Wave propagation in strongly anisotropic elastic materials (*)

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IN THIS PAPER the nature of wave propagation in a strongly transversely isotropic elastic material is examined. For the case considered the extensional modulus in the direction of the axis of transverse isotropy is much greater than that in any direction at right angles to this axis. The results for both the idealized inextensible and transversely isotropic materials are derived. The speeds of propagation and associated discontinuity vectors are obtained. Some generalizations to non-linear elastic materials are suggested.

W pracy zbadano charakter rozprzestrzeniania się fal w silnie poprzecznie izotropowym sprężystym materiale. W rozważanym przypadku moduł rozciągania w kierunkach osi poprzecznej izotropii jest znacznie większy niż w jakimkolwiek innym kierunku. Uzyskano rezultaty dla dwóch przypadków, mianowicie dla idealnej nierozciągliwości i dla poprzecznie izotropowego materiału. Zostały określone prędkości rozprzestrzeniania się fal i związane z nimi wektory nieciągłości. Zaproponowano pewne uogólnienia dotyczące nieliniowych materiałów sprężystych.

В работе исследован характер распространения волн в сильно поперечно изотропном, упругом материале. В рассматриваемом случае модель растяжения в направлениях оси поперечной изотропии значительно больше, чем в каком-нибудь другом направлении. Получены результаты для двух случаев: для идеальной нерастяжимости и для поперечно изотропного материала. Определены скорости распространения волн и связанные с ними векторы разрыва. Предложено некоторое обобщение, касающееся нелинейных упругих материалов.

1. Introduction

THE THEORY of the mechanical behaviour of a transversely isotropic elastic material which is inextensible in the direction of transverse isotropy has been developed in an attempt to model the behaviour of an isotropic elastic matrix reinforced by a family of parallel strong elastic fibres (see, e.g. SPENCER [1], ROGERS [2]). For such a material the extensional modulus in the fibre direction is much greater than that in a direction at right angles to the fibres and the constraint of inextensibility in the fibre direction is the idealization of this property. The additional constraint of incompressibility is frequently imposed on the idealized material since this leads to mathematically tractable boundary value problems. This constraint is not, however, an essential ingredient of the idealized theory.

CHEN and GURTIN [3] have examined the propagation of acceleration waves in inextensible non-linear elastic materials. Their results show that, in general, there are *two* possible speeds for the waves but that when the direction of propagation is orthogonal to the direction of inextensibility, then *three* wave speeds are possible. SCOTT [4] has examined acceleration waves in non-linear elastic materials subject to one or two internal

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constraints. Taking the inextensibility condition as an example of a single constraint, Scott remarks on the exceptional case of waves propagating at right angles to the direction of the constraint. In this paper we examine the nature of this exceptional case by considering wave propagation in a strongly transversely isotropic elastic material for which the extensional modulus in the direction of the axis of transverse isotropy is much greater than that in any direction at right angles to this axis. This approach was previously employed by EVERSTINE and PIPKIN [5] in order to interpret some of the results arising in the solution of static boundary value problems in the idealized theory.

For simplicity we restrict attention to plane acceleration discontinuities propagating in linear elastic materials for which the axis of transverse isotropy has the same direction at every point. In Sect. 2 we derive the results for the idealized inextensible material, obtaining the speeds of propagation and associated discontinuity vectors. For the general case of two wave speeds the associated discontinuities are orthogonal to the axis of transverse isotropy. In the exceptional case, two of the wave speeds and associated discontinuities correspond with the solution of the general case, the third wave speed is quite distinct and is associated with a discontinuity along the axis of transverse isotropy. The analysis in Sect. 3 relates to all transversely isotropic materials and shows that there are, in general, three speeds of propagation. One of these speeds is determined exactly and approximate expressions are obtained for the other two when the material is strongly anisotropic. It is shown that one of these wave speeds is, as a rule, very large compared with the other two but that this speed decreases rapidly and becomes comparable with the other two as the direction of propagation becomes orthogonal to the axis of transverse isotropy. The associated discontinuity is approximately in the direction of the axis of transverse isotropy becoming exactly so in the limit as the material becomes inextensible. In this limit the wave speed becomes infinite for all directions of propagation which are not orthogonal to the fibre direction, but remains finite when these directions are orthogonal. It is this which gives rise to the exceptional solution. Since the inextensible material is a mathematical idealization, the physical situation is modelled more realistically by the results for the strongly anisotropic material.

The stress-strain relations for the strongly anisotropic solid involve an elastic constant which becomes infinite in the limit as the material becomes inextensible. There is no difficulty in dealing with this for the problem considered here but the generalization to non-linear elastic materials is not straight-forward. An alternative approach, which may be more readily generalized, is to regard the material behaviour as a perturbation on the idealized inextensible material behaviour. The constitutive equations then involve a small parameter which tends to zero as the material becomes inextensible and we refer to the material as "almost inextensible". This approach is equivalent to that employed by EVERSTINE and PIPKIN [6] in developing their singular perturbation method. In Sect. 4 we adopt a suggestion due to PARKER [7] and introduce a Legendre transformation of the strain energy function for a strongly anisotropic material in order to derive the constitutive equations for the almost inextensible material. It is shown that with an appropriate choice of elastic constants the equations for wave propagation in this material are identical with those for the strongly anisotropic material of Sect. 3.

2. Inextensible materials

We consider a transversely isotropic elastic material which is inextensible in the direction of transverse isotropy. Let $u_i(x_k, t)$ denote the components of displacement at time t, relative to a Cartesian system of axes, of the material particle at the point with coordinates x_k. The components e_{ij} of the strain tensor are then given by

$$(2.1) \qquad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

and the components t_{ij} of the Cauchy stress tensor are related to these by the expressions

$$(2.2) \qquad t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + (T - \lambda e_{kk})a_i a_j.$$

In these expressions and throughout this paper the comma denotes partial differentiation so that $u_{i,j} = \partial u_i/\partial x_j$ and the summation convention is employed. The quantities λ, μ_T and μ_L are material constants, δ_{ij} is the Kronecker delta, a_i are the components of a (constant) unit vector in the direction of transverse isotropy and $T = T(x_k, t)$ is the reaction stress associated with the constraint of inextensibility. The deformation is restricted by the condition of inextensibility in the direction of the unit vector **a**, so that the strain components must be compatible with the constraint equation

$$(2.3) \qquad e_{ij}a_i a_j = 0.$$

In any deformation for which the body forces are zero, the stress components must satisfy the equations of motion

$$(2.4) \qquad t_{ij,j} = \varrho\ddot{u}_i,$$

where ϱ is the density of the material and the dot denotes differentiation with respect to time.

Consider a plane acceleration discontinuity surface with a unit normal **n** propagating with speed U through the material. The discontinuities in the second derivatives of u_i are given by the expressions (see, for example, TRUESDELL and TOUPIN [8])

$$(2.5) \qquad [\ddot{u}_i] = \sigma U^2 l_i, \qquad [\dot{u}_{i,j}] = -\sigma U l_i n_j, \qquad [u_{i,jk}] = \sigma l_i n_j n_k.$$

In Eqs. (2.5) the square brackets denote the jumps in the quantities enclosed, l_i are the components of a unit vector which gives the direction of the discontinuity and σ is a scalar which measures the strength of the discontinuity. Equations (2.4) are valid on each side of the discontinuity surface and taking the jump in these and using Eqs. (2.5) gives

$$(2.6) \qquad [t_{ij,j}] = \varrho\sigma U^2 l_i.$$

We assume the stress components t_{ij}, and therefore the reaction stress T, as continuous across the wave. In order to evaluate the left hand side of Eq. (2.6) we differentiate Eqs. (2.2) and take the jump in the resulting equation across the discontinuity surface. Using the definition (2.1) and Eqs. (2.5) to determine the jump in the strain derivatives gives

$$(2.7) \qquad [t_{ij,j}] = \sigma\{\lambda(n_i - a_i a_j n_j)l_k n_k + \mu_T(l_i + n_i l_k n_k)$$

$$+ (\mu_L - \mu_T)(a_i a_k l_k + a_i a_k n_k l_j n_j + a_j n_j a_k l_k n_i + a_j n_j a_k n_k l_i)\} + \tau a_i a_k n_k,$$

where τ is the strength of the discontinuity in the derivative of T which is given by

$$(2.8) \qquad [\partial T/\partial x_j] = \tau n_j, \qquad [\dot T] = -U\tau.$$

Eliminating the jump in the stress derivative between Eqs. (2.6) and (2.7) leads to the equations

$$(2.9) \quad \sigma\{[\mu_T\sin^2\phi+\mu_L\cos^2\phi-\varrho U^2]l_i+[(\lambda+\mu_T)l_kn_k+(\mu_L-\mu_T)\cos\phi a_kl_k]n_i$$
$$-\lambda\cos\phi l_kn_ka_i+(\mu_L-\mu_T)(a_kl_k+\cos\phi l_jn_j)a_i\}+\tau a_i\cos\phi = 0,$$

where we have put $a_kn_k = \cos\phi$, so that ϕ is the angle between $\mathbf{a}$ and $\mathbf{n}$. This is a system of three equations in the four unknowns σ, τ and two independent components of the unit vector $\mathbf{l}$. An additional equation is obtained by taking the jump in the derivative of the constraint equation (2.3) which leads to

$$(2.10) \qquad \sigma n_k a_i l_i \cos\phi = 0.$$

This is satisfied by any one of the conditions

$$(2.11) \qquad \sigma = 0, \quad a_il_i = 0, \quad \cos\phi = 0.$$

The first of these conditions when substituted into Eqs. (2.9) leads to the trivial result $\tau = 0$, except for the case $\cos\phi = 0$. In general, the direction of propagation $\mathbf{n}$ is not orthogonal to $\mathbf{a}$ so that $\cos\phi \neq 0$ and the second of the conditions (2.11) must hold, $a_il_i = 0$. The discontinuity is therefore orthogonal to $\mathbf{a}$ and Eqs. (2.9) become

$$(2.12) \quad \sigma\{[\mu_T\sin^2\phi+\mu_L\cos^2\phi-\varrho U^2]l_i+(\lambda+\mu_T)n_kl_kn_i$$
$$+(\mu_L-\mu_T-\lambda)\cos\phi l_jn_ja_i\}+\tau a_i\cos\phi = 0.$$

Multiplying these equations by a_i and contracting leads to

$$(2.13) \qquad (\sigma\mu_Ln_kl_k+\tau)\cos\phi = 0$$

and for $\cos\phi \neq 0$ this gives

$$(2.14) \qquad \tau = -\sigma\mu_Ln_kl_k.$$

Using Eq. (2.14) to eliminate τ from Eqs. (2.12) leads to the equations

$$(2.15) \quad \{[\mu_T\sin^2\phi+\mu_L\cos^2\phi-\varrho U^2]\delta_{ri}+(\lambda+\mu_T)n_in_r-(\lambda+\mu_T)\cos\phi a_in_r\}l_r\sigma = 0,$$

which, for $\sigma \neq 0$, have the two non-trivial solutions

$$(2.16) \qquad l_r^{(1)} = \frac{e_{rst}a_sn_t}{\sin\phi}, \qquad \varrho U_1^2 = \mu_L\cos^2\phi+\mu_T\sin^2\phi,$$

$$(2.17) \qquad l_r^{(2)} = \frac{n_r-a_r\cos\phi}{\sin\phi}, \qquad \varrho U_2^2 = \mu_L\cos^2\phi+(\lambda+2\mu_T)\sin^2\phi,$$

provided $\sin\phi \neq 0$. When $\sin\phi = 0$ the direction of propagation $\mathbf{n}$ coincides with the direction of inextensibility $\mathbf{a}$ and $\mathbf{l}$ is then any vector which is orthogonal to $\mathbf{a}$. The speed of propagation is given by $\varrho U^2 = \mu_L$ for all such vectors. This speed of propagation is clearly the limiting value of both Eqs. (2.16) and (2.17) as $\sin\phi \to 0$.

When the direction of propagation **n** is orthogonal to **a** the third of the conditions (2.11) hold, namely, $\cos\phi = 0$ and Eqs. (2.9) become

(2.18) $\qquad \{[\mu_T - \varrho U^2]\delta_{ir} + (\mu_L - \mu_T)a_i a_r + (\lambda + \mu_T)n_i n_r\}l_r \sigma = 0.$

These have *three* non-trivial solutions:

(2.19) $\qquad \begin{aligned} l_r^{(1)} &= e_{rst}a_s n_t, & \varrho U_1^2 &= \mu_T, \\ l_r^{(2)} &= n_r, & \varrho U_2^2 &= \lambda + 2\mu_T, \\ l_r^{(3)} &= a_r, & \varrho U_3^2 &= \mu_L. \end{aligned}$

The first two of these solutions correspond to the solutions (2.16) and (2.17) respectively in the limiting case as $\cos\phi \to 0$. The third solution only exists for $\cos\phi = 0$ and we have the exceptional case of three waves capable of propagating in any direction orthogonal to the direction of inextensibility whereas only two waves can propagate in any other direction. This is the result first obtained by CHEN and GURTIN [3] and remarked on by SCOTT [4].

3. Strongly anisotropic materials

The constitutive equation for a transversely isotropic material has the form (see SPENCER [1])

(3.1) $\qquad t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki})$

$$+ \alpha(a_k a_m e_{km}\delta_{ij} + e_{kk}a_i a_j) + \beta a_k a_m e_{km}a_i a_j,$$

where α and β are material constants. The constitutive equation (3.1) reduces to the form (2.2) in the limiting case as $\beta \to \infty$ and $a_k a_m e_{km} \to 0$ in such a way that their product is finite. In this limit the first term involving α vanishes and the second term in α may be incorporated into the term $Ta_i a_j$ of Eq. (2.2). Equation (3.1) may be employed to model a strongly anisotropic material by introducing the parameter $\varepsilon \ll 1$ defined by writing

(3.2) $\qquad\qquad\qquad\qquad\qquad \beta = \mu_L/\varepsilon.$

To examine the propagation of acceleration waves we differentiate Eqs. (3.1) with respect to x_j and take the jump in the resulting equation across the discontinuity surface. Taking the jump in the derivatives of Eqs. (2.1) and using the jump conditions (2.5) then gives

(3.3) $\qquad [t_{ij,j}] = \sigma\{(\mu_T \sin^2\phi + \mu_L \cos^2\phi)\delta_{ir} + (\lambda + \mu_T)n_i n_r$

$$+ (\mu_L - \mu_T + \alpha)\cos\phi(a_i n_r + n_i a_r) + (\mu_L - \mu_T + \beta\cos^2\phi)a_i a_r\}l_r.$$

Eliminating the discontinuity in the stress derivative by means of Eqs. (2.6) gives the equations

(3.4) $\qquad \sigma\{(\mu_T \sin^2\phi + \mu_L \cos^2\phi - \varrho U^2)\delta_{ir} + (\lambda + \mu_T)n_i n_r$

$$+ (\mu_L - \mu_T + \alpha)\cos\phi(a_i n_r + n_i a_r) + (\mu_L - \mu_T + \beta\cos^2\phi)a_i a_r\}l_r = 0.$$

Equations (3.4) are the well-known propagation conditions. These give a determinantal condition for the squared wave speed U^2 which, in general, has three roots. In the case when $\sin\phi = 0$, so that the direction of propagation $\mathbf{n}$ coincides with the direction of transverse isotropy $\mathbf{a}$, two of these roots coincide and the solutions are $\varrho U_1^2 = \varrho U_2^2 = \mu_L$ and $\varrho U_3^2 = \lambda + 4\mu_L - 2\mu_T + 2\alpha + \beta$. The first of these speeds corresponds to discontinuities orthogonal to $\mathbf{a}$ ($a_k l_k = 0$) and the second to a discontinuity parallel to $\mathbf{a}$ ($a_k l_k = 1$).

In order to solve the propagation conditions for $\sin\phi \neq 0$ it is convenient to replace Eqs. (3.4) by an equivalent system of equations. To do this we introduce the unit vector $\mathbf{m}$ which lies in the plane of $\mathbf{a}$ and $\mathbf{n}$ and is orthogonal to $\mathbf{a}$. The components of $\mathbf{m}$ are given by

$$(3.5) \qquad m_k = \frac{n_k - a_k \cos\phi}{\sin\phi},$$

from which we have $n_k l_k = m_k l_k \sin\phi + a_k l_k \cos\phi$. We multiply Eqs. (3.4) in turn by $e_{ijk} a_j m_k$, m_i and a_i and carry out the summations over i to obtain the equations

$$(\mu_T \sin^2\phi + \mu_L \cos^2\phi - \varrho U^2) e_{ijk} l_i a_j m_k = 0,$$

$$(3.6) \quad \{(\lambda + 2\mu_T)\sin^2\phi + \mu_L \cos^2\phi - \varrho U^2\} m_k l_k + (\lambda + \mu_L + \alpha)\sin\phi \cos\phi a_k l_k = 0,$$

$$(\lambda + \mu_L + \alpha)\sin\phi \cos\phi m_k l_k + \{\lambda \cos^2\phi + \mu_L(1 + 3\cos^2\phi)$$
$$- 2\mu_T \cos^2\phi + 2\alpha \cos^2\phi + \beta \cos^2\phi - \varrho U^2\} a_k l_k = 0.$$

One solution of these equations is obtained when $\mathbf{l}$ is orthogonal to $\mathbf{a}$ and $\mathbf{m}$ and, therefore, to the plane containing $\mathbf{a}$ and $\mathbf{n}$. The second and third of Eqs. (3.6) are then trivially satisfied and the first equation gives

$$(3.7) \qquad \varrho U_1^2 = \mu_T \sin^2\phi + \mu_L \cos^2\phi, \qquad l_i^{(1)} = e_{ist} a_s m_t.$$

The two remaining solutions are obtained when $\mathbf{l}$ lies in the plane of $\mathbf{a}$ and $\mathbf{m}$, $\mathbf{l} = \mathbf{a}r + \mathbf{m}s$ where r and s are parameters which must satisfy the condition

$$(3.8) \qquad r^2 + s^2 = 1$$

since $\mathbf{l}$ is a unit vector. The first of Eqs. (3.6) is then trivially satisfied and the remaining equations become

$$(3.9) \quad \begin{aligned} \{(\lambda + 2\mu_T)\sin^2\phi + \mu_L \cos^2\phi - \varrho U^2\} s + (\lambda + \mu_L + \alpha)\sin\phi \cos\phi r = 0, \\ (\lambda + \mu_L + \alpha)\sin\phi \cos\phi s + \{(\lambda + 3\mu_L - 2\mu_T + 2\alpha + \beta)\cos^2\phi + \mu_L - \varrho U^2\} r = 0. \end{aligned}$$

These two homogeneous equations have non-trivial solutions for r and s provided ϱU^2 is a root of the corresponding determinantal equation. We are particularly interested in the case where $\beta = \mu_L/\varepsilon$ and $\varepsilon \ll 1$. There exists the possibility that ϱU^2 is of order β and we accordingly assume that U, r and s have the forms

$$U^2 = \frac{1}{\varepsilon}({}^0U^2 + \varepsilon\,{}^1U^2 + \varepsilon^2\,{}^2U^2 + \ldots),$$

$$(3.10)$$
$$r = r_0 + \varepsilon r_1 + \varepsilon^2 r_2 + \ldots,$$

$$s = s_0 + \varepsilon s_1 + \varepsilon^2 s_2 + \ldots.$$

These expressions are substituted into the left hand sides of Eqs. (3.9) and the coefficients of each power of ε are then equated to zero. The terms of order $1/\varepsilon$ give the equations

(3.11)
$$\varrho \,^0U^2 s_0 = 0, \quad (\mu_L \cos^2\phi - \varrho \,^0U^2) r_0 = 0,$$

and the terms independent of ε give

(3.12)
$$\{(\lambda + 2\mu_T)\sin^2\phi + \mu_L \cos^2\phi - \varrho^1 U^2\} s_0 - \varrho^0 U^2 s_1 + (\lambda + \mu_L + \alpha)\sin\phi\cos\phi r_0 = 0,$$
$$(\lambda + \mu_L + \alpha)\sin\phi\cos\phi s_0 + \{(\lambda + 3\mu_L - 2\mu_T + 2\alpha)\cos^2\phi + \mu_L - \varrho^1 U^2\} r_0$$
$$+ (\mu_L \cos^2\phi - \varrho^0 U^2) r_1 = 0.$$

Carrying out the same procedure in Eqs. (3.8) and taking the terms independent of ε and the terms of order ε gives the equations

(3.13)
$$r_0^2 + s_0^2 = 1,$$
$$r_0 r_1 + s_0 s_1 = 0.$$

One solution of Eqs. (3.11) and the first of Eqs. (3.13) is

(3.14)
$$r_0 = 0, \quad s_0 = 1, \quad {}^0U^2 = 0,$$

and when these are substituted into Eqs. (3.12) and the second of Eqs. (3.13), we obtain

(3.15)
$$\varrho^1 U^2 = (\lambda + 2\mu_T)\sin^2\phi + \mu_L \cos^2\phi,$$
$$r_1 = -\frac{(\lambda + \mu_L + \alpha)}{\mu_L}\tan\phi, \quad s_1 = 0.$$

These give a second solution of Eqs. (3.6)

(3.16)
$$\varrho U_2^2 = (\lambda + 2\mu_T)\sin^2\phi + \mu_L \cos^2\phi + 0\left(\frac{1}{\beta}\right),$$
$$l_k^{(2)} = -\frac{(\lambda + \mu_L + \alpha)}{\beta}\tan\phi\, a_k + m_k + 0\left(\frac{1}{\beta^2}\right).$$

To obtain the third solution of Eqs. (3.6) we return to Eqs. (3.11) and (3.13) which have, in addition to Eqs. (3.14) the solutions

(3.17)
$$r_0 = 1, \quad s_0 = 0, \quad \varrho^0 U^2 = \mu_L \cos^2\phi.$$

These, when substituted into Eqs. (3.12), and the second of Eqs. (3.13) give

(3.18)
$$\varrho^1 U^2 = \mu_L + (\lambda + 3\mu_L - 2\mu_T + 2\alpha)\cos^2\phi,$$
$$r_1 = 0, \quad s_1 = \frac{(\lambda + \mu_L + \alpha)}{\mu_L}\tan\phi,$$

and the corresponding solution of Eqs. (3.6) is

(3.19)
$$\varrho U_3^2 = (\lambda + 3\mu_L - 2\mu_T + 2\alpha + \beta)\cos^2\phi + \mu_L + 0\left(\frac{1}{\beta}\right),$$
$$l_k^{(3)} = a_k + \frac{(\lambda + \mu_L + \alpha)}{\beta}\tan\phi\, m_k + 0\left(\frac{1}{\beta^2}\right).$$

In the limit as $\beta \to \infty$ the solutions (3.7) and (3.16) become the solutions (2.16) and (2.17) respectively for waves in the inextensible material. The solution (3.19) then gives

an infinite wave speed, associated with a discontinuity in the direction of inextensibility **a**, provided $\cos\phi$ is non-zero.

The method used to obtain the solutions (3.16) and (3.19) must be re-examined when $\cos\phi$ is close to zero. The solution (3.19) shows that the transition region is given by $\cos\phi = m\sqrt{\varepsilon}$ where m is a parameter of order one. We then write

$$U^2 = {}^0\bar{U}^2 + \sqrt{\varepsilon}\,{}^1\bar{U}^2 + \varepsilon^2\bar{U}^2 + \dots,$$

(3.20)
$$r = \bar{r}_0 + \bar{r}_1\sqrt{\varepsilon} + \bar{r}_2\varepsilon + \dots,$$

$$s = \bar{s}_0 + \bar{s}_1\sqrt{\varepsilon} + \bar{s}_2\varepsilon + \dots$$

and substitute into Eqs. (3.8) and (3.9). The terms independent of ε give

$$(\lambda + 2\mu_T - \varrho^0\bar{U}^2)s_0 = 0,$$

(3.21)
$$(\mu_L(1+m^2) - \varrho^0\bar{U}^2)\bar{r}_0 = 0,$$

$$\bar{r}_0^2 + \bar{s}_0^2 = 1,$$

and the terms of order $\sqrt{\varepsilon}$ give

$$-\varrho^1\bar{U}^2 s_0 + (\lambda + 2\mu_T - \varrho^0\bar{U}^2)\bar{s}_1 + (\lambda + \mu_L + \alpha)m\bar{r}_0 = 0,$$

(3.22)
$$(\lambda + \mu_L + \alpha)m\bar{s}_0 + (\mu_L(1+m^2) - \varrho^0\bar{U}^2)\bar{r}_1 - \varrho^1\bar{U}^2\bar{r}_0 = 0,$$

$$\bar{r}_0\bar{r}_1 + \bar{s}_0\bar{s}_1 = 0.$$

Equations (3.21) and (3.22) have the solutions

$$\bar{r}_0 = 0, \quad \bar{s}_0 = 1, \quad \varrho^0\bar{U}^2 = \lambda + 2\mu ,$$

(3.23)
$$\bar{r}_1 = \frac{(\lambda + \mu_L + \alpha)m}{\lambda + 2\mu_T - \mu_L(1+m^2)} , \quad \bar{s}_1 = 0, \quad \varrho^1\bar{U}^2 = 0$$

and

$$\bar{r}_0 = 1, \quad \bar{s}_0 = 0, \quad \varrho^0\bar{U}^2 = \mu_L(1+m^2),$$

(3.24)
$$\bar{r}_1 = 0, \quad \bar{s}_1 = -\frac{(\lambda + \mu_L + \alpha)m}{\lambda + 2\mu_T - \mu_L(1+m^2)} , \quad \varrho^1\bar{U}^2 = 0.$$

The corresponding solutions of Eqs. (3.6) are then

$$\varrho U_2^2 = \lambda + 2\mu_T + 0(\varepsilon),$$

(3.25)
$$l_k^{(2)} = m_k + \frac{(\lambda + \mu_L + \alpha)\cos\phi}{\lambda + 2\mu_T - \mu_L - \beta\cos^2\phi}\, a_k + 0(\varepsilon),$$

and

$$\varrho U_3^2 = \mu_L + \beta\cos^2\phi + 0(\varepsilon),$$

(3.26)
$$l_k^{(3)} = \frac{-(\lambda + \mu_L + \alpha)\cos\phi}{\lambda + 2\mu_T - \mu_L - \beta\cos^2\phi}\, m_k + a_k + 0(\varepsilon).$$

It is readily shown that the expressions (3.25) and (3.26) remain valid in the limit as $m \to 0$ with ε remaining small but non-zero, corresponding to $\cos\phi$ tending to zero while β remains large but finite.

The solutions (3.19) and (3.26) show that the third wave speed changes from a large value of order $\sqrt{\beta/\varrho}$ to a value comparable with the other two wave speeds as $\cos\phi$ changes from values of order one to values of order $(\mu_L/\beta)^{\frac{1}{2}}$. Further, all three wave speeds are finite and of comparable magnitude for values of ϕ satisfying $\frac{\pi}{2}-m(\mu_L/\beta)^{\frac{1}{2}}$ $< \phi < \frac{\pi}{2}+m(\mu_L/\beta)^{\frac{1}{2}}$ where m is of order one.

4. Almost inextensible materials

In Sect. 3 we considered the inextensible material as the limiting case of a strongly anisotropic material. Here we take a different approach in which we regard the strongly anisotropic material as a perturbation on the inextensible material. Let $W(e_{ij}, a_k)$ denote the strain energy density function of a transversely isotropic elastic material whose axis of transverse isotropy is in the direction of the unit vector $\mathbf{a}$. The stress is

$$(4.1) \qquad t_{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right).$$

Adopting the suggestion of PARKER [7] we define the function $\hat{W}(e_{ij}, a_k, T)$, where $T = t_{pq}a_p a_q$, by the Legendre transformation

$$(4.2) \qquad \hat{W}(e_{ij}, a_k, T) = W(e_{ij}, a_k) - Ta_l a_m e_{lm}.$$

We then have on differentiating that

$$(4.3) \qquad \begin{aligned} \frac{\partial \hat{W}}{\partial T} &= -a_l a_m e_{lm}, \\ \frac{1}{2}\left(\frac{\partial \hat{W}}{\partial e_{ij}} + \frac{\partial \hat{W}}{\partial e_{ji}}\right) &= t_{ij} - Ta_i a_j. \end{aligned}$$

Here the derivatives of $\hat{W}$ with respect to the e_{ij} are not independent since it follows from the definition of T that

$$(4.4) \qquad a_i a_j \frac{\partial \hat{W}}{\partial e_{ij}} = a_i a_j t_{ij} - T = 0.$$

From the first of Eqs. (4.3) it may be seen that the material is inextensible in the direction $\mathbf{a}$ if $\partial\hat{W}/\partial T = 0$. T is then the reaction stress associated with the constraint of inextensibility. The material will be termed "almost inextensible" if $\partial\hat{W}/\partial T \ll (e_{ij}e_{ij})^{\frac{1}{2}}$ for all e_{ij} and all finite T.

For a linear elastic material the most general form for $\hat{W}$ consistent with the constraint (4.4) is

$$(4.5) \qquad \hat{W} = c_1 e_{kk}(e_{ss} - 2a_i a_j e_{ij}) + c_2 e_{ij}e_{ji} + c_3 a_i a_j e_{ik}e_{kj}$$

$$+ (c_1 - c_2 - c_3)a_i a_j e_{ij}a_r a_s e_{rs} + \gamma T(e_{kk} - a_i a_j e_{ij}) + \frac{\delta}{2c_2}T^2,$$

where c_1, c_2 and c_3 are elastic moduli and γ and δ dimensionless constants. Using the expression (4.5) in Eqs. (4.3) gives

(4.6) $t_{ij} = 2c_1(e_{ss}-a_r a_s e_{rs})\delta_{ij} - 2c_1 e_{kk} a_i a_j + 2c_2 e_{ij} + c_3(a_i a_k e_{jk}$

$$+ a_k a_j e_{ki}) + 2(c_1 - c_2 - c_3)a_r a_s e_{rs} a_i a_j + \gamma T \delta_{ij} + (1-\gamma)Ta_i a_j,$$

and

(4.7) $$a_l a_m e_{lm} = -\frac{\partial \hat{W}}{\partial T} = (a_i a_j e_{ij} - e_{kk})\gamma - \frac{\delta}{c_2|}T.$$

It may be seen from Eq. (4.7) that the material is almost inextensible if $\gamma \ll 1$ and $\delta \ll 1$.

We differentiate Eqs. (4.6) with respect to x_j, take the jump across the discontinuity surface, and use Eqs. (2.5), (2.6) and (2.8) to obtain the equations

(4.8) $\sigma\left\{\left(c_2 + \dfrac{1}{2}c_3\cos^2\phi - \varrho U^2\right)\delta_{ir} + (2c_1 + c_2)n_i n_r - \left(2c_1 - \dfrac{1}{2}c_3\right)\cos\phi(a_i n_r\right.$

$$\left. + n_i a_r) + \left[\frac{1}{2}c_3 + 2(c_1 - c_2 - c_3)\cos^2\phi\right]a_i a_r\right\}l_r + \tau\{\gamma n_i + (1-\gamma)\cos\phi a_i\} = 0.$$

Equations (4.8) form a system of three equations in the four unknowns σ, τ and $\mathbf{l}$. In order to obtain the fourth equation we differentiate Eq. (4.7) with respect to time and take the jump across the discontinuity surface to obtain

(4.9) $$\sigma\{\gamma n_r + (1-\gamma)\cos\phi a_r\}l_r + \frac{\delta}{c_2}\tau = 0.$$

Eliminating τ between Eqs. (4.8) and (4.9) leads to the propagation conditions

(4.10) $\sigma\left\{\left(c_2 + \dfrac{1}{2}c_3\cos^2\phi - \varrho U^2\right)\delta_{ir} + (2c_1 + c_2(1-\gamma^2/\delta))n_i n_r\right.$

$$- \left(2c_1 - \frac{1}{2}c_3 + c_2\frac{\gamma}{\delta}(1-\gamma)\right)\cos\phi(a_i n_r + n_i a_r)$$

$$\left. + \left[\frac{1}{2}c_3 + 2(c_1 - c_2 - c_3)\cos^2\phi - c_2\frac{(1-\gamma)^2}{\delta}\cos^2\phi\right]a_i a_r\right\}l_r = 0.$$

Equations (4.10) become identical with Eqs. (3.4) if we put

$$c_2 = \mu_T, \quad c_3 = 2(\mu_L - \mu_T),$$

$$c_1 = \frac{1}{8}(\lambda + \gamma^2\mu_T/\delta),$$

(4.11)

$$\frac{\gamma}{\delta} = -\frac{(\lambda + \alpha)}{\mu_T},$$

$$\frac{1}{\delta} = -\frac{(\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)}{\mu_T}.$$

It may be seen that $\delta \to 0$ and $\gamma \to 0$ as $\beta \to \infty$ and in this limit the material becomes the inextensible solid of Sect. 2. For $\beta = \mu_L/\varepsilon$ and $\varepsilon \ll 1$ we have

$$(4.12) \qquad \delta = -\frac{\mu_T}{\mu_L}\varepsilon + 0(\varepsilon^2), \quad \gamma = -\frac{(\lambda+\alpha)}{\mu_L}\varepsilon + 0(\varepsilon^2)$$

and the material is almost inextensible. The solutions of Eqs. (4.10) in this case have been derived in Sect. 3. By using these solutions in Eqs. (4.9) it is possible to obtain some information on the strengths of the discontinuities. Introducing the vector **m** defined in Sect. 3, Eq. (4.9) may be rewritten as

$$(4.13) \qquad \tau = [\{(\lambda+\alpha)\sin\phi + 0(\varepsilon)\}m_r l_r + \beta\cos\phi a_r l_r].$$

The solution (3.7) for which $l_r^{(1)} = e_{rst}a_s m_t$ gives $\tau_1 = 0$, and the solution (3.16) gives

$$(4.14) \qquad \tau_2 = -\sigma_2(\mu_L\sin\phi + 0(\varepsilon)).$$

In the limit as $\varepsilon \to 0$ these results agree with those derived from Eq. (2.14) for the inextensible material. The third solution [Eqs. (3.19)] leads to

$$(4.15) \qquad \tau_3 = \sigma_3\beta\cos\phi$$

and for $\cos\phi \neq 0$ the discontinuity τ_3 becomes infinite as $\beta \to \infty$.

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