Archives of Mechanics • Archiwum Mechaniki Stosowanej • 30, 1, pp. 17-26, Warszawa 1978

Axisymmetric problem of the punch for the consolidating semi-space with mixed boundary permeability conditions

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IN THE PAPER we investigate the state of deformations and stresses in the consolidating semi-space, edged by the axially-symmetric punch and with mixed boundary permeability conditions. In particular, the contact stresses, liquid pressure and the punch displacement are determined. The results of the paper are the generalization of the author's earlier result for the fully permeable (or non-permeable) boundary.

W pracy zbadany jest stan naprężenia i deformacji w konsolidującej półprzestrzeni obciążonej osiowo-symetrycznym stemplem przy nieznanych warunkach brzegowych przepuszczalności. W szczególności wyznaczone są naprężenia kontaktowe, ciśnienie cieczy oraz przemieszczenie stempla. Praca jest uogólnieniem wcześniejszych badań Autorów, dotyczących całkowicie przepuszczalnego (lub nieprzepuszczalnego) brzegu.

Темой работы являются деформационное и напряженное состояния консолидирующего полупространства в условиях нажима осесимметричного штампа со смешанными условиями проницаемости границ. В частности работа посвящена определению контактных напряжений, давления жидкости под штампом и оседания штампа. Результаты работы являются обобщением более раннего результата авторов полученного для вполне проницаемой (или непроницаемой) границы.

1. Introduction

RECENTLY, considerable attention has been paid to the contact problems of the theory of consolidation [4, 5, 6, 12, 13, 14, 1, 15, 7, 10]. The mixed boundary permeability conditions have been analysed by GASZYŃSKI and KUMANIECKA [9] as well as by ARENICZ and GASZYŃSKI [2]; both papers investigated a given load for the whole boundary (stress boundary conditions). Thus far there has been no research into the influence of the mixed permeability conditions in the contact problem of the punch with a known zone of contact. In this paper we consider the axisymmetric state of deformation and stresses in a consolidating visco-elastic semi-space, loaded by a rigid punch under a given force $P(t)$. The semi-space boundary is non-permeable under the punch and permeable elsewhere.

The results of the paper are a direct generalization of those obtained by the authors in [10].

Considerations are carried out for the most general, linear model of a consolidating porous medium with the visco-elastic skeleton without aging. In such a case the constitutive relations of the skeleton have the form [10]

$$\sigma_{ij} = 2N\varepsilon_{ij} + M\varepsilon_{kk}\delta_{ij} - Ap\delta_{ij}, \tag{1.1}$$

where N, M, A are the Volterra integral operators of the second kind, σ_{ij} — coordinates of the stress tensor of the skeleton, ε_{ij} — coordinates of the strain tensor of the skeleton, p — liquid pressure in pores.

In addition, we should satisfy the equilibrium equations (with the body forces neglected)

$$\sigma_{ij,j} = 0, \tag{1.2}$$

the geometric relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}), \tag{1.3}$$

and the flow equation for the liquid in the porous deformable medium [15]

$$\frac{k}{\gamma}\Delta p = \frac{3n}{\alpha_w}\dot{p}+\dot{\varepsilon}, \tag{1.4}$$

where u_i — coordinates of the displacement vector in the skeleton, k — filtration coefficient, γ — density of the medium, n — porosity of the medium α_w — compressibility modulus of the liquid, Δ — Laplace operator, $(\dot{\cdot}) = \frac{\partial}{\partial t}$.

2. Formulation of the problem

Making use of the cylindrical reference frame $0, r, \varphi, z$, we consider the flat smooth circular punch, being edged into the semi-space $z \geqslant 0$ by the force $P(t)$ (Fig. 1).

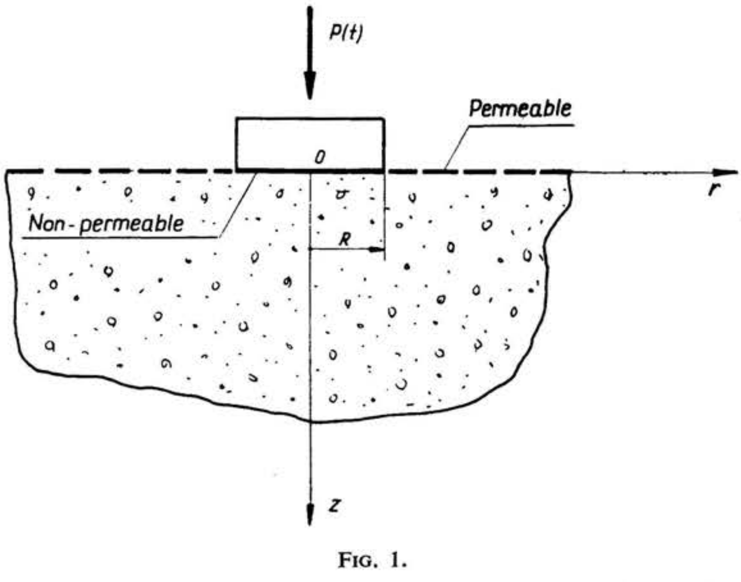

FIG. 1.

For the axial symmetry the governing set of displacement equations of the consolidation theory takes the well-known form [10]

$$\begin{aligned} N\left(\Delta u-\frac{u}{r^2}\right)+(N+M)\varepsilon_{,r} &= Ap_{,r},\\ N\Delta w+(N+M)\varepsilon_{,z} &= Ap_{,z},\\ \frac{k}{\gamma}\Delta p &= \frac{3n}{\alpha_w}\dot{p}+\dot{\varepsilon}, \end{aligned} \tag{2.1}$$

where $u = u(r, z, t)$, $w = w(r, z, t)$ — radial and vertical displacements, respectively.

The boundary conditions are as follows:

$$
\begin{aligned}
&w(r, 0, t) = c(t), && r < R,\\
&\sigma_z(r, 0, t) = 0, && r > R,\\
&\sigma_{rz}(r, 0, t) = 0, && r > 0,\\
&p_{,z}(r, z, t)_{z=0} = 0, && r < R,\\
&p(r, 0, t) = 0, && r > R.
\end{aligned} \tag{2.2}
$$

With respect to the infinite region of the medium, we adopt the conditions in infinity

$$
(u, w, p) \to 0 \quad \text{for} \quad \sqrt{r^2+z^2} \to \infty. \tag{2.3}
$$

Assuming an increment of the force from zero to a given finite value to be slow in time, we consider the homogeneous initial conditons

$$
u(r, z, 0) = w(r, z, 0) = p(r, z, 0) = 0, \tag{2.4}
$$

which satisfy the compatibility conditions for the set (2.1) [11].

3. General solution of Eqs. (2.1)

The method of a general solution of the set (2.1) has been presented in a few papers, e.g. [10 and 11]. Making use of the Hankel and Laplace integral transformations, according to the definitions

$$
\begin{aligned}
&\tilde{\bar{u}}(\omega, z, s) = \int_0^\infty \int_0^\infty u(r, z, t) r J_1(\omega r) e^{-st} dr dt,\\
&\begin{bmatrix} \tilde{\bar{w}}(\omega, z, s) \\ \tilde{\bar{p}}(\omega, z, s) \end{bmatrix} = \int_0^\infty \int_0^\infty \begin{bmatrix} w(r, z, t) \\ p(r, z, t) \end{bmatrix} r J_0(\omega r) e^{-st} dr dt,
\end{aligned} \tag{3.1}
$$

we arrive at the transforms of the inquired functions in the form

$$
\begin{aligned}
\tilde{\bar{u}} &= C_3 e^{-\omega z} + C_1 \frac{\bar{N}+\bar{M}}{2\bar{N}} \left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{\bar{N}+\bar{M}} \right) z e^{-\omega z} - C_2 \frac{A\omega}{s\bar{B}(2\bar{N}+\bar{M})} e^{-mz},\\
\tilde{\bar{w}} &= C_4 e^{-\omega z} + C_1 \frac{\bar{N}+\bar{M}}{2\bar{N}} \left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{\bar{N}+\bar{M}} \right) z e^{-\omega z} - C_2 \frac{Am}{s\bar{B}(2\bar{N}+\bar{M})} e^{-mz},\\
\tilde{\bar{p}} &= C_1 e^{-\omega z} + C_2 e^{-mz},
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
&[\bar{A}, \bar{N}, \bar{M}] = \int_0^\infty [A, N, M] e^{-st} dt,\\
&\bar{B} = \frac{\gamma}{k} \left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{2\bar{N}+\bar{M}} \right), \qquad m^2 = m^2(\omega, s) = \omega^2 + s\bar{B}.
\end{aligned} \tag{3.3}
$$

We find the transforms of the stresses, we are interested in by using the relations (1.1) and obtaining

$$
\begin{aligned}
\tilde{\bar{\sigma}}_z &= 2\bar{N}\tilde{\bar{\varepsilon}}_z + \bar{M}\tilde{\bar{\varepsilon}} - \bar{A}\tilde{\bar{p}}, \\
\tilde{\bar{\sigma}}_{rz} &= 2\bar{N}\tilde{\bar{\varepsilon}}_{rz}.
\end{aligned}
\tag{3.4}
$$

4. Reduction of mixed boundary conditions to the set of dual integral equations

Now we proceed to the analysis of the boundary conditions. Taking into account the relation $(3.4)_2$, we obtain from the condition $(2.2)_3$

$$
C_1\frac{\bar{N}+\bar{M}}{2\bar{N}}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{\bar{N}+\bar{M}}\right)+C_2\frac{2\bar{A}\omega m}{s\bar{B}(2\bar{N}+\bar{M})}-(C_3+C_4)\omega = 0. \tag{4.1}
$$

Furthermore, according to the method of solution of the set (2.1) (decoupling due to the heightening of the order of the filtration equation), we obtain an additional relation

$$
2\bar{N}\omega(C_3-C_4)+\left[\frac{3n}{\alpha_w}(3\bar{N}+\bar{M})+\bar{A}\right]C_1 = 0,
$$

which, together with the relation (4.1), yields the relations

$$
\begin{aligned}
C_3 &= -\frac{3n}{\alpha_w}\frac{1}{2\omega}C_1+\frac{\bar{A}m}{s\bar{B}(2\bar{N}+\bar{M})}C_2, \\
C_4 &= \frac{2\bar{N}+\bar{M}}{\bar{N}}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{2\bar{N}+\bar{M}}\right)\frac{1}{2\omega}C_1+\frac{\bar{A}m}{s\bar{B}(2\bar{N}+\bar{M})}C_2.
\end{aligned}
\tag{4.2}
$$

The remaining constants C_1 and C_2 follow from the mixed conditions $(2.2)_{1,2}$ and $(2.2)_{4,5}$. Taking into account the relations (3.2), (3.4) and (4.2), we obtain for those

$$
\begin{aligned}
&\int_0^\infty C_1J_0(\omega r)d\omega = \frac{2\bar{N}}{2\bar{N}+\bar{M}}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{2\bar{N}+\bar{M}}\right)^{-1}\bar{c}(s), \qquad r<R, \\
&\int_0^\infty \left[C_1(\bar{N}+\bar{M})\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{\bar{N}+\bar{M}}\right)+C_2\frac{2\bar{A}\bar{N}\omega(m-\omega)}{s\bar{B}(2\bar{N}+\bar{M})}\right]\omega J_0(\omega r)d\omega = 0, \qquad r>R, \\
&\int_0^\infty (C_1\omega+C_2m)\omega J_0(\omega r)d\omega = 0, \qquad r<R, \\
&\int_0^\infty (C_1+C_2)\omega J_0(\omega r)d\omega = 0, \qquad r>R.
\end{aligned}
\tag{4.3}
$$

Substituting

$$
\begin{aligned}
\Phi_1(\omega,s) &= \left[C_1(\bar{N}+\bar{M})\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{\bar{N}+\bar{M}}\right)+C_2\frac{2\bar{A}\bar{N}\omega(m-\omega)}{s\bar{B}(2\bar{N}+\bar{M})}\right]\omega, \\
\Phi_2(\omega,s) &= C_1\omega+C_2\omega,
\end{aligned}
\tag{4.4}
$$

we get

$$C_1 = \frac{1}{\omega M(\omega, s)} \Phi_1(\omega, s) - \frac{2\bar{A}\bar{N}\omega(m-\omega)}{s\bar{B}(2\bar{N}+\bar{M})} \frac{1}{\omega M(\omega, s)} \Phi_2(\omega, s),$$

$$C_2 = \frac{1}{\omega M(\omega, s)} \Phi_1(\omega, s) + (\bar{N}+\bar{M})\left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{\bar{N}+\bar{M}}\right) \frac{1}{\omega M(\omega, s)} \Phi_2(\omega, s),$$

where

$$M(\omega, s) = (\bar{N}+\bar{M})\left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{\bar{N}+\bar{M}}\right) - \frac{2\bar{A}\bar{N}\omega(m-\omega)}{s\bar{B}(2\bar{N}+\bar{M})}.$$

The above relations in the equations (4.3) are used and, after simple manipulations, we obtain

$$\int_0^\infty \left[\frac{\Phi_1(\omega, s)}{\omega M(\omega, s)} - \frac{2\bar{A}\bar{N}\omega(m-\omega)}{s\bar{B}(2\bar{N}+\bar{M})} \frac{\Phi_2(\omega, s)}{\omega M(\omega, s)}\right] J_0(\omega r)\,d\omega = \frac{2\bar{N}}{2\bar{N}+\bar{M}}\left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{2\bar{N}+\bar{M}}\right)^{-1} c(s), \quad r < R,$$

(4.5)
$$\int_0^\infty \Phi_1(\omega, s) J_0(\omega r)\,d\omega = 0, \quad r > R,$$

$$\int_0^\infty \left[\frac{m-\omega}{M(\omega, s)} \Phi_1(\omega, s) - (\bar{N}+\bar{M})\left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{\bar{N}+\bar{M}}\right) \frac{m-\omega}{M(\omega, s)} \Phi_2(\omega, s) - \omega\Phi_2(\omega, s)\right] \times J_0(\omega r)\,d\omega = 0, \quad r < R,$$

$$\int_0^\infty \Phi_2(\omega, s) J_0(\omega r)\,d\omega = 0, \quad r > R.$$

We write these equations in a form which is more suitable for further considerations:

$$\int_0^\infty \left\{\frac{1}{\omega}[A_{11} - B_{11} H(\omega, s)]\Phi_1(\omega, s) - \frac{1}{\omega}[A_{12} - B_{12} H(\omega, s)]\Phi_2(\omega, s)\right\} J_0(\omega r)\,d\omega = \frac{\bar{A}\bar{N}(\bar{N}+\bar{M})}{2\bar{N}+\bar{M}} \bar{c}(s), \quad r < R,$$

(4.6)
$$\int_0^\infty \Phi_1(\omega, s) J_0(\omega r)\,d\omega = 0, \quad r > R,$$

$$\int_0^\infty \left\{\frac{1}{\omega}[A_{21} - B_{21} H(\omega, s)]\Phi_1(\omega, s) - \frac{1}{\omega}[A_{22} - B_{22} H(\omega, s)]\Phi_2(\omega, s) - \omega\Phi_2(\omega, s)\right\} \times J_0(\omega r)\,d\omega = 0, \quad r < R,$$

$$\int_0^\infty \Phi_2(\omega, s) J_0(\omega r)\,d\omega = 0, \quad r > R,$$

where we denote

$$
(4.7)\qquad
\begin{aligned}
A_{11} &= 1, & B_{11} &= s\bar{B}\frac{\bar{A}\bar{N}}{2\bar{N}+\bar{M}},\\
A_{12} &= \frac{\bar{A}\bar{N}}{2\bar{N}+\bar{M}}, & B_{12} &= s\bar{B}\frac{\bar{A}\bar{N}(\bar{N}+\bar{M})}{2\bar{N}+\bar{M}}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{\bar{N}+\bar{M}}\right),\\
A_{21} &= s\frac{\gamma}{2k(\bar{N}+\bar{M})}, & B_{21} &= s^2\bar{B}\frac{\gamma}{2k}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{\bar{N}+\bar{M}}\right),\\
A_{22} &= s\frac{\gamma}{2k}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{\bar{N}+\bar{M}}\right), & B_{22} &= s^2\bar{B}\frac{\gamma}{2k}(\bar{N}+\bar{M})\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{\bar{N}+\bar{M}}\right)^2,
\end{aligned}
$$

$$H(\omega, s) = \frac{1}{(m+\omega)^2 M(\omega, s)}.$$

The set of equations (4.6) is a particular case of a more general set of dual integral equations

$$\int_0^\infty \{\omega^{-1}[\overset{*}{A}_{ij}+\overset{*}{H}_{ij}(\omega)]+\omega[\overset{**}{A}_{ij}+\overset{**}{H}_{ij}(\omega)]\}\Phi_j(\omega)J_0(\omega r)d\omega = f_i(r), \qquad r < R,$$

$$\int_0^\infty \Phi_i(\omega)J_0(\omega r)d\omega = 0, \qquad r > R,$$

$$i = 1, 2, \ldots, n,$$

which has been considered and solved in the paper [8]. A similar set has been obtained by the authors of the paper [9] who discussed the consolidation of the semi-space under a given loading of the boundary and for mixed permeability conditions for the boundary.

We seek the solution of the sets of dual integral equations in the form [8]

$$
(4.8)\qquad
\begin{aligned}
\Phi_1(\omega, s) &= \omega\int_0^R \varphi_1(\xi, s)\cos\omega\xi\, d\xi,\\
\Phi_2(\omega, s) &= \omega\int_0^R \varphi_2(\xi, s)\cos\omega\xi\, d\xi,
\end{aligned}
$$

with the boundary conditions

$$(4.9)\qquad \varphi_2(R, s) = \varphi_2'(0, s) = 0.$$

In such a case Eqs. $(4.6)_2$ and $(4.6)_4$ are satisfied identically. This follows from the properties of the Weber-Schafheitlin integral. Substituting the relations (4.8) into Eqs. $(4.6)_1$ and $(4.6)_3$ and taking into account the results of the paper [8], we finally obtain the set of integro-differential equations of the Fredholm type for the unknown functions $\varphi_1(\xi, s)$ and $\varphi_2(\xi, s)$:

$$
(4.10)\qquad A_{11}\varphi_1(r, s)-\frac{1}{\pi}B_{11}\int_0^R K(r, \xi, s)\varphi_1(\xi, s)d\xi - A_{12}\varphi_2(r, s)
+\frac{1}{\pi}B_{12}\int_0^R K(r, \xi, s)\varphi_2(\xi, s)d\xi = \frac{2}{\pi}\,\frac{\bar{A}\bar{N}(\bar{N}+\bar{M})}{2\bar{N}+\bar{M}}\,\bar{c}(s),
$$

$$A_{21}\varphi_1(r,s)-\frac{1}{\pi}B_{21}\int\limits_0^R K(r,\xi,s)\varphi_1(\xi,s)d\xi-A_{22}\varphi_2(r,s)$$
$$+\frac{1}{\pi}B_{22}\int\limits_0^R K(r,\xi,s)\varphi_2(\xi,s)d\xi+\varphi_2''(r,s)=0,$$

where

$$K(r,\xi,s)=\int\limits_0^\infty H(\omega,s)[\cos\omega(\xi+r)+\cos\omega(\xi-r)]d\omega. \tag{4.11}$$

The complex structure of kernels of these equations does not allow to find the exact solutions. However, we can easily analyse the properties of solutions because both kernels in Eq. (4.11) and the right-hand sides of Eqs. (4.10) are continuous functions, integrable with their squares. From these properties we conclude that they pass to the functions $\varphi_1(r,s)$ and $\varphi_2(r,s)$. The construction of resolvents for the kernels of Eqs. (4.10) as well as the construction of the solution itself have been discussed in details in the papers [9 and 10]. Further, we present a qualitative analysis of the solution, written in the following form:

$$\varphi_i(r,s)=\left[\alpha_i-\beta_i\int\limits_0^R R(r,\xi,s)d\xi\right]\frac{2}{\pi}\frac{\bar{A}\bar{N}(\bar{N}+\bar{M})}{2\bar{N}+\bar{M}}\bar{c}(s),\quad i=1,2, \tag{4.12}$$

where $R(r,\xi,s)$ — resolvent of the kernel $K(r,\xi,s)$, α_i,β_i — coefficients depending on A_{ij} and B_{ij}.

Denoting for the sake of brevity

$$R_i^*(r,s)=\alpha_i-\beta_i\int\limits_0^R K(r,\xi,s)d\xi,\quad i=1,2, \tag{4.13}$$

we have

$$\varphi_i(r,s)=\frac{2}{\pi}\frac{\bar{A}\bar{N}(\bar{N}+\bar{M})}{2\bar{N}+\bar{M}}\bar{c}(s)R_i^*(r,s),\quad i=1,2. \tag{4.14}$$

5. Distribution of contact stresses and porous pressure in the contact region and of the punch displacement

Using Eqs. $(4.6)_2$ and $(4.8)_1$, we easily find the distribution of contact stresses under the punch. We have for the Laplace transforms of these stresses

$$\bar{q}(r,s)=\bar{\sigma}_z(r,s)_{r<R}=\int\limits_0^\infty \Phi_1(\omega,s)J_0(\omega r)d\omega=\int\limits_0^\infty \omega J_0(\omega r)\int\limits_0^R \varphi_1(\xi,s)\cos\omega\xi\, d\xi\, d\omega$$
$$=\int\limits_0^\infty\left[\sin\omega\xi\varphi_1(\xi,s)|_0^R-\int\limits_0^R\varphi_1'(\xi,s)\sin\omega\xi\, d\xi\right]J_0(\omega r)d\omega,$$

and, after integration,

$$\bar{q}(r,s) = \frac{\varphi_1(R,s)}{\sqrt{R^2-r^2}} - \int\limits_r^R \frac{\varphi_1'(\xi,s)}{\sqrt{\xi^2-r^2}}\,d\xi. \tag{5.1}$$

In addition, for the punch edged by the force $P(t)$ the following equilibrium condition must be satisfied:

$$2\pi \int\limits_0^R q(r,t)\,dr = P(t).$$

It follows for the Laplace transform

$$2\pi \int\limits_0^R \bar{q}(r,s)\,dr = \bar{P}(s), \qquad \bar{P}(s) = \int\limits_0^\infty P(t)e^{-st}\,dt. \tag{5.2}$$

Taking into account the formula (4.14) in Eqs (5.1) and (5.2), we obtain

$$\frac{L\bar{A}\bar{N}(\bar{N}+\bar{M})}{2\bar{N}+\bar{M}}\,\bar{c}(s)L(s)\bar{P}(s), \tag{5.3}$$

where

$$L(s) = \int\limits_0^R \left[\frac{R_1^*(R,s)}{\sqrt{R^2-r^2}} - \int\limits_0^R \frac{dR_1^*(\xi,s)}{d\xi}\,\frac{d\xi}{\sqrt{\xi^2-r^2}}\right]dr.$$

From Eq. (5.3) we obtain the transform of the punch displacement

$$\bar{c}(s) = \frac{2\bar{N}+\bar{M}}{4\bar{A}\bar{N}(\bar{N}+\bar{M})}\cdot\frac{\bar{P}(s)}{L(s)}. \tag{5.4}$$

The inverse transform of the punch displacement is therefore given by the formula

$$c(t) = \frac{1}{2\pi i}\int\limits_s \frac{2\bar{N}+\bar{M}}{4\bar{A}\bar{N}(\bar{N}+\bar{M})}\,\frac{\bar{P}(s)}{L(s)}e^{st}\,ds. \tag{5.5}$$

Substituting the expression (5.4) into Eqs. (4.14) and (5.1), we get after the inversion of the Laplace transformation the following formula for the contact stresses:

$$q(r,t) = \frac{1}{2\pi i}\int\limits_s \frac{\bar{P}(s)}{2\pi L(s)}\left[\frac{R_1^*(R,s)}{\sqrt{R^2-r^2}} - \int\limits_r^R \frac{dR_1^*(\xi,s)}{d\xi}\,\frac{d\xi}{\sqrt{\xi^2-r^2}}\right]e^{st}\,ds. \tag{5.6}$$

In a similar way we obtain from the relations $(4.6)_4$ and $(4.8)_4$ the expression for the Laplace transform of the porous pressure

$$\bar{p}_0(r,s) = \bar{p}(r,s)_{r<R} = \frac{\varphi_2(R,s)}{\sqrt{R^2-r^2}} - \int\limits_r^R \frac{\varphi_2'(\xi,s)}{\sqrt{\xi^2-r^2}}\,d\xi,$$

or, applying the boundary condition (4.9),

$$\bar{p}_0(r,s) = -\int\limits_r^R \frac{\varphi_2'(\xi,s)}{\sqrt{\xi^2-r^2}}\,d\xi. \tag{5.7}$$

Taking into account the relations (4.14) and (5.4), we obtain after the inversion of the Laplace transformation

$$(5.8) \qquad p_0(r,t) = \frac{1}{2\pi i}\int_{\mathfrak{s}} \frac{\bar{p}(s)}{2\pi L(s)} \int_r^R \frac{dR_1^*(\xi,s)}{d\xi}\,\frac{d\xi}{\sqrt{\xi^2-r^2}}\,e^{st}ds.$$

The formulae (5.5), (5.6) and (5.8) are the essence of the solution of the problem. In order to find the numerical values of the interesting quantities, the burdensome integrations (inverse transforms) should be carried out; however they do not in the least lead to any substantial difficulties, as proved in the papers [10 and 19]. The Krylov method of inversion of the transforms, successfully applied by the authors, can be recommended. With respect to the known procedure of this method [9 and 10], we do not present it here. Hence, we can treat the problem as fully solved.

It should be noted (the formula (5.8)) that the mixed boundary permeability conditions do not change the character of singularity of the contact stresses — the differences are only quantitative, when compared to the case of the permeable (or non-permeable)

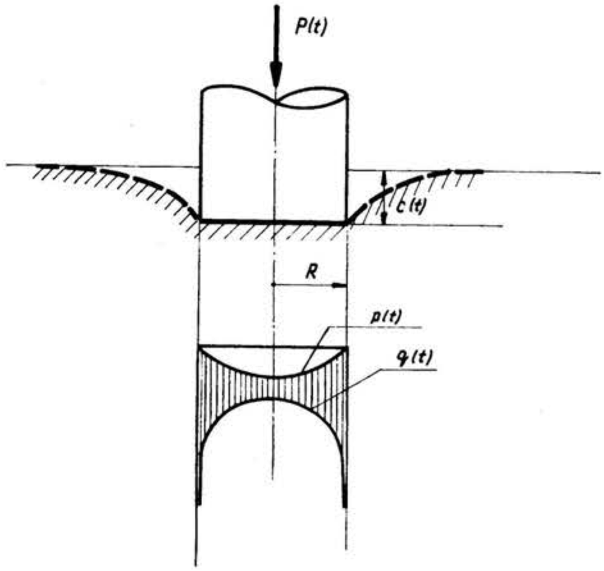

FIG. 2.

boundary. This is due to the continuity of the liquid pressure, taking the zero value on the edge of the punch (the function (5.8) is continuous). The character of the distribution of stresses and liquid pressure in pores is shown in Fig. 2.

The results obtained are the generalization of these presented in [10].

Simultaneously, we should point to the results of the paper due to AGBEZUGE and DERESIEWICZ [1], in which the authors consider a problem similar to ours for the spherical punch. However, with respect to the approximate method of solution (an assumed constant external force P, the time-independence of contact stresses, etc.), applied in [1], our results are not a particular case of these delivered in [1]. In the solution presented above we pay special attention to the qualitative features of the contact problem (in this case to the problem of a known zone of contact, implying immediately the singularity of the solution, not appearing in the results of [1]). In our opinion this problem is exactly and consistently taken into account.

References

1. L. K. AGBEZUGE, H. DERESIEWICZ, *On the indentation of a consolidating half-space*, Isr. J. Techn., 12, 1974.
2. R. ARENICZ, J. GASZYŃSKI, *Konsolidacja półprzestrzeni lepkosprężystej z mieszanymi warunkami przepuszczalności brzegu — zagadnienie płaskie*, Rozpr. Inżyn., **25**, 3, 1977.
3. M. A. BIOT, *General theory of three-dimensional consolidation*, J. Appl. Phys., 12, 1941.
4. J. B. BOOKER, *A numerical method for the solution of Biot's consolidation theory*, Quart. J. Mech. Appl. Math., **26**, 4, 1973.
5. W. DERSKI, *On application of dual integral equations in theory of consolidation* [in Polish], II Symp. Phed., Wrocław 1966.
6. J. DOMSKI, G. SZEFER, *Consolidation of elastic half-plane loaded by punch* [in Polish], Arch. Hydrotech., **21**, 2, 1974.
7. J. GASZYŃSKI, *On a certain solution of dual integral equations and its applications to contact problems of consolidation*, Arch. Mech., **28**, 1, 1976.
8. J. GASZYŃSKI, *Some solution of set of dual integral equations with Bessel kernels*, Eng. Trans., **24**, 4, 1976.
9. J. GASZYŃSKI, A. KUMANIECKA, *Consolidation of visco-elastic half-space with mixed boundary condition for permeability*, Eng. Trans., **25**, 4, 1977.
10. J. GASZYŃSKI, G. SZEFER, *Axisymmetric punch problem under condition of consolidation*, Arch. Mech., **27**, 3, 1975.
11. J. GASZYŃSKI, G. SZEFER, *Consolidation of visco-elastic half-space loaded by concentrated force*, Eng. Trans., **23**, 3, 1975.
12. В. И. КЕРЧМАН, *Контактная задача теории консолидации водонасыщенной среды*, Изд. АН СССР, МТТ, 3, 1974.
13. В. З. ПАРТОН, *Дуальные интегральные уравнения в контактной задаче консолидации*, Изд. М.В. и ССО 1968.
14. В. З. ПАРТОН, Б. А. КУДРАВЦЕВ, *Контактная задача механики деформации пористих вязкоупрсгих сред*, Пробл. мех. Тверд. деф. тела.Судпромгиз 1971.
15. Ю. К. ЗАРЕЦКИЙ, *Теория консолидации грунтов*, Изд. Наука 1967.

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Received November 29, 1976.