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Optimum reliability and structure in reliability-based structural design (*)

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A PROBLEM is considered to determine simultaneously optimum reliability and structure when the costs caused by the failure of the structure are given. For this purpose the expected total cost is defined; it is taken as the sum of the structural costs and the costs due to the failure of the structure. An algorithmic procedure which applies stochastic programming and the uni-dimensional search technique is developed and a numerical example is presented.

Rozpatrzono problem jednoczesnego wyznaczania optymalnej niezawodności konstrukcji oraz zaprojektowania konstrukcji optymalnej, gdy straty spowodowane jej katastrofą są z góry dane. Funkcją celu jest wtedy suma kosztu konstrukcji i strat wywołanych jej zniszczeniem. Wykorzystując zasady programowania stochastycznego, opracowano algorytm procedury rozwiązującej oraz technikę rozwiązania w przypadku jednowymiarowym. Przedstawiono również przykład numeryczny.

Рассмотрена проблема одновременного определения оптимальной надежности конструкции и проектирования оптимальной конструкции, когда потери, вызванные ее катастрофой, заранее заданы. Функцией цели является тогда сумма стоимостей конструкции и потерь, вызванных ее разрушением. Используя принципы стохастического программирования, разработан алгоритм решающей процедуры и техника решения в одномерном случае. Представлен тоже численный пример.

1. Introduction

LOADS acting on the structures and the strength of the structural elements are sometimes subject to random variations. In such a case structural reliability or, alternatively, the probability of failure has been used as a criterion for structural safety. By applying reliability analysis, optimum design problems have been studied [1-10] to determine the structure for minimizing the structural weight or cost. However, due to the lack of an efficient method for computing the failure probability of a multi-element or multi-mode structure, considering the statistical dependence, either correlation was ignored or an approximate method for calculating correlation was proposed using a Gaussian distribution for all the joint probability distributions. A method was developed [11] for calculating the multi-dimensional Gaussian distribution, using the Hermite polynomial expansion methods. By the use of this method the authors [10] treated a problem to determine the optimum structure for minimizing the structural cost or weight under the specified failure probability of the structure. Applying stochastic programming, an efficient algorithmic procedure was proposed for solving the problem.

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In this paper, a problem is considered to determine simultaneously the optimum failure probability and the structure when the cost caused by the failure of the structure is specified. For this purpose the expected total cost is defined; it is taken as the sum of the structural cost and the cost due to the failure of the structure. An algorithmic procedure is developed and a numerical example is presented.

2. Statement of problem

Consider a structural system in which the structural elements or the failure modes are described by a linear combination of the resistances of the elements and the loads acting on the structure. Hence, the reserve strengths Z_i of the structural elements or the failure modes are given by

$$Z_i = \sum_{j=1}^{n} a_{ij} R_j - \sum_{j=1}^{l} b_{ij} L_j \quad (i = 1, 2, \ldots, m), \tag{2.1}$$

where R_j — structural resistance of the j-th element, L_j — the j-th load acting on the structure, a_{ij} — resistance coefficient determined by the position and condition of the j-th element or failure mode related to the i-th element or failure mode, b_{ij} — load coefficient determined by the position and magnitude of the j-th load on the structure related to the i-th element or failure mode, n — number of structural resistances, l — number of loads, m — number of elements or failure modes.

Failure of the structure occurs when any value of Z_i $(i = 1, 2, \ldots, m)$ is negative, i.e. any one of the elements or failure modes fails. When the structural resistances R_j and the loads L_j exhibit statistical variations and are thus treated as random variables, the reserve strengths Z_i also become random variables. Hence the reliability of the structure must be evaluated by the failure probability of the structure. Let F_i be the event of failure of element or mode i and $\overline{F}_i$ the survival of element or mode i. The failure probability of the structure can be written as follows:

$$P_f = \mathrm{Prob}(F_1) + \mathrm{Prob}(\overline{F}_1 \cap F_2) + \mathrm{Prob}(\overline{F}_1 \cap \overline{F}_2 \cap F_3) + \ldots + \mathrm{Prob}(\overline{F}_1 \cap \overline{F}_2 \cap \ldots \cap \overline{F}_{m-1} \cap F_m) = 1 - \mathrm{Prob}(\overline{F}_1 \cap \overline{F}_2 \cap \ldots \cap \overline{F}_m). \tag{2.2}$$

The structural resistance R_j is a function of design variables A_j such as the cross-sectional area and the strength of the materials C_{yj} to be used, both of which are, in general, random variables. However, only C_{yj} are treated as random variables in this paper as the design variables. The dimensions of the structural elements are adopted and their values are assumed to be determined by the mean values of the structural resistances and the strengths of the materials, $\overline{R}_j$ and $\overline{C}_{yj}$, i.e.

$$A_j = A_j(\overline{R}_j, \overline{C}_{yj}). \tag{2.3}$$

The structural cost H_C is a function of the dimensions of the structural elements when the materials to be used are specified and thus, using Eq. (2.3), it can be written as follows:

$$H_C = H_C(\overline{R}_1, \overline{R}_2, \ldots, \overline{R}_n). \tag{2.4}$$

Now consider the case where the failure probability P_f is determined by specifying only the mean values of the structural resistances R_j when the probabilistic natures of the loads L_j are given. An example of such a case is the one where R_j are Gaussian random variables with known coefficients of variation.

Let C_f denote the cost caused by the failure of the structure. The expected total cost H_T is given by

$$H_T = H_C + C_f P_f. \tag{2.5}$$

The total cost may, in general, be affected by the support conditions [13] and the connecting members [14, 15]. However, they are not considered here for the sake of simplicity.

The problem to be considered is

PROBLEM. *Given the configuration of the structure and the materials to be used, determine the structural resistances* $\bar{R}_j$ *to minimize the expected total cost.*

By solving the problem, the optimum failure probability or, alternatively, the optimum reliability of the structure is determined together with the optimum structure.

3. Solution of problem

It is time-consuming to calculate the multi-dimensional probability distribution functions for evaluating the failure probability of the structure P_f in Eq. (2.2). Further, the probability thus evaluated is approximate using any method developed so far for calculating multi-dimensional probability distribution functions. Thus, it is desirable to make use of a search method to attain the optimum solution without employing the derivative of P_f, which requires much processing time and may result in the accumulation of errors. For this purpose consider a subproblem.

SUBPROBLEM. *With a specified probability level* P_{fa}, *determine the optimum structure* $\bar{R}_j$ *to minimize the structural cost* H_C *under the constraint*

$$P_f \leqslant P_{fa}. \tag{3.1}$$

The subproblem is the problem treated in the previous paper [10] and the outline of the algorithmic procedure solving the solution is presented in Appendix 1. For the optimum structural design problem it is shown in Appendix 2 that the solution to the subproblem is attained in general on the boundary $P_f = P_{fa}$. Consequently, the solution to the original problem is obtained by solving sequentially the subproblem. The algorithmic procedure is given as follows:

Step 1: Specify the initial value of P_{fa}.

Step 2: For the given value of P_{fa}, solve the subproblem and calculate the expected total cost corresponding to the optimum solution thus obtained. If the optimality condition for the original problem is satisfied, stop the calculation. Otherwise, go to Step 3.

Step 3: Applying the uni-dimensional search technique [12], the value of P_{fa} must be adjusted so as to minimize the expected total cost H_T. Return to Step 2.

The procedure is repeated until the optimality condition is satisfied. The flow chart for the above procedure is given in Fig. 1, and the mathematical background is given in Appendix 2.

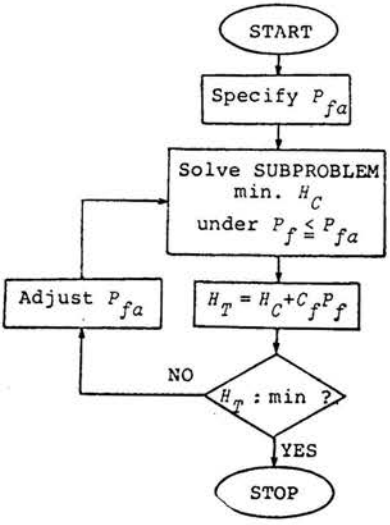

FIG. 1. Algorithmic procedure using uni-dimensional search.

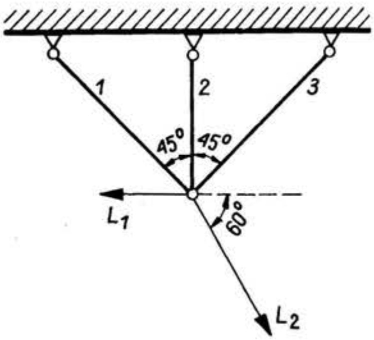

FIG. 2. 3-member truss structure.

4. Numerical example

Consider a plastic design of an indeterminate 3-member truss structure as shown in Fig. 2. The failure of the structure occurs when any two members among three collapse. Thus the following three failure modes are considered and their reserve strenghts Z_i are given as follows:

i) Members 1 and 2 collapse:

$$Z_1 = R_1 + \frac{\sqrt{2}}{2} R_2 + \frac{\sqrt{2}}{2} L_1 - \frac{\sqrt{2}(\sqrt{3}+1)}{4} L_2; \tag{4.1}$$

ii) Members 2 and 3 collapse:

$$Z_2 = \frac{\sqrt{2}}{2} R_2 + R_3 - \frac{\sqrt{2}}{2} L_1 - \frac{\sqrt{2}(\sqrt{3}-1)}{4} L_2; \tag{4.2}$$

and iii) Members 1 and 3 collapse:

$$Z_3 = \frac{\sqrt{2}}{2} R_1 + \frac{\sqrt{2}}{2} R_3 - L_1 + \frac{L_2}{2}. \tag{4.3}$$

The failure probability of the structure is calculated as follows:

$$\begin{aligned} P_f &= \mathrm{Prob}(Z_1 < 0) + \mathrm{Prob}(Z_1 \geqslant 0 \cap Z_2 < 0) + \mathrm{Prob}(Z_1 \geqslant 0 \cap Z_2 \geqslant 0 \cap Z_3 < 0) \\ &= 1 - \mathrm{Prob}(Z_1 \geqslant 0 \cap Z_2 \geqslant 0 \cap Z_3 \geqslant 0). \end{aligned} \tag{4.4}$$

In this example, the strength of the members R_j are related to their respective cross sectional area A_j and yield stress C_{yj} as follows:

(4.5) $$R_j = C_{yj}A_j.$$

The structural cost is given by

(4.6) $$H_C = \sum_{j=1}^{3} C_{mj}d_j l_j A_j,$$

where C_{mj} — material cost of the j-th member per unit weight, d_j — specific weight of the j-th member and l_j — length of the j-th member.

Consider the case where the resistances of the member R_j and the loads L_j are independent Gaussian random variables and the coefficients of variations CV_{Rj} and CV_{Lj} and the means of the loads $\bar{L}_j$ are given. Thus the failure probability of the structure is determined by specifying the mean value of the strengths $\bar{R}_j$.

For example, when the cross sectional area A_j are deterministic variables and the yield stresses C_{yj} are Gaussian random variables with the known means $\bar{C}_{yj}$ and coefficients of variations CV_{yj}, the resistances of the j-th members become Gaussian random variables with known coefficients of variations as seen from Eq. (4.5). Considering the case mentioned above, A_j are taken as $\bar{R}_j/\bar{C}_{yj}$.

Further, the reserve strengths given by Eqs. (4.1)–(4.3) become Gaussian random variables as well, and thus to evaluate the failure probability P_f three-dimensional Gaussian distribution functions need to be calculated. For this purpose the method developed in Ref. [11] and outlined in Appendix 3 is used. The data concerned are as follows:

$$\bar{C}_{yj} = 40 \text{ ksi} \quad (j = 1, 2, 3), \quad \bar{L}_1 = 100 \text{ kips}, \quad \bar{L}_2 = 150 \text{ kips},$$
$$CV_{yj} = 0.05 \quad (j = 1, 2, 3), \quad CV_{Lj} = 0.20 \quad (j = 1, 2),$$
$$l_1 = l_3 = \sqrt{2}l_2, \quad l_2 = 60 \text{ in}, \quad C_{mj}d_j = 0.03 \text{ \$/in}^3, \quad j = 1, 2, 3.$$

Figure 3 illustrates a search procedure in Steps 2 and 3 given in Section 3, using the quadratic approximation [12] for the case of $C_f = 10^3$ \$. A brief explanation of the quadratic approximation method is given in Appendix 4.

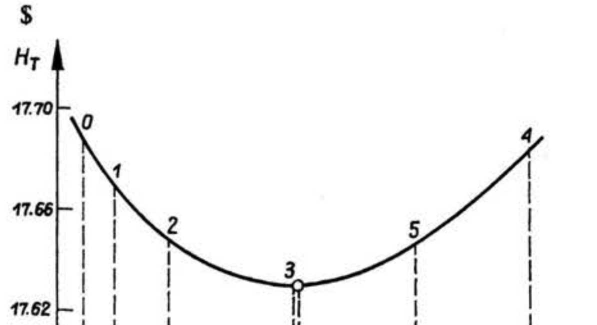

FIG. 3. A procedure searching for optimum failure probability by the use of the quadratic approximation method ($C_f = 10^3$ \$).

Table 1. Optimum solutions corresponding to various values of the failure cost C_f.

C_f \$	A_1 in²	A_2 in²	A_3 in²	P_f	H_C \$	H_T \$
10^2	1.75	3.30	1.56	9.64×10^{-3}	14.71	15.67
10^3	2.23	3.50	1.76	7.79×10^{-4}	16.85	17.63
10^4	2.63	3.67	1.92	6.79×10^{-5}	18.61	19.29
10^5	2.98	3.82	2.06	6.16×10^{-6}	20.15	20.77
10^6	3.30	3.95	2.18	5.71×10^{-7}	21.55	22.13

The optimum solutions are listed in Table 1 for various values of C_f. As the value of C_f becomes large, that is, the cost due to failure of the structure becomes large, the

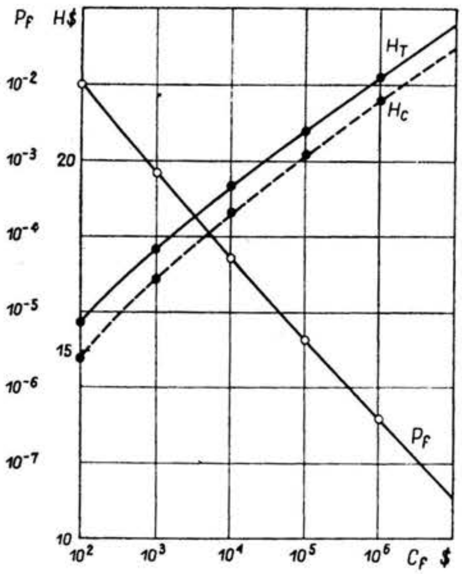

FIG. 4. Effect of failure cost C_f on optimum solution.

optimum failure probability becomes small, while the structural cost becomes large. This fact is also schematically shown in Fig. 4.

Table 2. Optimum solutions by a procedure directly calculating the gradient of H_T.

C_f \$	A_1 in²	A_2 in²	A_3 in²	P_f	H_C \$	H_T \$
10^3	2.07	3.44	1.82	1.22×10^{-3}	16.49	17.71
10^5	2.95	3.71	2.14	7.06×10^{-6}	20.10	20.81

The optimum solutions to the original problem are tentatively searched by directly calculating the gradient of the total expected cost H_T, and some results are given in Table 2. Comparing the results to those of Table 1, it is seen that the proposed algorithmic procedure gives us better solutions within the shorter processing time.

5. Conclusion

An optimum structural design problem is treated for determining simultaneously the optimum failure probability (or reliability) and structure when the cost due to the failure of the structure is given. An efficient method is proposed to solve the problem using stochastic programming and the uni-dimensional search technique. A numerical example is presented to illustrate the procedure.

Appendix 1. Algorithmic procedure for solving subproblem

For the solution of subproblem consider the subproblem:

SUBPROBLEM A. *Determine the optimum value of the structural resistances $\overline{R}_j$ to minimize the structural weight or cost H_C under the constraints on the allowable failure probability of each element or failure mode*

$$P_{fi} \leqslant P_{fai} \quad (i = 1, 2, \ldots, m),$$

where P_{fi} and P_{fai} are the failure probability and the specified allowable failure probability of mode i, respectively.

Subproblem A is a typical stochastic programming [1], [2] and reduced to the following equivalent convex programming when R_j and L_j are independent Gaussian random variables:

$$\min H_C$$

subject to

$$\sum_{j=1}^{n} a_{ij}\overline{R}_j + \sum_{j=1}^{l} b_{ij}\overline{L}_j + \psi(P_{fai})\left[\sum_{j=1}^{n} a_{ij}^2 CV_{Rj}^2 \overline{R}_j^2 + \sum_{j=1}^{l} b_{ij}^2 CV_{Lj}^2 \overline{L}_j^2\right]^{1/2} \geqslant 0,$$

where

$$\int_{-\infty}^{\psi(P_{fai})} \frac{1}{\sqrt{2\pi}} e^{-t^2} dt = P_{fai}$$

and CV_{Rj} and CV_{Lj} are the coefficients of variations of the strengths and the loads.

Thus, an algorithmic procedure is proposed to attain the optimum solution to subproblem by solving sequentially subproblem A. The procedure constists of the following steps:

Step 1: Specify the initial values of $P_{fai}^{(k)} (i = 1, 2, \ldots, m)$. For example, $P_{fai}^{(0)} = P_{fa}$, $(i = 1, 2, \ldots, m)$.

Step 2: Setting $P_{fai} = P_{fai}^{(k)}$, solve subproblem A and then calculate the failure probability of the structure using the optimum solution thus obtained. If $P_f = P_{fa}$, go to Step 4. Otherwise, go to step 3.

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(1) Y. MUROTSU, et al., *A study on stochastic nonlinear programming problems*, Trans. Japan Society of Instrument and Control Engineers, **18**, 3, 341–347, 1972.

(2) Y. MUROTSU, et al., *On a determination of allowance for control variables in stochastic linear programming problems*, Journal of Japan Association of Automatic Control Engineers, **18**, 3, 219–225, 1974.

Step 3: Change the allowable failure probabilities of the active failure modes, i.e.

$$P_{fai'}^{(k+1)} = P_{fai'}^{(k)} - \gamma_{i'}(P_f - P_{fa}) \quad \text{for} \quad i' \varepsilon \{i | P_{fi} = P_{fai}\},$$

where $\gamma_{i'}$ is a parameter to be adjusted to assure the convergence of the algorithm and satisfies the inequalities

$$\frac{P_{fai}^{(k)} - 0.5}{P_f - P_{fa}} \leqslant \gamma_{i'} < \frac{P_{fai}^{(k)}}{P_f - P_{fa}}$$

considering the condition $0 < P_{fai}^{(k+1)} \leqslant 0.5$. Go to Step 2.

Step 4: Search for the optimum solution along the boundary of the probability constraint $P_f = P_{fa}$. The following procedure is used for the search in Step 4.

Step 4-1: At a boundary point $\bar{\mathbf{R}}^0$, find a feasible direction $\mathbf{s}$ satisfying the following relations:

$$\nabla H_C \cdot \mathbf{s} \leqslant 0 \quad \text{and} \quad \nabla P_f \cdot \mathbf{s} \leqslant 0.$$

If there is a feasible direction, go to Step 4-2. Otherwise, the point $\bar{\mathbf{R}}^0$ is optimum.

Step 4-2: At a feasible point $\bar{\mathbf{R}}$: $\bar{\mathbf{R}} = \bar{\mathbf{R}}^0 + \alpha\mathbf{s}$, calculate the failure probabilities P_{fi}. α is a constant to be taken so that the point $\mathbf{R}$ may be feasible.

Step 4-3: Resetting the allowable failure probability of each failure mode such that $P_{fai} = P_{fi}$, go to Step 2. The flow chart is given in Fig. A-1.

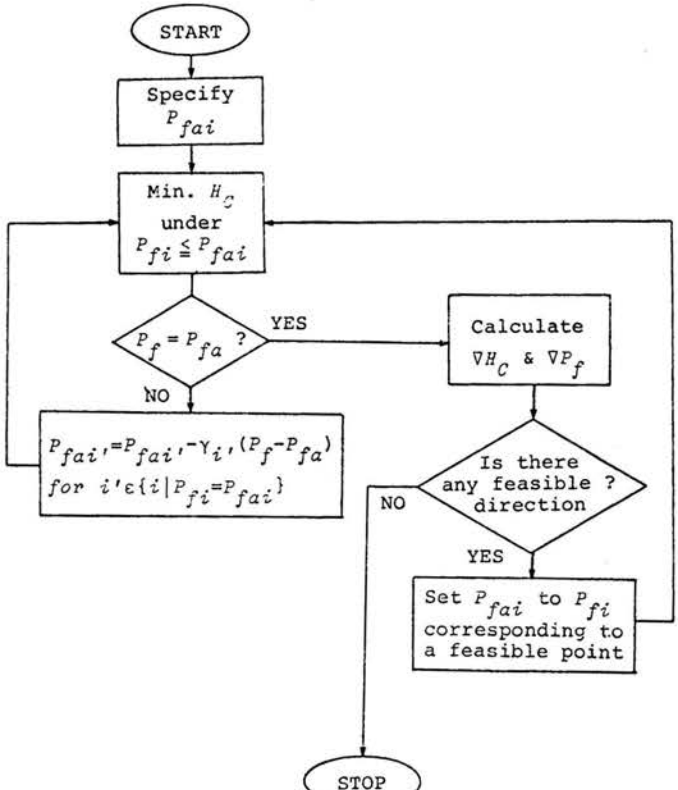

FIG. A-1. Algorithmic procedure for solving Subproblem.

Appendix 2. Mathematical background of algorithmic procedure

Let the design variables be expressed by the n-dimensional vector $\mathbf{X}$ and its design space Γ be a subspace of n-dimensional Euclidean space E^n, i.e. $\Gamma \subset E^n$. The structural cost H_C and the failure probability P_f is a function of the design vector $\mathbf{X}$ and are thus written as

$$H_C = H_C(\mathbf{X}), \quad P_f = P_f(\mathbf{X}) \quad \text{for} \quad \mathbf{X} \in \Gamma.$$

In the structural systems the structural costs increase in general as the design variables are taken to be large, while the failure probabilities decrease for the cases considered. Hence the following conditions are satisfied in general:

(C1) $H_C(\mathbf{X})$ is componentwise increasing, i.e. for some $j \in \{1, 2, \ldots, n\}$ and for any $\mathbf{X}^1$ and $\mathbf{X}^2 \in \Gamma$ such that $\mathbf{X}^2 - \mathbf{X}^1 = (X_j^2 - X_j^1)\mathbf{e}_j > \mathbf{0}$, $H_C(\mathbf{X})$ is increasing along $[\mathbf{X}^1, \mathbf{X}^2]$. $\mathbf{e}_j$ is a n-dimensional unit vector with the j-th element of unit and all others of zero.

(C2) $P_f(\mathbf{X})$ is componentwise decreasing, i.e. $P_f(\mathbf{X})$ is decreasing along $[\mathbf{X}^1, \mathbf{X}^2]$ as defined in (C1). The following lemma holds for the solution to subproblem.

LEMMA 1: The solution to subproblem is attained on the boundary of the probability constraint, i.e. $P_f(\mathbf{X}^*) = P_{fa}$.

Proof. For any vector $\mathbf{X}^1$ contained in an open set

$$G \overset{\Delta}{=} \{\mathbf{X} | P_f(\mathbf{X}) < P_{fa}\}$$

i.e. $\mathbf{X}^1 \in G$, there exists a number $\varepsilon > 0$ which defines the neighbourhood

$$O_\varepsilon(\mathbf{X}^1) \overset{\Delta}{=} \{\mathbf{X} | \, ||\mathbf{X}^1 - \mathbf{X}|| < \varepsilon\} \subset G.$$

Consider a vector $\mathbf{X}^0$ whose elements X_j^0 are identical with those of $\mathbf{X}^1$ except the i-th element, i.e.

$$X_i^0 = X_i^1 - \varepsilon/2, \quad X_j^0 = X_j^1 \quad (j = 1, 2, \ldots, n, \; j \neq i)$$

and which satisfies

$$\mathbf{X}^0 \varepsilon O_\varepsilon(\mathbf{X}^1) \subset G.$$

Considering the condition (C1), the following inequality holds between the structural costs corresponding to $\mathbf{X}^0$ and $\mathbf{X}^1$:

$$H_C(\mathbf{X}^1) > H_C(\mathbf{X}^0).$$

Consequently, $\mathbf{X}^1$ can not be an optimum to subproblem (q.e.d.).

As to the expected total cost, the following lemma holds.

LEMMA 2: When the failure probability P_f is specified to be P_{fa}, the expected total cost H_T is minimum for the solution to subproblem.

Proof. For the specified value of P_f, the second term of H_T is constant, i.e. $C_f P_f = C_f P_{fa} =$ constant. Then

$$(H_T^*)_{P_f = P_{fa}} \overset{\Delta}{=} \min_{\mathbf{X} \subset \Gamma, P_f = P_{fa}} (H_C + C_f P_f) = \min_{\mathbf{X} \subset \Gamma, P_f = P_{fa}} H_C + C_f P_{fa}$$

$$= \min_{\mathbf{X} \subset \Gamma, P_f \leqslant P_{fa}} H_C + C_f P_{fa}.$$

The last relation follows from Lemma 1 (q.e.d.).

From Lemmas 1 and 2, the following lemma concerning the solution of Problem holds.

LEMMA 3: The solution to Problem is obtained by sequentially solving subproblem.

Proof. From Lemmas 1 and 2, the following relation results:

$$\min_{\mathbf{X}\subset\Gamma} H_T = \min_{P_{fa}}(H_T^*)_{P_f-P_{fa}} = \min_{P_{fa}}\{\min_{\mathbf{X}\subset\Gamma, P_f=P_{fa}} (H_C+C_f P_f)\}$$

$$= \min_{P_{fa}}\{\min_{\mathbf{X}\subset\Gamma, P_f=P_{fa}} H_C+C_f P_f)\} = \min_{P_{fa}}\{\min_{\mathbf{X}\subset\Gamma, P_f\leqslant R_{fa}} H_C+C_f P_f\} \quad \text{(q.e.d.)}.$$

Denote the structural cost corresponding to the optimum solution of subproblem for a specified value of P_{fa} as $H_C^0(P_{fa})$ i.e.

$$H_C^0(P_{fa}) = \min_{\mathbf{X}\subset\Gamma, P_f(\mathbf{X})\leqslant P_{fa}} H_C(\mathbf{X}).$$

The following lemma holds:

LEMMA 4: $H_C^0(P_{fa})$ is a decreasing function of P_{fa}.

Proof. For the specified values of the allowable failure probability such that $P_{fa}^1 < P_{fa}^2$, consider the corresponding feasible regions:

$$G^1 \triangleq \{\mathbf{X}|P_f(\mathbf{X}) \leqslant P_{fa}^1\}, \quad G^2 \triangleq \{\mathbf{X}|P_f(\mathbf{X}) \leqslant P_{fa}^2\}.$$

The conditions (C2) yields

$$G^1 \subset G^2.$$

Hence

$$H_C^0(P_{fa}^1) > H_C^0(P_{fa}^2)$$

from Lemma 1 (q.e.d.).

Finally, it is clear from Lemma 3 that the following proposition holds concerning the algorithmic procedure for solving Problem.

PROPOSITION: The solution to Problem is obtained by the procedure given in Sect. 3, using the uni-dimensional search with respect to the allowable failure probability.

It should be remarked here that the algorithm does not always work well when the expected total cost H_T is not unimodal with respect to the allowable failure probability P_{fa}. In that case the optimization should be started from a number of initial values of P_{fa}, and search for the global minimum since the solution from any one initial value may be a local minimum.

Appendix 3. Calculation of multi-dimensional Gaussian distribution function

Let X_i and x_i $(i = 1, 2, \ldots, k)$ be the random variables and their realizations. Using Hermite polynomials, the k-dimensional probability density function $p(x_1, x_2, \ldots, x_k)$ is expanded into the following form:

$$p(x_1, x_2, \ldots, x_k)$$
$$= \left\{\prod_{i=1}^{k} \phi\left(\frac{x_i-\mu_i}{\sigma_i}\right)\right\}\left\{1 + \sum_{m_1,m_2,\ldots,m_k\geqslant 0}' A(m_1, m_2, \ldots, m_k)\cdot \prod_{i=1}^{k} H_{m_i}\left(\frac{x_i-\mu_i}{\sigma_i}\right)\right\},$$

where $\phi(x) = (1/\sqrt{2\pi})\exp(-x^2/2)$, $H_{m_i}(x)$: the m_i-th order Hermite polynomial,

$$A(m_1, m_2, \dots, m_k) = \left\langle \prod_{i=1}^{k} \frac{1}{m_i!} H_{m_i}\left(\frac{x_i-\mu_i}{\sigma_i}\right)\right\rangle,$$

$\langle(\cdot)\rangle$: mean of $(\cdot)$, μ_i, σ_i: mean, standard deviation of X, and Σ' represents the summation excluding the case where $m_1 = m_2 = \dots = m_k = 0$. In the following, the random variables X_i are used in the sense of the standardized random variables $(X_i-\mu_i)/\sigma_i$.

In the case of Gaussian distribution, the probability distribution function $P(x_1, x_2, \dots \dots, x_k)$ is expressed in the following form, using the property of Hermite polynomials.

$$P(x_1, x_2, \dots, x_k) = \sum_{N=0}^{\infty} \sum_{[m_{ij}]_N} \frac{\varrho_{12}^{m_{12}} \varrho_{13}^{m_{13}} \dots \varrho_{(k-1)k}^{m_{(k-1)k}}}{m_{12}! m_{13}! \dots m_{(k-1)k}!} \cdot \prod_{j=1}^{k} (-1) H_{m_j-1}(x_j)\phi(x_j),$$

where

$$H_{-1}(x_j)\phi(x_j) = \Phi(x_j) = \int_{-\infty}^{x_j} \phi(t)\,dt.$$

$\sum_{[m_{ij}]N}$ denotes the summation taken over all sets of non-negative values of the m_{ij} which satisfy the following relation for the given N:

(*) $$N = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} m_{ij},$$

ϱ_{ij}: correlation coefficient between X_i and X_j

$$m_j = \sum_{i=1}^{j-1} m_{ij} + \sum_{i=j+1}^{k} m_{ji}.$$

Using the above relations, the following algorithmic procedure is given for calculating the multi-dimensional Gaussian probability distribution functions taking account of the moment terms to any specified order:

Step 1. Specify the dimension (k), the order of the moment terms retained (NMT) and the value of x_i to calculate the probability distribution function.

Step 2. Set $P_0 = \prod_{j=1}^{k} \Phi(x_j)$ and $N = 0$.

Step 3. Set $N = N+1$ and perform the summation

$$\Delta P_{2N} = \sum_{[mij]_N} \frac{\varrho_{12}^{m_{12}} \varrho_{13}^{m_{13}} \dots \varrho_{(k-1)k}^{m_{(k-1)k}}}{m_{12}! m_{13}! \dots m_{(k-1)k}!} \cdot \prod_{j=1}^{k} (-1) H_{m_j-1}(x_j)\phi(x_j)$$

for all possible sets of non-negative values of the m_{ij} which satisfy Eq. (*) for the given N. Putting $P_{2N} = P_{2N-2} + \Delta P_{2N}$, go to Step 4.

Step 4. If $N = NMT$, stop the calculation. Otherwise, go to Step 3.

The flow chart is given in Fig. A-2 which illustrates the computational procedure mentioned above.

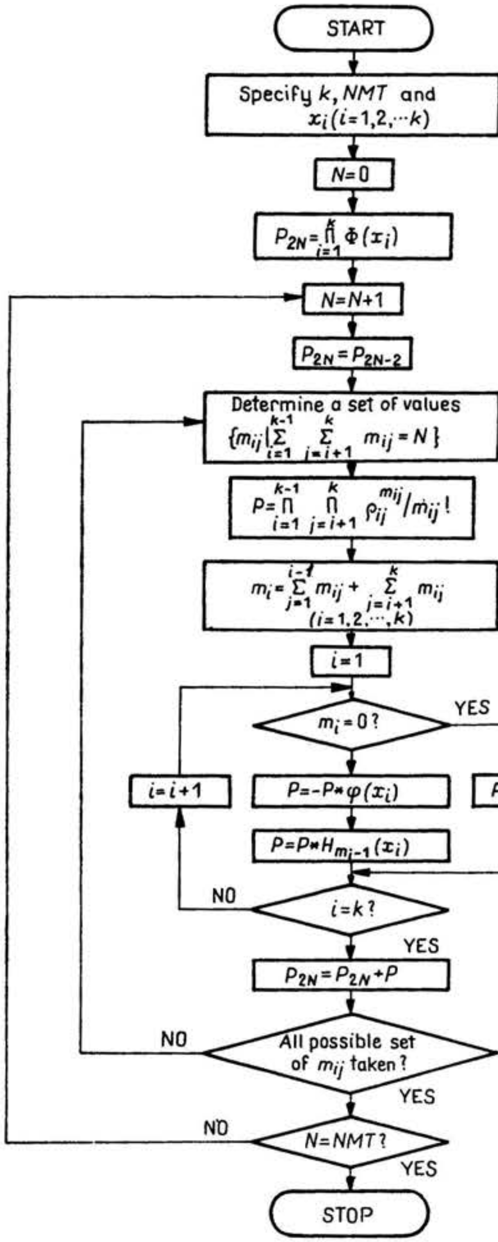

FIG. A-2. Algorithmic procedure for calculating the multi-dimensional Gaussian probability distribution function.

In order to examine the contribution of the moment terms of various order, partial sums of the series ΔP_{2N} are calculated and listed in Table A-1 for two-dimensional Gaussian probability distribution functions with variances of unit and some correlation coefficients ϱ_{12}.

Table A-1. Contribution of moment terms ΔP_{2N} $(x_1, x_2) = (1, 1)$

N \ ϱ_{12}	0.1	0.3	0.5	0.7	0.9
1	0.5854981E-02	0.1756494E-01	0.2927490E-01	0.4098487E-01	0.5269483E-01
2	0.2927491E-03	0.2634741E-02	0.7318726E-02	0.1434470E-01	0.2371267E-01
3	0.	0.	0.	0.	0.
4	0.9758305E-06	0.7904225E-04	0.6098939E-02	0.2342968E-02	0.6402421E-02
5	0.1951661E-07	0.4742535E-05	0.6098939E-04	0.3280155E-03	0.1152436E-02
6	0.2927492E-08	0.2134141E-05	0.4574204E-04	0.3444163E-03	0.1555788E-02
7	0.2973960E-09	0.6504048E-06	0.2323405E-04	0,2449183E-03	0.1422435E-02
8	0.5808517E-11	0.3810966E-07	0.2268951E-05	0.3348492E-04	0.2500374E-03
9	0.2811322E-11	0.5533523E-07	0.5490860E-05	0.1134469E-03	0.1089163E-02
10	0.1264966E-14	0.7469494E-10	0.1235318E-10	0.3573213E-06	0.4410660E-05
11	0.2168889E-13	0.3842119E-08	0.1059027E-05	0.4288598E-04	0.6806197E-03
12	0.1070880E-15	0,5691089E-10	0.2614450E-07	0.1482234E-05	0.3024477E-04
13	0.1455078E-15	0.2319863E-09	0.1776217E-06	0.1409810E-04	0.3698610E-03
14	0.3763462E-17	0.1800051E-10	0.2297033E-07	0.2552464E-05	0.8609584E-04
15	0.8532704E-18	0.1224349E-10	0.2603972E-07	0.4050949E-05	0.1756806E-03
16	0.6167317E-19	0.2654825E-11	0.9410568E-08	0.2049577E-05	0.1142815E-03
17	0.4220690E-20	0.5450601E-12	0.3220127E-08	0.9818600E-06	0.7038911E-04
18	0.7593919E-21	0.2942036E-12	0.2896846E-08	0.1236602E-05	0.1139805E-03
19	0.1578400E-22	0.1834511E-13	0.3010555E-08	0.1799199E-06	0.2132181E-04
20	0.	0.2783640E-13	0.7613563E-09	0.6370134E-06	0.9705947E-04

Table A-2. Effect of dimensions and moment terms retained on computer processing time (sec)

N \ dimension	2	3	4	6
2		0.00964	0.02632	0.08268
5	0.00715	0.02729	0.20533	7.97935
10	0.01146	0.10571	3.25794	....
15	0.01646	0.27620	21.63036	....
20	0.02214	0.58323	90.96636	....

The computer processing times for various dimensions are listed in Table A-2 against the moment terms retained to calculate the multi-dimensional probability distribution functions. The processing time becomes large as the order of the moment terms retained is increased and the dimensions become large. The computations are processed with the use of the TOSBAC-5600 Model 120 computer system at the Computer Center of the University of Osaka Prefecture.

Appendix 4. Quadratic approximation

An algorithmic procedure comprises of bracketing a region containing the minimum and carrying out the quadratic approximation by using the smallest three points:

Step 1: Evaluate $H_T(P_{fa})$ at $P_{fa}^{(0)}$ and $P_{fa}^{(0)}+h$ by solving subproblem. If $H_T(P_{fa}^{(0)}+ +h) \leqslant H_T(P_{fa}^{(0)})$, go to Step 2. Otherwise, let $h = -2h$ and go to Step 2.

Step 2: Set $P_{fa}^{(k+1)} = P_{fa}^{(k)}+h$, and evaluate $H_T(P_{fa}^{(k+1)})$ by solving subproblem. Go to Step 3.

Step 3: If $H_T(P_{fa}^{(k+1)}) \leqslant H_T(P_{fa}^{(k)})$, double h and return to Step 2 with $k = k+1$. Otherwise, denote $P_{fa}^{(k+1)}$ by $P_{fa}^{(m)}$, $P_{fa}^{(k)}$ by $P_{fa}^{(m-1)}$, and $P_{fa}^{(k-1)}$ by $P_{fa}^{(m-2)}$. Reduce h by one-half, and go to Step 4.

Step 4: Set $P_f^{(m+1)} = P_f^{(m)}-h$, and evaluate $H_T(P_{fa}^{(m+1)})$ by solving subproblem. Of the four equally-spaced values of P_{fa} in the set $(P_{fa}^{(m+1)}, P_{fa}^{(m)}, P_{fa}^{(m-1)}, P_{fa}^{(m-2)})$, discard either $P_{fa}^{(m)}$ or $P_{fa}^{(m-2)}$, whichever is farthest from the P_{fa} corresponding to the smallest value of $H_T(P_{fa})$ in the set. Let the remaining three values of P_{fa} be denoted by $P_{fa}^{(a)}$, $P_{fa}^{(b)}$, and $P_{fa}^{(c)}$, where $P_{fa}^{(b)}$ is the center point, i.e. $P_{fa}^{(a)} = P_{fa}^{(b)}-h$ and $P_{fa}^{(c)} = P_{fa}^{(b)}+h$. Go to Step 5.

Step 5: Carry out a quadratic approximation of the optimum failure probability P_{fa}^*

$$P_{fa}^* \simeq P_{fa}^{(b)} + \frac{h[H_T(P_{fa}^{(a)}) - H_T(P_{fa}^{(c)})]}{2[H_T(P_{fa}^{(a)}) - 2H_T(P_{fa}^{(b)}) + H_T(P_{fa}^{(c)})]}.$$

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