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BRIEF NOTES

Laminar hydromagnetic pipe flow with heat transfer

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VELOCITY and magnetic field distributions having dependence on both radial and axial directions have been obtained by using perturbation technique for incompressible viscous flow through a porous-walled circular pipe in the presence of a uniform radial magnetic field. The heat transfer problem has been reduced to a boundary value problem in ordinary differential equations and has been treated numerically by Galerkin's method for determining the temperature distribution for different sets of values of parameters characteristic of flow conditions.

Introduction

THE PROBLEM of laminar forced convection MHD heat transfer in the presence of a transverse magnetic field has been studied for both flat plate and channel flows with constant temperature or uniform flux conditions at the walls by HWANG, KNIEPER and FAN [1], C. W. TAN and K. SUH [2] and L. F. GENIN and A. K. PODSHIBYACIN [3]. MITTAL [4] has analysed the problem of MHD heat transfer in a circular pipe with the wall kept at a constant temperature gradient in the direction of flow while GARDNER [5] has considered the problem of laminar MHD pipe flow in a transverse magnetic field with prescribed uniform flux at the wall.

In this paper, the problem of heat transfer in a two-dimensional laminar steady-state flow of incompressible viscous and electrically-conducting fluid in a porous circular pipe in the presence of a uniform radially transverse magnetic field has been studied. The effect of wall porosity has been taken into account by a suitable choice for divergence free velocity and magnetic fields (BERMAN [6, 7]). The perturbation method has been adopted to solve the coupled non-linear ordinary differential equations of flow for small values of the cross-flow Reynolds number and then the velocity profile and magnetic field distributions thus obtained have been utilized in constructing an approximate numerical solution, using the Galerkin method, of the energy equation for different sets of values of parameters characteristic of the flow conditions. The combined effect of wall porosity and magnetic field on the possibility of flow reversal has also been studied.

1. Basic equations and boundary conditions

We use cylindrical polar coordinates (r, ϕ, z) with the z-axis coincident with the central axis of symmetry. Let a be the radius of the pipe and v_0, H_0 be the uniform cross-flow (radial) velocity (suction for $v_0 > 0$ and injection for $v_0 < 0$) and constant radially-transverse magnetic field. Then the boundary conditions satisfied by the radial and axial components of velocity and magnetic field, i.e. $(v_r(r, z), 0, v_z(r, z))$ and $(H_r(r, z), 0, H_z(r, z))$ for the axis-symmetric flow are

$$
\begin{aligned}
&v_r(a, z) = v_0, \\
&v_z(a, z) = 0, \\
&v_r(0, z) = 0 \quad \text{and} \quad \frac{\partial v_z(r, z)}{\partial r}\Big|_{r=0} = 0, \\
&H_r(a, z) = H_0, \\
&H_z(a, z) = 0.
\end{aligned} \tag{1.1}
$$

The boundary condition $(1.1)_3$ follows from the symmetry of the flow while the conditions $(1.1)_{4,5}$ follow from continuity of the magnetic field components.

An appropriate choice for the components of velocity and magnetic fields satisfying the divergence relation is

$$
\begin{aligned}
v_r &= \frac{2av_0 F(r)}{r}, \quad v_z = -\frac{2av_0 zF'(r)}{r}, \\
h_r &= \frac{2ah_0 G(r)}{r}, \quad h_z = -\frac{2ah_0 zG'(r)}{r},
\end{aligned} \tag{1.2}
$$

in which $F(r)$ and $G(r)$ are non-dimensional unknown functions to be found and prime denotes the derivative with respect to r; the components $(h_r, h_z) = 1/\sqrt{4\pi\varrho}\,(H_r, H_z)$ etc. have dimensions of velocity.

Introducing the following non-dimensional quantities

$\xi = \dfrac{2z}{a}$ axial distance coordinate,

$\eta = \left(\dfrac{r}{a}\right)^2$ radial distance coordinate,

$\overline{\omega}(\xi, \eta) = \dfrac{p(\xi, \eta)}{\frac{1}{2}\varrho \overline{v}_z^2(0)}$ pressure in which $\overline{v}_z(0)$ is the average axial component of velocity at $z = 0$,

$R = \dfrac{av_0}{\nu}$ cross-flow Reynolds number; ν kinematic viscosity,

$R_e = \dfrac{av_z(0)}{4\nu}$ entrance Reynolds number,

$\delta = \dfrac{h_0}{v_0}$ parameter characterizing conditions of flow,

$\varepsilon = \dfrac{\nu}{\nu_H}$ fluid characteristic parameter,

$\nu_H = \dfrac{1}{4\pi\sigma}$ magnetic diffusivity, σ being electrically-conductive, the hydromagnetic momentum equations (in electromagnetic units) in absence of free charges and displacement currents become

$$(1.3)\qquad \begin{aligned} \frac{1}{\xi}\frac{\partial\tilde{\omega}}{\partial\xi} &= -\frac{1}{2}\frac{R}{R_e^2}(\eta F'')' - \frac{1}{2}\left(\frac{R}{R_e}\right)^2\{\delta^2 GG'' + (F'^2 - FF'')\},\\ \frac{\partial\tilde{\omega}}{\partial\eta} &= \frac{1}{2}\frac{R}{R_e^2}F'' - \frac{1}{4}\left(\frac{R}{R_e}\right)^2\left\{\delta^2\xi^2(G'^2)' + \left(\frac{F^2}{\eta}\right)'\right\}, \end{aligned}$$

which, on eliminating $\overline{\omega}$, yield

$$(1.4)\qquad (\eta F'')' + R\{(F'^2 - FF'') - \delta^2(G'^2 - GG'')\} = k,$$

k being the integration constant, while the induction equation becomes

$$(1.5)\qquad \varepsilon R(FG' - F'G) = \eta G''$$

in which prime now denotes the derivative with respect to η.

The boundary conditions (1.1) now take the form

$$(1.6)\qquad \begin{aligned} &F(0) = 0, \quad F(1) = \frac{1}{2}, \quad F'(1) = 0, \quad \lim_{\eta\to 0}\eta^{\frac{1}{2}}F''(\eta) = 0,\\ &G(0) = 0, \quad G(1) = \frac{1}{2}. \end{aligned}$$

The solution of the coupled non-linear ordinary differential equations (1.4), (1.5) subject to the boundary conditions (1.6) determines the velocity profile and magnetic field distribution.

2. Velocity field and magnetic field distributions

The system of equations (1.4), (1.5) and (1.6) is solved approximately for small values of the cross-flow Reynolds number R by assuming a solution of the form

$$(2.1)\qquad \begin{aligned} F(\eta) &= F_0(\eta) + F_1(\eta)R + F_2(\eta)R^2 + \dots,\\ G(\eta) &= G_0(\eta) + G_1(\eta)R + G_2(\eta)R^2 + \dots,\\ k &= k_0 + k_1R + k_2R^2 + \dots, \end{aligned}$$

where F_n, G_n are independent of R and k_n are constants.

On using Eqs. (2.1), the equations (1.4) and (1.5) lead to a set of decoupled linearized system of equations which, on solving subject to the boundary conditions $(1.6)_{1,2}$, yield up to and including terms involving R^2,

$$(2.2)\qquad \begin{aligned} F(\eta) &= \left(\eta - \frac{1}{2}\eta^2\right) + \frac{R}{72}(4\eta - 9\eta^2 + 6\eta^3 - \eta^4)\\ &\quad + \frac{R^2}{10\,800}(166\eta - 380\eta^2 + 275\eta^3 - 75\eta^4 + 15\eta^5 - \eta^6),\\ G(\eta) &= \frac{1}{2}\eta + \frac{R}{24}(\eta^3 - \eta) + \frac{R^2}{2880}[(-13\eta + 30\eta^3 - 20\eta^4 + 3\eta^5)\\ &\quad + \varepsilon(-7\eta - 10\eta^3 + 20\eta^4 - 3\eta^5)],\\ k &= -1 + \left(0.75 - \frac{\delta^2}{4}\right)R + \left(0.0407 + \frac{\varepsilon\delta^2}{24}\right)R^2. \end{aligned}$$

Having determined the functions $F(\eta)$ and $G(\eta)$ as in Eq. (2.2), the components of velocity and magnetic field are then readily obtained from Eqs. $(1.2)_1$ and $(1.2)_2$ respectively.

Pressure distribution

On integrating Eqs. $(1.3)_{1,2}$, the axial and radial pressure variations in non-dimensional form are given respectively by

$$\tilde{\omega}(0,\eta)-\omega(\xi,\eta)=\frac{1}{4}\frac{R}{R_e^2}\xi^2(k+\delta^2G'^2R) \tag{2.3}$$

and

$$\tilde{\omega}(\xi,0)-\tilde{\omega}(\xi,\eta)=\frac{1}{4}\frac{R}{R_e^2}\left[\frac{F^2}{\eta}R-2\{F'(\eta)-F'(0)\}\right]. \tag{2.4}$$

It may be noticed from Eq. (2.3) that axial pressure drop is not uniform across a cross-section of the pipe. In fact, this non-uniformity is due to the presence of a magnetic field.

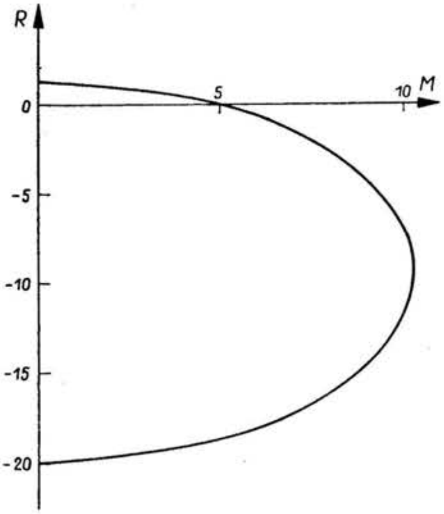

FIG. 1. *R versus M* for flow reversal.

It is further observed that the necessary condition for occurrence of flow reversal is that the value of the expression $k+\delta^2G'^2R$ should change sign.

The average value of $k+\delta^2G'^2R$ across the entire cross-section of the pipe, on using the expression for $G(\eta)$ as given in Eq. (2.2) is

$$\langle(k+\delta^2G'^2R)\rangle=0.0407R^2+0.75R+\left(\frac{M^2}{24}-1\right) \tag{2.5}$$

in which the Hartmann number $M=[\delta^2\in R^2]^{\frac{1}{2}}=\dfrac{ah_0}{\nu\nu_H}$ has been introduced.

The values of the cross-flow Reynolds number R for which flow reversal will occur are given by the roots of the equation

$$R^2+18.40R+1.0227(M^2-24)=0 \tag{2.6}$$

which has been shown graphically in Fig. 1.

It is interesting to observe that Eq. (2.6) possesses real roots only for values of M which are less than 10.33 and for values of M ranging from 0 to 4.9, one root is positive (i.e. suction Reynolds number) while the other root is negative (i.e. injection Reynolds number). For values of M belonging to the range $4.9 < M < 10.33$, both roots are negative.

When $M = 0$ (i.e. absence of a magnetic field) the value of the positive root comes out to be 1.25 which agrees with the result obtained by BERMAN [7] for porous pipe flow in the absence of a magnetic field.

When the fluid is being withdrawn through the wall of the pipe, the flow region will be restricted to a finite length of the pipe. It can be readily seen from the equation of contituity that the range of allowable values of ξ is $0 \leqslant \xi \leqslant 4\dfrac{R_e}{R}$.

3. Temperature field

For the flow under study, the relevant form of the energy equation and the boundary conditions to be satisfied by temperature T are

$$\varrho c\left(v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z}\right) = \varkappa\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \phi(r, z) \tag{3.1}$$

in which the dissipation function ϕ (including both viscous and Joule's dissipation) is given by

$$\phi = \varrho\nu\left[2\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\frac{v_r^2}{r^2} + \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2\right] + \varrho\nu_H\left(\frac{\partial h_r}{\partial z} - \frac{\partial h_z}{\partial r}\right)^2,$$

c being specific heat at constant pressure and $\varkappa$ the coefficient of thermal conductivity, and

$$\begin{aligned} T &= T_1 \quad \text{at} \quad r = a, \\ \frac{\partial T}{\partial r} &= 0 \quad \text{at} \quad r = 0. \end{aligned} \tag{3.2}$$

Introducing the non-dimensional temperature $\pi(\xi, \eta)$ by

$$\pi(\xi, \eta) = \frac{(T - T_1)c}{v_0^2}$$

and using Eqs. (1.2a,b), the energy equation (3.1) and the boundary conditions (3.2) in non-dimensional form become

$$\frac{\partial^2 \pi}{\partial \xi^2} + \eta\frac{\partial^2 \pi}{\partial \eta^2} + \frac{\partial \pi}{\partial \eta} - RP_r\left(F\frac{\partial \pi}{\partial \eta} - \xi F'\frac{\partial \pi}{\partial \xi}\right) + \Psi(\xi, \eta) = 0, \tag{3.3}$$

$$\pi(\xi, 1) = 0, \quad \lim_{\eta \to 0} \eta^{\frac{1}{2}}\frac{\partial \pi}{\partial \eta} = 0, \tag{3.4}$$

where $P_r = \frac{\varrho \nu c}{\varkappa}$ is the Prandtl number, and

$$\Psi(\xi, \eta) = P_r \left[8F'^2 + \frac{4}{\eta^2}(F^2 - 2\eta F F') + 4\xi^2 \eta (F'')^2 + \frac{\delta^2}{\varepsilon}(G'')^2 \right\}\Bigg]$$

in which $F(\eta)$ and $G(\eta)$ are given by Eq. (2.2).

If the wall of the pipe is maintained at a constant axial temperature gradient A (prescribed), an appropriate form for the temperature function $\pi(\xi, \eta)$ is

$$\pi(\xi, \eta) = A\xi + \theta(\eta)$$

and Eqs. (3.3) and (3.4) reduce to

$$L[\theta(\eta)] = Q(\xi, \eta) \tag{3.5}$$

in which, $L = \eta \frac{d^2}{d\eta^2} + (1 - RP_r F)\frac{d}{d\eta}$, $Q(\xi, \eta) = -ARP_r F'\xi - \Psi(\xi, \eta)$

and

$$\theta(1) = 0 \quad \text{and} \quad \lim_{\eta \to 0} \eta^{\frac{1}{2}} \frac{\partial \theta}{\partial \eta} = 0, \tag{3.6}$$

where A is given by

$$A = -\frac{\int\limits_0^1 [\eta\theta'' + (1 - RP_r F)\theta' + \Psi] d\eta}{\int\limits_0^1 RP_r F' \xi d\eta} \text{ as in [8].} \tag{3.7}$$

Let $\{\phi_k(\eta)\}$, $k = 0, 1, 2, \ldots, n$, be a finite set of linearly independent continuous functions which satisfy the boundary conditions (3.6). For the particular problem under study we choose

$$\phi_k(\eta) = (1 - \eta^2)\eta^k, \quad k = 0, 1, 2, \ldots, n. \tag{3.8}$$

An approximate solution $\theta^*(\eta)$ of Eq. (3.5) can be constructed by a linear combination of $\phi_k(\eta)$ as

$$\theta^*(\eta) = \sum_{j=0}^{n} a_j \phi_j(\eta). \tag{3.9}$$

Substituting from Eq. (3.9) in Eq. (3.5), we get

$$L[\theta^*(\eta)] = L[\Sigma a_j \phi_j(\eta)] = \Sigma a_j L[\phi_j(\eta)] = Q(\xi, \eta). \tag{3.10}$$

Since L is a linear differential operator and ϕ_k are known functions of the selected set (3.8), Eq. (3.10) is a functional relation in terms of unknown constants a_j. Multiplying succesively Eq. (3.10) by each function of the set $\phi_k(\eta)$ and integrating over the range of η, we get the following set of $n+1$ linear algebraic equations in $n+1$ unknowns a_j, $j = 0, 1, 2, \ldots, n$

$$\sum_{j=0}^{n} a_j \int\limits_0^1 L[\phi_j(\eta)]\phi_k(\eta) d\eta = \int\limits_0^1 Q(\xi, \eta)\phi_k(\eta) d\eta, \quad k = 0, 1, 2, \ldots, n. \tag{3.11}$$

After evaluating the integrals in Eqs. (3.11), the above system can be written in the form

$$\sum_{j=0}^{n} a_j\{A_0(j,k)+A_1(j,k)+\ldots+A_8(j,k)\} = B_1+B_2+\ldots+B_{10}, \quad k = 0,1,2,\ldots,n \tag{3.12}$$

in which $A_0, A_1, \ldots, A_8$ and $B_1, B_2, \ldots, B_{10}$ are known in terms of expressions involving the parameters R, P_r and M.

4. Conclusions

The set of linear algebraic equations (3.12) have been solved numerically for the unknown coefficients a_j by means of the Gaussian reduction method. Figure 2 gives the temperature profiles for different values of physical parameters at two different cross-

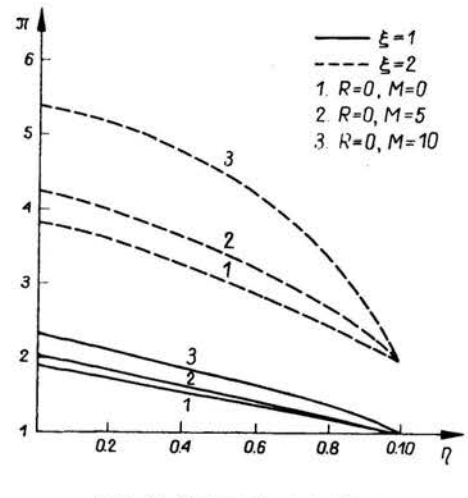

FIG. 2. Temperature profiles.

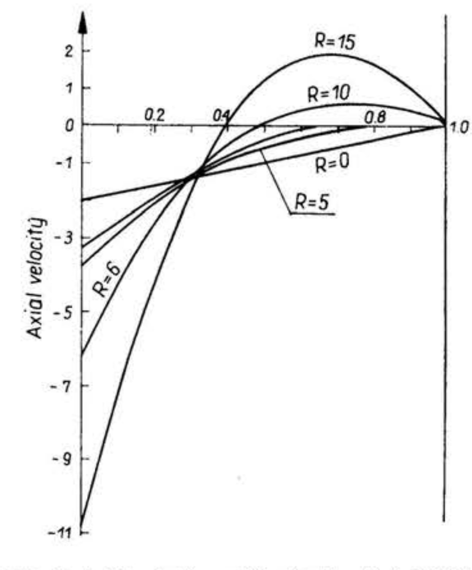

FIG. 3. Axial velocity profiles for $R = 0, 5, 10, 15$.

sections of the channel. It is noticed that temperature increases with the increase in the values of the Hartmann number M and cross-flow Reynolds number R at every cross-section; furthermore, for fixed M and R, temperature increases down-stream in the flow region restricted by $\theta \leqslant \xi \leqslant 4\dfrac{R_e}{R}$. Figure 3 gives the axial velocity profiles for different values of the cross-flow Reynold number R. It is interesting to note that the velocity profiles for which $R \leqslant 6$ maintain the same direction across the entire cross-section of the channel while those for which $R > 6$ suffer reversal in direction, the point of flow reversal approaching closer to the central axis ($\eta = 0$) as R increases.

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