X}_{(\alpha)}(\mathbf{x}, t). \tag{2.2}$$

Existence of such an inverse motion implies that

$$J_{(\alpha)} \equiv \det(\partial x^k/\partial X_K^{(\alpha)}) \neq 0, \quad \alpha = 1, 2, \ldots, n,$$

except at some singular points, lines and surfaces.

The velocity $v^k_{(\alpha)}$ of a material point $\mathbf{X}_{(\alpha)}$, occupying the space point $\mathbf{x}$ at time t, is definedy by:

$$v^k_{(\alpha)} \equiv \left.\frac{\partial x^k}{\partial t}\right|_{\mathbf{X}_{(\alpha)}}. \tag{2.3}$$

The acceleration $\mathbf{a}_{(\alpha)}$ of the αth component at $(\mathbf{x}, t)$ is defined by:

$$\dot{v}^k_{(\alpha)} \equiv a^k_{(\alpha)} \equiv \left.\frac{\partial v^k_{(\alpha)}}{\partial t}\right|_{\mathbf{X}_{(\alpha)}} = \frac{D^{(\alpha)}}{Dt} v^k_{(\alpha)}. \tag{2.4}$$

The material time derivative of any tensor function $\psi_{(\alpha)}$, following the motion of the αth component, is given by:

$$\dot{\psi}^{(\alpha)} \equiv \frac{D^{(\alpha)}}{Dt}\psi_{(\alpha)} \equiv \left.\frac{\partial \psi_{(\alpha)}}{\partial t}\right|_{\mathbf{x}} + v^k_{(\alpha)}\psi_{(\alpha);k}. \tag{2.5}$$

Assuming the existence of a partial mass density $\varrho_{(\alpha)}(\mathbf{x}, t)$ of the th continuum, the density of the whole mixture is given by:

$$\varrho \equiv \sum_{\alpha} \varrho_{(\alpha)}. \tag{2.6}$$

The average value ψ of the tensor field $\psi_{(\alpha)}$ is defined by

$$\varrho\psi \equiv \sum_{\alpha} \varrho_{(\alpha)}\psi_{(\alpha)}. \tag{2.7}$$

By means of (2.7), various average fields associated with the whole mixture can be calculated. For example, if we take $\psi_{(\alpha)} \equiv \mathbf{v}_{(\alpha)}$, we obtain the barycentric velocity $\mathbf{v}$:

$$\varrho\mathbf{v} \equiv \sum_{\alpha} \varrho_{(\alpha)}\mathbf{v}^{(\alpha)}. \tag{2.8}$$

The integral balance laws of multicomponent media may be expressed in the general form:

$$\sum_{\alpha}\left[\frac{D^{(\alpha)}}{Dt}\int_{v_{(\alpha)}-\sigma_{(\alpha)}} \phi_{(\alpha)}dv_{(\alpha)} - \int_{S_{(\alpha)}-\sigma_{(\alpha)}} \tau_k^{(\alpha)}da_k^{(\alpha)} - \int_{v_{(\alpha)}-\sigma_{(\alpha)}} g_{(\alpha)}dv_{(\alpha)}\right] = 0, \tag{2.9}$$

where $\phi_{(\alpha)}$ is a field of αth continuum over the body B at time t, having material volume $v_{(\alpha)}$ excluding the discontinuity surface $\sigma_{(\alpha)}$, which may be sweeping the body at velocity $u_{(\alpha)}$; $g_{(\alpha)}$ is the source of $\phi_{(\alpha)}$; and $\tau^k_{(\alpha)}$ is its influx through the surface $S_{(\alpha)}$ of $v_{(\alpha)}$, excluding those points of $\sigma_{(\alpha)}$ which intersect $S_{(\alpha)}$.

By means of the generalized Green-Gauss theorem, (2.9) may be converted into (c.f. ERINGEN [3]):

$$\sum_{\alpha}\left\{\int_{v_{(\alpha)}-\sigma_{(\alpha)}}\left[\frac{\partial\phi_{(\alpha)}}{\partial t} + (\phi^{(\alpha)}v_k^{(\alpha)})_{,k} - \tau_{k,k}^{(\alpha)} - g^{(\alpha)}\right]dv_{(\alpha)} + \int_{\sigma_{(\alpha)}}[\phi_{(\alpha)}(v^k_{(\alpha)} - u^k_{(\alpha)}) - \tau^k_{(\alpha)}]n_k^{(\alpha)}da^{(\alpha)}\right\} = 0, \tag{2.10}$$

where the square bracket [ ] denotes the jump across $\sigma_{(\alpha)}(t)$.

In classical mixtures, it is posited that (2.10) is valid for every part of the body and, therefore, the integrands of (2.10) are set equal to zero. As a result, we obtain the local laws of mixture. In nonlocal continuum physics, we do not impose this severe restriction. Localization may still be accomplished by writing (2.10) in equivalent forms:

$$\frac{\partial\phi_{(\alpha)}}{\partial t} + (\phi_{(\alpha)}v_k^{(\alpha)})_{,k} - g_{(\alpha)} - \tau_{k,k}^{(\alpha)} = \hat{g}^{(\alpha)} \quad \text{in} \quad v_{(\alpha)} - \sigma_{(\alpha)}, \tag{2.11}$$

$$[\phi_{(\alpha)}(v^k_{(\alpha)} - u^k_{(\alpha)}) - \tau^k_{(\alpha)}]n_k^{(\alpha)} = \hat{G}_{(\alpha)} \quad \text{on} \quad \sigma_{(\alpha)}, \tag{2.12}$$

such that

$$\sum_{\alpha}\left[\int_{v_{(\alpha)}-\sigma_{(\alpha)}} \hat{g}_{(\alpha)}dv_{(\alpha)} + \int_{\sigma_{(\alpha)}} \hat{G}_{(\alpha)}da^{(\alpha)}\right] = 0. \tag{2.13}$$

The new fields $\hat{g}_{(\alpha)}$ and $\hat{G}_{(\alpha)}$ introduced are called the "localization residuals" or "nonlocal interactions". Determination of the nonlocal interactions is, of course, an integral part of nonlocal continuum physics.

The Eqs. (2.11)–(2.13) are the master balance laws of nonlocal mixtures. We now employ these equations to obtain special balance equations for mass, momentum, moment of momentum, and energy.

i) Conservation of mass

If we take $\phi_{(\alpha)} = \varrho_{(\alpha)}$, $\tau^k_{(\alpha)} = g_{(\alpha)} = 0$, where $\varrho_{(\alpha)}$ is the partial mass density of αth continuum in the mixture, then (2.11)–(2.13) give:

$$\begin{aligned}
&\frac{\partial \varrho_{(\alpha)}}{\partial t} + (\varrho_{(\alpha)} v^k_{(\alpha)})_{,k} = \hat{c}_{(\alpha)} \quad \text{in} \quad v_{(\alpha)} - \sigma_{(\alpha)},\\
&[\varrho_{(\alpha)}(v^k_{(\alpha)} - u^k_{(\alpha)})] n^{(\alpha)}_k = \hat{C}_{(\alpha)} \quad \text{on} \quad \sigma_{(\alpha)},\\
&\sum_\alpha \Big[\int_{v_{(\alpha)}-\sigma_{(\alpha)}} \hat{c}_{(\alpha)} dv_{(\alpha)} + \int_{\sigma_{(\alpha)}} \hat{C}_{(\alpha)} da_{(\alpha)} \Big] = 0,
\end{aligned} \tag{2.14}$$

where $\hat{c}_{(\alpha)}$ and $\hat{C}_{(\alpha)}$ are the rate of mass production of αth constituent in $v_{(\alpha)} - \sigma_{(\alpha)}$ and on $\sigma_{(\alpha)}$, respectively. These quantities may have local characters as well as their nonlocal natures.

ii) Balance of momentum

By setting $\phi_{(\alpha)} = \varrho_{(\alpha)} \mathbf{v}_{(\alpha)}$, $\tau^k_{(\alpha)} = \mathbf{t}^k_{(\alpha)}$, $g_{(\alpha)} = \varrho_{(\alpha)} \mathbf{f}^{(\alpha)}$ in (2.11)–(2.13), we obtain the nonlocal balance law for momentum:

$$\begin{aligned}
&\mathbf{t}^{(\alpha)}_{k,k} + \varrho_{(\alpha)}(\mathbf{f}^{(\alpha)} - \mathbf{v}'_{(\alpha)})_{,k} = \hat{c}_{(\alpha)} \mathbf{v}_{(\alpha)} - \hat{\mathbf{r}}_{(\alpha)} \quad \text{in} \quad v_{(\alpha)} - \sigma_{(\alpha)},\\
&[\varrho_{(\alpha)} \mathbf{v}^{(\alpha)}(v^k_{(\alpha)} - u^k_{(\alpha)}) - \mathbf{t}^k_{(\alpha)}] n^{(\alpha)}_k = \hat{\mathbf{R}}_{(\alpha)} \quad \text{on} \quad \sigma_{(\alpha)},\\
&\sum_\alpha \Big[\int_{v_{(\alpha)}-\sigma_{(\alpha)}} \tilde{\mathbf{r}}_{(\alpha)} dv_{(\alpha)} + \hat{\mathbf{R}}_{(\alpha)} da_{(\alpha)} \Big] = 0,
\end{aligned} \tag{2.15}$$

where $\hat{\mathbf{r}}_{(\alpha)}$ and $\hat{\mathbf{R}}_{(\alpha)}$ are the rates of linear momentum transfer, or production, within the αth constituent, due to interactions with other species in the mixture, in $v_{(\alpha)} - \sigma_{(\alpha)}$ and on $\sigma_{(\alpha)}$, respectively. Here $\mathbf{f}_{(\alpha)}$ is the body force per unit mass. The above equation may be written in component form as:

$$\begin{aligned}
&t^{(\alpha)}_{kl,k} + \varrho_{(\alpha)}(f^{(\alpha)}_l - v'^{(\alpha)}_l) = \hat{c}_{(\alpha)} v^{(\alpha)}_l - \hat{r}^{(\alpha)}_l \quad \text{in} \quad v_{(\alpha)} - \sigma_{(\alpha)},\\
&[\varrho_{(\alpha)} v^{(\alpha)}_l (v^{(\alpha)}_k - u^{(\alpha)}_k) - t^{(\alpha)}_{kl}] n^{(\alpha)}_k = \hat{R}^{(\alpha)}_l \quad \text{on} \quad \sigma_{(\alpha)},
\end{aligned} \tag{2.16}$$

where $t^{(\alpha)}_{kl}$ is the partial stress tensor of the αth component of the mixture.

iii) Moment of momentum

To obtain the nonlocal balance law for the moment of momentum, in (2.11)–(2.13) we set $\phi_{(\alpha)} = \mathbf{x} \times \varrho_{(\alpha)} \mathbf{v}_{(\alpha)}$, $\tau^k_{(\alpha)} = \mathbf{x} \times \mathbf{t}^k_{(\alpha)}$, $g_{(\alpha)} = \mathbf{x} \times \varrho_{(\alpha)} \mathbf{f}_{(\alpha)}$, and use the Eqs. (2.14) and (2.16) to arrive at:

$$\begin{aligned}
&\varepsilon_{ijk}(x_j \hat{r}^{(\alpha)}_k - t^{(\alpha)}_{jk}) = \hat{m}^{(\alpha)}_i \quad \text{in} \quad v_{(\alpha)} - \sigma_{(\alpha)},\\
&\varepsilon_{ijl} x_j [\varrho_{(\alpha)} v^l_{(\alpha)} (v^k_{(\alpha)} - u^k_{(\alpha)}) - t^{kl}_{(\alpha)}] n^{(\alpha)}_k = \hat{M}^{(\alpha)}_i \quad \text{on} \quad \sigma_{(\alpha)},\\
&\sum_\alpha \Big[\int_{v_{(\alpha)}-\sigma_{(\alpha)}} \hat{m}^{(\alpha)}_i dv_{(\alpha)} + \int_{\sigma_{(\alpha)}} \hat{M}^{(\alpha)}_i da_{(\alpha)} \Big] = 0.
\end{aligned} \tag{2.17}$$

At this point, it will be very useful to decompose the $\hat{m}_i^{(\alpha)}$ as

(2.18) $$\hat{m}_i^{(\alpha)} \equiv \hat{m}_i'^{(\alpha)} + \hat{m}_i''^{(\alpha)},$$

such that

(2.19) $$\hat{m}_i''^{(\alpha)} = \varepsilon_{ijk} x_j \hat{r}_k^{(\alpha)} \quad \text{in} \quad v_{(\alpha)} - \sigma_{(\alpha)}.$$

Such a decomposition is the result of the consideration that part of the Eq. (2.17) should be invariant under the translational motions of the spatial frame of reference. Employing (2.18) and (2.19) in the Eq. (2.17) we obtain:

(2.20) $$\varepsilon_{ijk} t_{jk}^{(\alpha)} = \hat{m}_i'^{(\alpha)} \quad \text{in} \quad v_{(\alpha)} - \sigma_{(\alpha)},$$

subject to

(2.21) $$\sum_\alpha \int_{v_{(\alpha)}-\sigma_{(\alpha)}} \hat{m}_l'^{(\alpha)} dv_{(\alpha)} = -\sum_\alpha \Big[\int_{v_{(\alpha)}-\sigma_{(\alpha)}} \varepsilon_{ijk} x_j \hat{r}_k^{(\alpha)} dv_{(\alpha)} + \int_{\sigma_{(\alpha)}} \varepsilon_{ijl} x_j [\varrho_{(\alpha)} v_{(\alpha)}^l (v_{(\alpha)}^k - u_{(\alpha)}^k) - t_{(\alpha)}^{kl}] n_k^{(\alpha)} da^{(\alpha)} \Big].$$

As is known from the classical theory of mixtures, the partial stress tensor is not symmetric.

iv) Conservation of energy

The nonlocal law of energy conservation is obtained if in (2.11)–(2.13) we set:

$$\phi_{(\alpha)} = \varrho_{(\alpha)} \varepsilon_{(\alpha)} + \frac{1}{2} \varrho_{(\alpha)} \mathbf{v}_{(\alpha)}^2, \quad \tau_{(\alpha)}^k = \mathbf{t}_{(\alpha)}^k \cdot \mathbf{v}_{(\alpha)} + q_{(\alpha)}^k,$$

$$g_{(\alpha)} = \varrho_{(\alpha)} \mathbf{f}_{(\alpha)} \cdot \mathbf{v}_{(\alpha)} + \varrho_{(\alpha)} h_{(\alpha)}$$

and make use of (2.14) and (2.16). Here, $\varepsilon_{(\alpha)}$ is the partial internal energy density per unit mass, $q_{(\alpha)}^k$ is the surface energy influx (e.g. the heat vector), and $h_{(\alpha)}$ is the energy source per unit mass. Thus,

(2.22) $$\varrho_{(\alpha)} \varepsilon'_{(\alpha)} = t_{(\alpha)}^{kl} v_{l,k}^{(\alpha)} + q_{(\alpha),k}^k + \varrho_{(\alpha)} h_{(\alpha)} - \hat{r}_l^{(\alpha)} v_l^{(\alpha)} + \hat{c}^{(\alpha)} \left(\frac{1}{2} \mathbf{v}^{(\alpha)} \cdot \mathbf{v}^{(\alpha)} - \varepsilon_{(\alpha)} \right) + \hat{e}_{(\alpha)} \quad \text{in} \quad v_{(\alpha)} - \sigma_{(\alpha)},$$

$$\left[\left(\varrho_{(\alpha)} \varepsilon_{(\alpha)} + \frac{1}{2} \varrho_{(\alpha)} \mathbf{v}_{(\alpha)}^2 \right) (v_{(\alpha)}^k - u_{(\alpha)}^k) - t_{(\alpha)}^{kl} v_l^{(\alpha)} - q_{(\alpha)}^k \right] n_k^{(\alpha)} = \hat{E}_{(\alpha)} \quad \text{on} \quad \sigma_{(\alpha)},$$

such that

$$\sum_\alpha \Big[\int_{v_{(\alpha)}-\sigma_{(\alpha)}} \hat{e}_{(\alpha)} dv_{(\alpha)} + \int_{\sigma_{(\alpha)}} \hat{E}_{(\alpha)} da_{(\alpha)} \Big] = 0,$$

where $\hat{e}_{(\alpha)}$ and $\hat{E}_{(\alpha)}$ are the rates of energy production in $v_{(\alpha)} - \sigma_{(\alpha)}$ and on $\sigma_{(\alpha)}$. The Eqs. (2.14), (2.16), (2.20), (2.21) and (2.22) are the fundamental balance laws of nonlocal continuum mixtures. They are valid for all types of bodies (fluids, solids, viscoelastic materials, etc.) irrespective of their geometry and constitutions.

3. Second law of thermodynamics

Studies on thermodynamics of mixtures are meager, and have not gone beyond certain propositions whose physical bases are yet unknown. Most authors who are dealing with either continuum or statistical approaches use the entropy inequality for the whole system (c.f. COHEN [5], MÜLLER [6]). On the other hand, ERINGEN and INGRAM [7] have proposed the entropy inequalities for each species. Such a formulation provides sufficient restrictions on possible thermodynamical states of the body in such a way that each individual species in the mixture has a stable equilibrium. In the present work, we also postulate the entropy inequalities for each species. We do this by replacing the sign $(=)$ by $(\geqslant)$ and setting

$$\phi_{(\alpha)} = \varrho_{(\alpha)}\eta_{(\alpha)}, \quad \tau^k_{(\alpha)} = q^k_{(\alpha)}/\theta_{(\alpha)}, \quad g_{(\alpha)} = \varrho_{(\alpha)}h_{(\alpha)}/\theta_{(\alpha)},$$

where $\eta_{(\alpha)}$ and $\theta_{(\alpha)}$ are respectively the entropy density and absolute temperature of αth species subject to

$$\begin{aligned}
&\varrho_{(\alpha)}\eta'_{(\alpha)} + \hat{c}_{(\alpha)}\eta_{(\alpha)} - (q^k_{(\alpha)}/\theta_{(\alpha)})_{,k} - \varrho_{(\alpha)}h_{(\alpha)}/\theta_{(\alpha)} - \hat{n}_{(\alpha)} \geqslant 0 \quad \text{in} \quad v_{(\alpha)} - \sigma_{(\alpha)},\\
&[\varrho_{(\alpha)}\eta_{(\alpha)}(v^k_{(\alpha)} - u^k_{(\alpha)}) - q^k_{(\alpha)}/\theta_{(\alpha)}]n^{(\alpha)}_k - \hat{N}_{(\alpha)} \geqslant 0 \quad \text{on} \quad \sigma_{(\alpha)}, \qquad (3.1)\\
&\sum_\alpha \Big[\int_{v_{(\alpha)}-\sigma_{(\alpha)}} \hat{n}_{(\alpha)} dv_{(\alpha)} + \int_{\nu_{(\alpha)}} \hat{N}_{(\alpha)} da_{(\alpha)} \Big] = 0.
\end{aligned}$$

Here $\hat{n}_{(\alpha)}$ and $\hat{N}_{(\alpha)}$ are the volume and surface nonlocal rates of the entropy productions.

Now, we transform (3.1) into a more convenient form by introducing the Helmholtz free energy function:

$$\psi_{(\alpha)} \equiv \varepsilon_{(\alpha)} - \theta_{(\alpha)}\eta_{(\alpha)}, \qquad (3.2)$$

and eliminating the $h_{(\alpha)}$ between $(2.22)_1$ and (3.1). Thus we have:

$$\begin{aligned}
-\frac{\varrho_{(\alpha)}}{\theta_{(\alpha)}}(\psi'_{(\alpha)} + \eta_{(\alpha)}\theta'_{(\alpha)}) + \frac{1}{\theta_{(\alpha)}} t^{kl}_{(\alpha)} v^{(\alpha)}_{l,k} + \frac{1}{\theta^2_{(\alpha)}} q^k_{(\alpha)}\theta^{(\alpha)}_{,k} - \frac{c_{(\alpha)}}{\theta_{(\alpha)}}\left(\psi_{(\alpha)} - \frac{1}{2}v^2_{(\alpha)}\right)& \qquad (3.3)\\
-\frac{\hat{r}^{(\alpha)}_l}{\theta_{(\alpha)}} v^{(\alpha)}_l + (\hat{e}^{(\alpha)}/\theta_{(\alpha)} - \hat{n}_{(\alpha)}) \geqslant 0 \quad \text{in} \quad v_{(\alpha)} - \sigma_{(\alpha)}.&
\end{aligned}$$

The inequality (3.3) is fundamental for nonlocal thermodynamic processes and it is somewhat of a generalization of Clausius-Duhem inequality. We shall use this inequality to obtain certain specific forms for the constitutive equations.

4. Constitutive equations of a binary mixture

The balance laws formulated in Sect. 2 are inadequate for the determination of motions of nonlocal mixtures. The nature of the medium must be characterized by means of a set of constitutive equations. We need to construct equations for the constitutive dependent variables $t^{(\alpha)}_{kl}$, $q^{(\alpha)}_k$, $\psi_{(\alpha)}$ and $\eta_{(\alpha)}$. In addition, the nonlocal residuals must be determined. We assume that the constitutive dependent variables are functionals of the following independent state variables:

$$\varrho_{(\beta)}, v^{(\beta)}_k, \theta_{(\beta)}, x_k; \varrho'_{(\beta)}, v'^{(\beta)}_k, \theta'_{(\beta)}, x'_k \quad (\alpha = 1, 2). \qquad (4.1)$$

As is seen from the above list of independent variables, we are dealing with nonlocal mixtures of two non-heat conducting inviscid fluids. Thus the constitutive functional should have the following form

$$\psi_{(\alpha)} = \psi_{(\alpha)}(\varrho_{(\beta)}, \theta_{(\beta)}, v_k^{(\beta)}, x_k; \varrho'_{(\beta)}, \theta'_{(\beta)}, v_k'^{(\beta)}, x'_k). \tag{4.2}$$

Similar forms of constitutive functionals are valid for the other dependent variables.

Now let us construct the time rate of $\psi_{(\alpha)}$. Bearing in mind the definition of the Fréchet derivative of a functional, we have:

$$\psi'_{(\alpha)} = \sum_\beta \Bigg\{ (\dot{\varrho}_{(\beta)} + u_k^{(\alpha\beta)} \varrho_{,k}^{(\beta)}) \left[\frac{\partial \psi_{(\alpha)}}{\partial \varrho_{(\beta)}} + \int\limits_{v_{(\beta)}-\sigma_{(\beta)}} \left(\frac{\partial \psi_{(\alpha)}}{\partial \varrho'_{(\beta)}} \right)^* dv'_{(\beta)} \right] + (\dot{\theta}_{(\beta)} + u_k^{(\alpha\beta)} \theta_{,k}^{(\beta)}) \Bigg\},$$

$$\left[\frac{\partial \psi_{(\alpha)}}{\partial \theta_{(\beta)}} + \int\limits_{v_{(\beta)}-\sigma_{(\beta)}} \left(\frac{\delta \psi_{(\alpha)}}{\delta \theta'_{(\beta)}} \right)^* dv'_{(\beta)} \right] + (v_l'^{(\beta)} + u_k^{(\alpha\beta)} v_{l,k}^{(\beta)}) \left[\frac{\partial \psi_{(\alpha)}}{\partial v_k^{(\beta)}} \right. \tag{4.3}$$

$$\left. + \int\limits_{v_{(\beta)}-\sigma_{(\beta)}} \left(\frac{\delta \psi_{(\alpha)}}{\delta v'_{(\beta)}} \right)^* dv'_{(\beta)} \right] + v_k^{(\alpha)} \left[\frac{\partial \psi_{(\alpha)}}{\partial \chi_k^{(\beta)}} + \int\limits_{v_{(\alpha)}-\sigma_{(\alpha)}} \left(\frac{\delta \psi_{(\alpha)}}{\delta x'_k} \right)^* dv'_{(\alpha)} \right]$$

$$+ \int\limits_{v_{(\alpha)}-\sigma_{(\alpha)}} (\Phi_{(\alpha)} - \Phi^*_{(\gamma)}) dv_{(\alpha)},$$

where the quantities $u_k^{(\alpha\beta)}$ and $\Phi_{(\alpha)}$ are defined by:

$$u_k^{(\alpha\beta)} \equiv v_k^{(\alpha)} - v_k^{(\beta)}, \tag{4.4}$$

$$\Phi_{(\alpha)} \equiv \frac{\delta \psi_{(\alpha)}}{\delta x'_k} v_k'^{(\alpha)} + \sum_\beta \frac{\delta \psi_{(\alpha)}}{\delta \varrho'_{(\beta)}} (\dot{\varrho}_{(\beta)} + u_k'^{(\alpha\beta)} \varrho_{,k}'^{(\beta)}) + \frac{\delta \psi_{(\alpha)}}{\delta \theta'_{(\beta)}} (\dot{\theta}'_{(\beta)} \tag{4.5}$$

$$+ u_k'^{(\alpha\beta)} \theta_{,k}'^{(\beta)}) + \frac{\delta \psi_{(\alpha)}}{\delta v_l'^{(\beta)}} (\dot{v}_l'^{(\beta)} + u_k'^{(\alpha\beta)} v_{l,k}'^{(\beta)}).$$

Here, the symbol $\delta/\delta(\ )'$ is used to denote the Fréchet (or functional) gradient, and $\Phi^*_{(\alpha)}$ is a function obtained by interchanging the primed and unprimed variables. We also note that

$$\iint\limits_{v_{(\alpha)}-\sigma_{(\alpha)}} (\phi_{(\alpha)} - \phi^*_{(\alpha)}) dv_{(\alpha)} dv'_{(\alpha)} \equiv 0. \tag{4.6}$$

For convenience in the subsequent analysis, we introduce the following quantities:

$$N_{(\alpha\beta)} \equiv \frac{\varrho_{(\alpha)}}{\theta_{(\alpha)}} \left[\frac{\partial \psi_{(\alpha)}}{\partial \theta_{(\beta)}} + \int\limits_{v_{(\beta)}-\sigma_{(\beta)}} \left(\frac{\delta \psi_{(\alpha)}}{\delta \theta'_{(\beta)}} \right)^* dv'_{(\beta)} \right], \tag{4.7}$$

$$\pi_{(\alpha\beta)} \equiv \frac{\varrho_{(\alpha)} \varrho_{(\beta)}}{\theta_{(\alpha)}} \left[\frac{\partial \psi_{(\alpha)}}{\partial \varrho_{(\beta)}} + \int\limits_{v_{(\beta)}-\sigma_{(\beta)}} \left(\frac{\delta \psi_{(\alpha)}}{\delta \varrho'_{(\beta)}} \right)^* dv_{(\beta)} \right], \tag{4.8}$$

$$T_l^{(\alpha\beta)} \equiv \frac{\varrho_{(\alpha)}}{\theta_{(\alpha)}} \left[\frac{\partial \psi_{(\alpha)}}{\partial v_l^{(\beta)}} \int\limits_{v_{(\beta)}-\sigma_{(\beta)}} \left(\frac{\delta \psi_{(\alpha)}}{\delta v_l'^{(\beta)}} \right)^* dv_{(\beta)} \right], \tag{4.9}$$

$$\hat{R}_l^{(\alpha)} \equiv \varrho_{(\alpha)} \left[\frac{\partial \psi_{(\alpha)}}{\partial x_l} + \int\limits_{v_{(\alpha)}-\sigma_{(\alpha)}} \left(\frac{\delta \psi_{(\alpha)}}{\delta x'_l} \right)^* dv_{(\alpha)} \right]. \tag{4.10}$$

Employing these definitions in (4.3), the material derivative of $\psi_{(\alpha)}$ may be expressed as:

$$\text{(4.11)} \quad \dot{\psi} = \sum_{\beta} \left[\frac{\pi_{(\alpha\beta)}\theta_{(\alpha)}}{\varrho_{(\alpha)}\varrho_{(\beta)}} \left(\dot{\varrho}_{(\beta)} + u_k^{(\alpha\beta)} \varrho_{,k}^{(\beta)} \right) + \frac{N_{(\alpha\beta)}\theta_{(\alpha)}}{\varrho_{(\beta)}} \left(\dot{\theta}_{(\beta)} + u_k^{(\alpha\beta)} \theta_{,k}^{(\beta)} \right) \right] + \left[\frac{T_l^{(\alpha\beta)}\theta_{(\alpha)}}{\varrho_{(\alpha)}} (\dot{v}_l^{(\beta)} + u_k^{(\alpha\beta)} v_{l,k}^{(\beta)}) \right] + \frac{\hat{R}_l^{(\alpha)}\theta_{(\alpha)}}{\varrho_{(\alpha)}} v_l^{(\alpha)} + \int_{v_{(\alpha)}-\sigma_{(\alpha)}} (\Phi_{(\alpha)} - \Phi^*_{(\alpha)})\, dv_{(\alpha)}.$$

Introducing (4.11) into (3.3), and assuming that there is no mass transfer (locally and nonlocally) among the species, we obtain:

$$\text{(4.12)} \quad -\left(\frac{\varrho_{(\alpha)}}{\theta_{(\alpha)}} \eta_{(\alpha)} + N_{(\alpha\alpha)} \right) \dot{\theta}_{(\alpha)} + \left[\frac{t_{kl}^{(\alpha)}}{\theta_{(\alpha)}} + \pi_{(\alpha\alpha)} \delta_{kl} \right] v_{l,k}^{(\alpha)} + \sum_{\beta \neq \alpha} \left(\pi_{(\alpha\beta)} \delta_{kl} - T_l^{(\alpha\beta)} u_k^{(\alpha\beta)} \right) v_{l,k}^{(\beta)} + \frac{q_k^{(\alpha)}}{\theta_{(\alpha)}^2} \theta_{,k}^{(\alpha)} - \sum_{\beta \neq \alpha} N_{(\alpha\beta)} (\dot{\theta}_{(\beta)} + u_k^{(\alpha\beta)} \theta_{,k}^{(\beta)}) - \sum_{\beta} \pi_{(\alpha\beta)} u_k^{(\alpha\beta)} \varrho_{,k}^{(\beta)} - T_l^{(\alpha\beta)} \dot{v}_l^{(\beta)} - \frac{1}{\theta_{(\alpha)}} (\hat{R}_l^{(\alpha)} + \hat{r}_l^{(\alpha)}) v_l^{(\alpha)} + \frac{\hat{e}_{(\alpha)}}{\theta_{(\alpha)}} - \hat{n}_{(\alpha)} + \int_{v_{(\alpha)}-\sigma_{(\alpha)}} (\Phi_{(\alpha)} - \Phi^*_{(\alpha)})\, dv_{(\alpha)} > 0.$$

If we integrate the inequality (4.12) over the volume of the body, the last term in the expression disappears (see Eq. (4.6)). The inequality is linear in $\dot{\theta}_{(\beta)}$, $\dot{v}_k^{(\beta)}$, $v_{l,k}^{(\beta)}$, $\varrho_{,k}^{(\beta)}$ and $\theta_{,k}^{(\beta)}$. In order that this inequality may be valid for all independent variations of these variables, the coefficients of these quantities must vanish. Thus, the following relations are obtained:

$$\text{(4.13)} \quad \eta_{(\alpha)} = -\frac{\theta_{(\alpha)}}{\varrho_{(\alpha)}} N_{(\alpha\alpha)} \quad (\alpha = 1, 2),$$

$$\text{(4.14)} \quad t_{kl}^{(\alpha)} = -\theta_{(\alpha)} \pi_{(\alpha\alpha)} \delta_{kl} \quad (\alpha = 1, 2),$$

$$\text{(4.15)} \quad \pi_{(\alpha\beta)} = N_{(\alpha\beta)} = 0 \quad (\alpha \neq \beta),$$

$$\text{(4.16)} \quad T_l^{(\alpha\beta)} = \varrho q_l^{(\alpha)} = 0 \quad (\alpha, \beta = 1, 2).$$

The remaining parts of the inequality take the following form:

$$\text{(4.17)} \quad \int_{v_{(\alpha)}-\sigma_{(\alpha)}} \left[\frac{\hat{e}_{(\alpha)}}{\theta_{(\alpha)}} - \hat{n}_{(\alpha)} - \frac{1}{\theta_{(\alpha)}} (\hat{R}_l^{(\alpha)} + \hat{r}_l^{(\alpha)}) v_l^{(\alpha)} \right] dv_{(\alpha)} \geqslant 0.$$

The constitutive equations are further restricted by the principle of objectivity. This principles requires that the free energy functional should have the following form:

$$\text{(4.18)} \quad \psi_{(\alpha)} = \psi_{(\alpha)}(\varrho_{(\beta)}, \theta_{(\beta)}, u_k^{(\alpha\beta)}; \varrho'_{(\beta)}, \theta'_{(\beta)}, r'_k, u_k'^{(\alpha\beta)}),$$

where

$$\text{(4.19)} \quad u_k^{(\alpha\beta)} \equiv v_k^{(\beta)} - v_k^{(\alpha)}, \quad u_k'^{(\alpha\beta)} \equiv v_k'^{(\beta)} - v_k'^{(\alpha)}, \quad r'_k \equiv x'_k - x_k,$$

and $\psi_{(\alpha)}$ is an isotropic functional (spatially) in its vectorial variables.

The Eqs. (4.15) and (4.16) place certain restrictions on the free energy functions. These stipulations lead us to some integro-differential equations whose solutions are extremely difficult. From (4.15) and (4.16), some of these restrictions are given bellow:

$$\frac{\partial\psi_{(\alpha)}}{\partial\varrho_{(\beta)}} + \int\limits_{v_{(\alpha)}-\sigma} \left(\frac{\delta\psi_{(\alpha)}}{\delta\varrho'_{(\beta)}}\right)^* dv' = 0 \quad (\alpha \neq \beta),$$

(4.20)
$$\frac{\partial\psi_{(\alpha)}}{\partial\theta_{(\beta)}} + \int\limits_{v_{(\alpha)}-\sigma} \left(\frac{\delta\psi_{(\alpha)}}{\delta\theta'_{(\beta)}}\right)^* dv' = 0 \quad (\alpha \neq \beta),$$

$$\frac{\partial\psi_{(\alpha)}}{\partial u_k^{(\alpha\beta)}} + \int\limits_{v_{(\alpha)}-\sigma} \left(\frac{\delta\psi_{(\alpha)}}{\delta u_k'^{(\alpha\beta)}}\right)^* dv' = 0 \quad (\alpha \neq \beta).$$

From the Eqs. (4.18) and (4.20), it is seen that, by contrast with the results of local (classical) mixture theories, in nonlocal theories, the free energy function of a particular constituent of the mixture depends on the densities, temperatures and the diffusion velocities of the other species as well as its own. However, this dependence is not arbitrary, but rather restricted by the Eqs. (4.20). One of the particular solutions of these nonlinear integro-differential equations is obtained by selecting the free energy densities of the form:

(4.21)
$$\psi_{(\alpha)} = \psi_{(\alpha)}(\varrho_{(\alpha)}, \theta_{(\alpha)}; \varrho'_{(\alpha)}, \theta'_{(\alpha)}, r'_k) \quad (\alpha = 1, 2).$$

This form of free energy function automatically satisfies the Eqs. (4.20). For our future purposes, this form of free energy will be used in the remaining part of the work.

Since $\psi_{(\alpha)}$ is independent of the diffusion velocity, and has the form of (4.21), it may be convenient to express the rate of momentum transfer vector as:

(4.22)
$$\hat{r}_l^{(\alpha)} = \varrho_{(\alpha)} \int\limits_{v_{(\alpha)}-\sigma_{(\alpha)}} \left[\left(\frac{\delta\psi_{(\alpha)}}{\delta r'_l}\right)^* - \left(\frac{\delta\psi_{(\alpha)}}{\delta r'_l}\right)\right] dv'_{(\alpha)} + {}_D\hat{r}_l^{(\alpha)},$$

where ${}_D\hat{r}_l^{(\alpha)}$ is the diffusive part of the momentum transfer vector subject to

(4.23)
$${}_D\hat{r}_l^{(\alpha)}\big|_{\mathbf{u}^{(\beta\gamma)}=0} = 0.$$

Introducing (4.22) into (4.17), the entropy inequality becomes:

(4.24)
$$\int\limits_{v_{(\alpha)}-\sigma_{(\alpha)}} \left(\frac{\hat{e}_{(\alpha)}}{\theta_{(\alpha)}} - \hat{n}_{(\alpha)} - \frac{1}{\theta_{(\alpha)}}\, {}_D\hat{r}_l^{(\alpha)} v_l^{(\alpha)}\right) dv_{(\alpha)} \geqslant 0.$$

The principle of objectivity further requires that the free energy should have the fol owing form:

(4.25)
$$\psi_{(\alpha)} = \psi_{(\alpha)}(\varrho_{(\alpha)}, \theta_{(\alpha)}; \varrho'_{(\alpha)}, \theta'_{(\alpha)}, r'),$$

with

$$r' = |\mathbf{x}' - \mathbf{x}|.$$

From this general formulation, we can obtain nonlinear constitutive equations of various orders for nonlocal mixtures. In what follows, we shall give the constitutive relations which are linear in the diffusion velocities.

Constitutive equations linear in diffusion velocities

In this special case, the form of the rate of linear momentum transfer should read:

$$ {}_D\hat{r}_l^{(\alpha)} = \sum_{\beta} \Big(\nu_{(\alpha\beta)} u_l^{(\alpha\beta)} + \int_{v_{(\alpha)}} \nu'_{(\alpha\beta)} u_l'^{(\alpha\beta)} dv' \Big), \tag{4.26} $$

where $\nu_{(\alpha\beta)}(\varrho_{(\gamma)})$ and $\nu_{(\alpha\beta)}(r')$ are respectively the local and nonlocal collision frequencies of α and β species.

Moreover, the partial stresses have the following form:

$$ t_{kl}^{(\alpha)} = -\pi_{(\alpha)} \delta_{kl} \quad (\alpha = 1, 2) \tag{4.27} $$

with

$$ \pi_{(\alpha)}(\varrho_{(\alpha)}, \varrho'_{(\alpha)}, r') \equiv \varrho_{(\alpha)}^2 \left[\frac{\partial \psi_{(\alpha)}}{\partial \varrho_{(\alpha)}} + \int_{v_{(\alpha)}} \left(\frac{\delta \psi_{(\alpha)}}{\delta \varrho'_{(\alpha)}} \right)^* dv' \right]. \tag{4.28} $$

Introducing the Eqs. (4.22), (4.26), (4.27) and (4.28) into (2.16), the following field equations are obtained:

$$ \frac{\partial \varrho_{(\alpha)}}{\partial t} + (\varrho_{(\alpha)} v_k^{(\alpha)})_{,k} = 0, \tag{4.29} $$

$$ \frac{\partial \pi^{(\alpha)}}{\partial x_l} + \varrho_{(\alpha)}(f_l^{(\alpha)} - v_l'^{(\alpha)}) + {}_E\hat{r}_l^{(\alpha)} + \sum_{\beta} \Big(\nu_{(\alpha\beta)} u_l^{(\alpha\beta)} + \int_{v_{(\alpha)}} \nu'_{(\alpha\beta)} u_l'^{(\alpha\beta)} dv' \Big) = 0 \tag{4.30} $$

where, in the interests of brevity, we have defined

$$ {}_E\hat{r}_l^{(\alpha)} \equiv -\varrho_{(\alpha)} \int_{v_{(\alpha)}} \left[\left(\frac{\delta \psi_{(\alpha)}}{\delta r'} \right)^* + \frac{\delta \psi_{(\alpha)}}{\delta r'} \right] \frac{r'_l}{r'} dv' \quad (\alpha = 1, 2). \tag{4.31} $$

By means of these field equations and properly posed boundary conditions, one can, above all, solve certain problems which may be of some practical and physical interest to workers in this domain. As a very special case, let us assume that the body forces are zero and the body is at rest i.e. $v_l^{(\alpha)} \equiv v_l'^{(\alpha)} \equiv 0$. Thus, the field equations take the following simple forms:

$$ \frac{\partial_0 \pi^{(\alpha)}}{\partial x_l} + \varrho_{(\alpha)}^0 \int_{v_{(\alpha)}} \left[\left(\frac{\delta \psi_{(\alpha)}}{\delta r'} \right)^* + \frac{\delta \psi_{(\alpha)}}{\delta r'} \right] \frac{r'_l}{r'} dv' = 0, \tag{4.32} $$

where ${}_0\pi^{(\alpha)}$ is the hydrostatic pressure and $\varrho_{(\alpha)}^0$ is the initial mass density of the αth component.

As is seen from the Eq. (4.32), by contrast with classical theories of mixtures, the hydrostatic pressure for nonlocal mixtures is not a constant; it must rather be determined from the solution of the Eq. (4.32). If the initial mass distribution throughout the body is uniform, then, one of the solutions of the Eq. (4.32) is given by

$$ {}_0\pi^{(\alpha)} = 2\varrho_{(\alpha)}^0 \psi_{(\alpha)}^0(r'), \tag{4.33} $$

where $\psi_{(\alpha)}^0(r') = \psi_{(\alpha)}(\varrho_{(\alpha)}; \varrho'_{(\alpha)}, r')\big|_{\rho_{(\alpha)} = \rho_{(\alpha)}^0}$, which is not necessarily a constant, but is rather a functional in r'.

Diffusion of nonlocal gas mixtures

As another application of the theory, here we formulate the nonlocal diffusion problem (or generalized Fick's law). To this end, we assume that:

i) the body and inertia forces of the body are negligible;

ii) the total mass density of the mixture is constant — i.e., $\varrho = \varrho_{(1)}+\varrho_{(2)} = \varrho_0 = $ constant.

Under these assumptions, the field equations for each species become:

$$\frac{\partial \varrho_{(1)}}{\partial t} + (\varrho_{(1)} v_k^{(1)})_{,k} = 0, \tag{4.34}$$

$$\frac{-\partial \pi^{(1)}}{\partial x_l} + {}_E\hat{r}_l^{(1)} + \nu u_l^{(12)} + \int_v \nu' u_l'^{(12)} dv' = 0, \tag{4.35}$$

$$\frac{\partial \varrho_{(2)}}{\partial t} + (\varrho_{(2)} v_k^{(2)})_{,k} = 0, \tag{4.36}$$

$$-\frac{\partial \pi^{(2)}}{\partial x_l} + {}_E\hat{r}_l^{(2)} - \nu u_l^{(12)} - \int_v \nu' u_l'^{(12)} dv' = 0, \tag{4.37}$$

where we have defined ν and ν' as

$$\nu_{(12)} \equiv -\nu_{(21)} \equiv \nu, \quad \nu'_{(12)} \equiv -\nu'_{(21)} \equiv \nu'_{(r')}. \tag{4.38}$$

Let $\hat{\varrho}_{(\alpha)}$ be the deviation of mass density of the species from its initial value — i.e., $\hat{\varrho}_{(\alpha)} \equiv \varrho_{(\alpha)} - \varrho_{(\alpha)}^0$; then, in the accoustical approximation the above equations become:

$$\frac{\partial \hat{\varrho}_{(1)}}{\partial t} + \varrho_{(1)}^0 v_{k,k}^{(1)} = 0, \tag{4.39}$$

$$-K_{(1)} \frac{\partial \varrho_{(1)}}{\partial x_l} - \int_v \frac{\partial K'_{(1)}}{\partial x_l} \varrho'_{(1)} dv' + \nu u_l^{(12)} + \int_v \nu u_l^{(12)} dv' = 0, \tag{4.40}$$

$$\frac{\partial \hat{\varrho}_{(2)}}{\partial t} + \varrho_{(2)}^0 v_{k,k}^{(2)} = 0, \tag{4.41}$$

$$-K_{(2)} \frac{\partial \hat{\varrho}_{(2)}}{\partial x_l} - \int_v \frac{\partial K'_{(2)}}{\partial x_l} \hat{\varrho}'_{(2)} dv' - \nu u_l'^{(12)} - \int_v \nu' u_l'^{(12)} dv' = 0. \tag{4.42}$$

In obtaining these equations, we have made the following assumptions:

$$\pi^{(\alpha)} = {}_0\pi^{(\alpha)} + K_{(\alpha)} \hat{\varrho}_{(\alpha)} + \int_v K'_{(\alpha)} \hat{\varrho}'_{(\alpha)} dv', \tag{4.43}$$

where

$$K'_{(\alpha)} = K'_{(\alpha)}(r'), \quad K_{(\alpha)} = \text{constant},$$

and ${}_0\pi^{(\alpha)}$ is the solution of

$$\frac{\partial {}_0\pi^{(\alpha)}}{\partial x_l} + \varrho_{(\alpha)}^0 \int_v \left[\left(\frac{\delta \psi_{(\alpha)}}{\delta r'} \right)^* + \frac{\delta \psi_{(\alpha)}}{\delta r'} \right] \frac{r'_l}{r'} dv' = 0 \tag{4.44}$$

Noting the relation $\partial K'_{(\alpha)}/\partial x_l = -\partial K'_{(\alpha)}/\partial x'_l$, and making use of the Green-Gauss theorems in (4.40) and (4.42), we obtain:

$$(4.45)\qquad -K_{(1)}\frac{\partial\hat{\varrho}'_{(1)}}{\partial x_l}+\int_v K'_{(1)}\frac{\partial\hat{\varrho}_{(1)}}{\partial x'_l}dv'+\nu u_l^{(12)}+\int_v \nu' u_l'^{(12)}dv'-\oint_S K'_{(1)}\hat{\varrho}'_{(1)}n'_l da'=0,$$

$$(4.46)\qquad -K_{(2)}\frac{\partial\hat{\varrho}_{(2)}}{\partial x_l}+\int_v K'_{(2)}\frac{\partial\hat{\varrho}'_{(2)}}{\partial x'_l}dv'-\nu u_l^{(12)}-\int_v \nu' u_l'^{(12)}dv'-\oint_S K'_{(2)}\hat{\varrho}'_{(2)}n'_l da'=0.$$

Here n'_l is the unit exterior normal to the material surface S, and da'_l is the infinitesimal area of the surface at $\mathbf{x}'$.

In this work, we are interested in an infinite body, so that the surface effects may be disregarded. Keeping these facts in mind, and taking the divergence of the Eqs. (4.45) and (4.46), we have:

$$(4.47)\qquad -K_{(1)}\nabla^2\hat{\varrho}_{(1)}-\int_v K'_{(1)}\nabla'^2\hat{\varrho}'_{(1)}dv'+\nu(v_{l,l}^{(1)}-v_{l,l}^{(2)})-\int_v \nu'(v_{l,l}'^{(1)}-v_{l,l}'^{(2)})dv'=0,$$

$$(4.48)\qquad -K_{(2)}\nabla^2\hat{\varrho}_{(2)}-\int_v K'_{(2)}\nabla'^2\hat{\varrho}'_{(2)}dv'-\nu(v_{l,l}^{(1)}-v_{l,l}^{(2)})+\int_v \nu'(v_{l,l}'^{(1)}-v_{l,l}'^{(2)})dv'=0,$$

where ∇^2 is the Laplacian operator in space variables. Making use of (4.39) and (4.41) in the Eqs. (4.47) and (4.48), and eliminating $v_{l,l}^{(\alpha)}$ among these equations, we obtain

$$(4.49)\qquad \int_v D'_{(1)}(|\mathbf{x}'-\mathbf{x})|\nabla'^2\hat{m}'_{(1)}dv'+\int_v \chi'(|x'-x|)\frac{\partial\hat{m}_{(1)}}{\partial t}dv'=0,$$

$$(4.50)\qquad \int_v D'_{(2)}(|\mathbf{x}'-\mathbf{x}|)\nabla'^2\hat{m}'_{(2)}dv'+\int_v \chi'(|x'-x|)\frac{\partial\hat{m}'_{(2)}}{\partial t}dv'=0,$$

where $\delta(\mathbf{x}'-\mathbf{x})$ is the Delta function, and we have defined:

$$(4.51)\qquad \begin{aligned} D'_{(\alpha)} &\equiv [K'_{(\alpha)}+K_{(\alpha)}\delta(x'-x)]/\varrho^0,\\ \chi' &\equiv \frac{1}{\varrho^0}\left(\frac{1}{\varrho^0_{(1)}}+\frac{1}{\varrho^0_{(2)}}\right)[\nu\delta(\mathbf{x}'-\mathbf{x})-\nu'(|\mathbf{x}'-\mathbf{x}|)],\\ \hat{m}_{(\alpha)} &\equiv \hat{\varrho}_{(\alpha)}/\varrho^0 \qquad \text{(mass concentration deviation)}, \end{aligned}$$

and made use of the fact that

$$(4.52)\qquad \sum_\alpha \hat{m}_{(\alpha)}\equiv 0.$$

If the condition (4.52) — i.e. $\hat{m}_{(1)}=-\hat{m}_{(2)}$ — is employed in the Eqs. (4.49) and (4.50), we obtain the following relation:

$$(4.53)\qquad D'_{(1)}=D'_{(2)} \quad\text{or}\quad K_{(1)}=K_{(2)} \quad\text{and}\quad K'_{(1)}=K'_{(2)}.$$

In fact, such a result is to be expected as a consequence of the incompressibility of the mixture.

If the contribution of nonlocal diffusion velocity on the rate of momentum transfer is disregarded — e.g. ν' is negligibly small — the diffusion equations take the following form:

$$\int_v D'(|\mathbf{x}'-\mathbf{x}|)\nabla'^2 m' dv' + \frac{\partial m}{\partial t} = 0, \tag{4.54}$$

where, in the interests of brevity, we have set:

$$D'(|\mathbf{x}'-\mathbf{x})| \equiv \frac{1}{\nu}[K'_{(1)} + K_{(1)}\delta(\mathbf{x}'-\mathbf{x})]\left(\frac{1}{\varrho^0_{(1)}} + \frac{1}{\varrho^0_{(2)}}\right)^{-1}, \tag{4.55}$$

$$\hat{m}_{(1)} \equiv m.$$

This integro-differential equation is the generalization of classical Fick's law.

The Eq. (4.54) and properly posed boundary and initial conditions may be used to solve the problem of mass diffusion of a gas in a mixture. Yet, the form of the Kernel function $D'(|\mathbf{x}'-\mathbf{x}|)$ is not known to us. In solving certain practical problems, it might be a good approximation to select this function as an exponential in $|\mathbf{x}'-\mathbf{x}|$ or $|\mathbf{x}'-\mathbf{x}|^2$. The former is valid for somewhat strong interactions (c.f. ERINGEN [11]), while the latter is for a weaker nonlocal interaction.

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